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QUOTIENTS OF RAIKOV-COMPLETE TOPOLOGICAL GROUPS (*)

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A topological group X is called Raikov-complete if the two sides uniformity on X, that is the supremum $\mathcal{L} \vee \mathcal{R}$ of the left uniformity $\mathcal{L}$ and the right uniformity $\mathcal{R}$ on $\mathcal{X}$, is complete. It will be proved that the quotient $(X/G, (\mathcal{L}/G) \vee (\mathcal{R}/G))$ of an inframetrizable Raikov-complete topological group X with a neutral subgroup G is complete. (If G is normal (and therefore neutral), the completeness of $(X/G, (\mathcal{L}/G) \vee (\mathcal{R}/G))$ is equivalent to the Raikov-completeness of the quotient group X/G). The proof consist in an intricate lifting of $(\mathcal{L}/G) \vee (\mathcal{R}/G)$-Cauchy filters. Where it is possible, the results on topological groups will be derived from results on uniform spaces.

It is well-known that quotients of complete locally convex topological vector spaces are not necessarily complete ([7]). Because of [14, Proposition 11.1] there are many examples of complete Hausdorff abelian topological groups which have incomplete Hausdorff quotients. In this paper we shall give sufficient conditions for the completeness of quotients of Raikov-complete topological groups.

A topological group X is called *Raikov-complete* if the two-sides uniformity on X, that is the supremum $\mathcal{L} \vee \mathcal{R}$ of the left uniformity $\mathcal{L}$ and the right uniformity $\mathcal{R}$ on $\mathcal{X}$, is complete.

The Russian school developed the completeness concept «absolutely closed» (cf. the survey article of Nummela [10]). A topological space is called *absolutely closed* if it is closed in every extending space. In [1] A.D. Alexandroff proved that every topological group can be extended to an absolutely closed topological group. Raikov gave a special construction of completions of topological groups and he proved that the class of Raikov-complete topological groups is identical with the class of absolutely closed topological groups [13].

Because of [10, Theorem 5], the class of absolutely closed topological groups is equal to the class of Raikov-complete topological groups.

In this paper we shall use the concept of inframetrizable uniform spaces, which was introduced in [14, Definition 11.13].

A uniform space $(X, \mathcal{V})$ is called *inframetrizable* if there exists a pseudometrizable uniformity $\mathcal{V}_0 \subset \mathcal{V}$ such that

$$\forall M \in \mathcal{V} \, \exists N \in \mathcal{V}_0 \forall x \in X \, \exists E \subset X, E \text{ finite} : N[x] \subset M[E].$$

A topological group X is called inframetrizable if $(X, \mathcal{L})$, or equiv. $(X, \mathcal{R})$, is inframetrizable.

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(*) Proofs not corrected by the author.

Obviously, every uniformly locally precompact uniform space and every pseudometrizable uniform space is inframetrizable.

In the proof of Theorem 2 we shall need the following fact about inframetrizable uniform spaces (which is easy to prove): Every $\mathcal{V}_0$-Cauchy filter which is an ultrafilter is a $\mathcal{V}$-Cauchy filter ($\mathcal{V}_0$ and $\mathcal{V}$ as above).

Inframetrizable uniform space are closely related to hypometrizable uniform spaces and to almost metrizable topological spaces.

Hypometrizable uniform spaces were introduced by Kholodovskii [5] (under the name of «almost metrizable») to study the completeness of quotients of uniform spaces. Every hypometrizable uniform space is inframetrizable, but it is not known whether the converse holds (cf. [14, Remark after Definition 11.13]).

A topological space X is called *almost metrizable* if there exists a compact subset K of X which has a countable basis of neighbourhoods (Pasynkov [11, p. 404]).

A topological group X is almost metrizable if and only if there exists a compact subgroup K of X such that X/K is metrizable [14, Remark 13.19].

By [14, Proposition 13.20], every almost metrizable topological group is inframetrizable; conversely, by [14, Remark after 13.14], a topological group X is almost metrizable if it is inframetrizable and Raikov-complete.

A topological space is called Čech-complete if it is a G_δ-subset (i.e. a contable intersection of open subsets) of a compact Hausdorff space. Čech-complete topological groups are closely related to Raikov-complete topological groups. By [4] (or by [3, Theorem 1 and Corollary 3]) a Hausdorff topological group X is Čech-complete (which is the same as «topologically complete» in the terminology of Brown [3] and Pettis [12]) if and only if it is Raikov-complete and almost metrizable. If G is a closed subgroup of a Čech-complete topological group X, then, by [3, Theorem 2], the quotient X/G is also Čech-complete. From these statements the following theorem can easily be deduced (cf. [14, 13.34(b)] or [12, statement (H'), p. 39]):

(*) *If X is an almost metrizable Raikov-complete topological group and if G is a normal subgroup of X, then the quotient group X/G is also Raikov-complete.*

In [14, Theorem 11.21] we find another theorem about the completeness of quotients of Raikov-complete topological groups:

(**) *If X is an inframetrizable Raikov-complete topological group and if G is a neutral subgroup of X which has a group completion, then the quotient $(X/G, (\mathcal{L}/G) \vee (\mathfrak{R}/G))$ is complete.*

(*Neutral:* see below; every normal subgroup is neutral.

Note that if G is a normal subgroup of X, the completeness of $(X/G,(\mathcal{L}/G)\vee(\mathcal{R}/G))$ is equivalent to the Raikov-completeness of the quotient group X/G.)
The proof of (**) is based on a lifting property of $(\mathcal{L}/G)\vee(\mathcal{R}/G)$-Cauchy filters on X/G, which in turn is derived from the general theory of $\mathcal{L}/G-$(resp. $\mathcal{R}|G-$) Cauchy filters on X/G.

In this paper we shall give a generalization of both theorem (*) and (**) (by deleting «which has a group completion» in Theorem (**)). As a first step, Theorem 1 will treat the case of a pseudometrizable topological group; we shall present a more direct proof using a special lifting of $(\mathcal{L}/G)\vee(\mathcal{R}/G)$ Cauchy sequences (this answers the open question found in [14, remark after 13.34]). To further refine this technique we shall investigate in Theorem 2 the completeness of the quotient $(X/R,(\mathcal{V}/R)\vee(\mathcal{W}/R))$, where $\mathcal{V}$ and $\mathcal{W}$ are inframetrizable uniformities on a set X, and R is an equivalence relation on X. As a corollary of this theorem, we get the desired improvement of Theorems (*) and (**) in Theorem 3.

Our notation of group multiplication is multiplicative. The filter of neighbourhoods of the neutral element e of a topological group X is denoted by $\mathcal{U}_e(X)$. $\mathbb{N}$ is defined as the set $\{1,2,3,\ldots\}$.

We shall use the following compatibility concept between a uniformity and an equivalence relation (cf. [14, Lemma-definition 4.10]):

An equivalence relation R on a uniform space $(X,\mathcal{V})$ is called *compatible* with $\mathcal{V}$ if

$$\forall M\in\mathcal{V}\,\exists N\in\mathcal{V}: R\circ N\subset M\circ R.$$

In this case, by [14, 4.10], the sets $(q\times q)(M)(=\{(s,t)\in(X/R)\times(X/R):t\in q(M[s])\})(M\in\mathcal{V})$ form a basis of $\mathcal{V}/R$.

Let X be a topological group and G a subgroup of X. By [14, Theorem 5.21], the right uniformity $\mathcal{R}$ on X is alxais compatible with the equivalence relation $R_G:=\{(x,y)\in X\times X: x^{-1}y\in G\}$. The left uniformity $\mathcal{L}$ on X is compatible with R_G iff

$$\forall U\in\mathcal{U}_e(X)\,\exists V\in\mathcal{U}_e(X): GV\subset UG;$$

in this case G is called a *neutral* subgroup of X (cf. [14, Definition 5.29]). For example every normal, every open, and every compact subgroup of a topological group is neutral. In general R_G is not compatible with the supremum $\mathcal{L}\vee\mathcal{R}$ even if G is a normal subgroup of a Lie group (cf. [14, Example 6.21]). If R_G is compatible with $\mathcal{L}\vee\mathcal{R}$, then G is neutral (by an unpublished complement to [14, Proposition 5.31(b)] by W. Roelcke (see appendix)). Further information about neutral subgroups may be found in [9].

Let $(X,\mathcal{V})$ be a uniform space, R an equivalence relation on X, and $q: X\to X/R$ the quotient map. We say *a sequence* $(s_n)_{n\in\mathbb{N}}$ *in* X/R *can be lifted w.r.t.* q *to a* $\mathcal{V}$*-Cauchy sequence* $(x_n)_{x\in\mathbb{N}}$ if there exists a $\mathcal{V}$-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_n\in s_n$ for all $n\in\mathbb{N}$. We say *a filter* G *on* X/R *can be lifted w.r.t.* q *to a* $\mathcal{V}$*-Cauchy filter* $\mathcal{F}$ if there exists a $\mathcal{V}$-Cauchy filter $\mathcal{F}$ on X such that $q(\mathcal{F})\supset\mathcal{G}$.

Theorem 1. *Let X be a pseudometrizable topological group and G a neutral subgroup of X. Then every $(\mathcal{L}/G) \vee (\mathcal{R}/G)$-Cauchy sequence in X/G can be lifted w.r.t. the quotient map $q : X \to X/G$ to an $\mathcal{L} \vee \mathcal{R}$-Cauchy sequence in X. Consequently (cf. [14, Remark after 11.7]), if X is Raikov-complete, then $(X/G, (\mathcal{L}/G) \vee (\mathcal{R}/G))$ is complete.*

Remark. The Remark after Lemma 4 shows that Theorem 1 can be deduced from Theorem 3. As for Theorem 3, it is a Corollary of Theorem 2, which is proved indipendently from Theorem 1.

We shall now give a direct proof of theorem 1 which illustrates lucidly the special lifting technique for $(\mathcal{L}/G) \vee (\mathcal{R}/G)$-Cauchy sequences.

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be an $(\mathcal{L}/G) \vee (\mathcal{R}/G)$-Cauchy sequence in X/G and let $(U_n)_{n\in\mathbb{N}}$ be a basis of $\mathcal{U}_e(X)$ such that

(1) $U_{n+1}^2 \subset U_n (n \in \mathbb{N})$.

By induction, we shall construct sequences $(V_n)_{n\in\mathbb{N}}, (W_n)_{n\in\mathbb{N}}$ in $\mathcal{U}_e(X), (k_n)_{n\in\mathbb{N}}$, $(l_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, and $(x_{k_n})_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ in X such that we have for every $n \in \mathbb{N}$:

$(2)_n$ $V_n, W_n \subset U_n$, $x_{k_n} \in s_{k_n}$, $k_n < l_n \leq k_{n+1}$,

$(3)_n$ $x_{k_n} \in z_{n-1} V_{n-1}$ if $n \geq 2$,

$(4)_n$ $z_n \in W_n x_{k_n}$,

$(5)_n$ $W_n z_{n-1} \subset z_{n-1} U_n$ if $n \geq 2$,

$(6)_n$ $x_{k_n} V_n \subset U_n x_{k_n}$,

$(7)_n$ $s_i \in z_n . q(V_n)$ $(i \geq l_n)$

(with the operation $. : X \times (X/G) \to X/G, x.q(y) := q(xy))$.

Induction. Let $n \in \mathbb{N}$ and assume that $V_1, \ldots, V_{n_1}, W_1, \ldots, W_{n-1}, k_1, \ldots, k_{n-1}, l_1, \ldots, l_{n-1}$, $x_{k_1}, \ldots, x_{k_{n-1}}$ and $z_1, \ldots, z_{n-1}$ have already been constructed satisfyng $(2)_i, \ldots, (7)_i$ for $i < n$. Choose $W_n \in \mathcal{U}_e(X), W_n \subset U_n$, such that $(5)_n$ is satisfied. Since $(s_n)_{n\in\mathbb{N}}$ is (in particular) an $\mathcal{R}/G$-Cauchy sequence and R_G is compatible with $\mathcal{R}$, there exists $k_n \in \mathbb{N}$, with $k_n \geq l_{n-1}$ if $n \geq 2$, such that

(8) $s_i \in W_n . s_{k_n}$ $(i \geq k_n)$.

If $n = 1$, then let x_{k_1} be an arbitrary element of s_{k_1}. If $n > 1$, we can choose by $(7)_{n-1}$ and $k_n \geq l_{n-1}$ an element $k_{k_n} \in z_{n-1} V_{n-1}$. Thus $(3)_n$ is satisfied. Next xe choose $V_n \in \mathcal{U}_e(X), V_n \subset U_n$, such that $(6)_n$ holds. As G is a neutral subgroup of X, there exists $V \in \mathcal{U}_e(X)$ satisfyng

(9) $GV \subset V_n G$.

Since $(s_n)_{n\in\mathbb{N}}$ is an $\mathcal{L}/G$-Cauchy filter and G is neutral in X, we can find $l_n > k_n$ such that

(10) $s_i \in s_{l_n}.q(V)\ (i \geq l_n)$.

Because of (8), $x_{k_n} \in s_{k_n}$ and $l_n \geq k_n$, we can find $z_n \in s_{l_n}$ such that $z_n \in W_n x_{k_n}$; thus $(4)_n$ is satisfied. Since $s_i \overset{(10)}{\subset} z_n GVG \overset{(9)}{\subset} z_n V_n G$ for $i \geq l_n$, the relation $(7)_n$ is also satisfied.

For $n \in \mathbb{N}$ we have

$x_{k_n+1} \overset{(3,4)}{\in} W_n x_{k_n} V_n \overset{(6,2)}{\subset} U_n^2 x_{k_n}$,

$z_{n+1} \overset{(4,3)}{\in} W_{n+1} z_n V_n \overset{(5,2)}{\subset} z_n U_n^2$.

By (1), this implies that for $n \in \mathbb{N}$

(11) $x_{k_i} \in U_n^4 x_{k_n}\ (i \geq n)$,

(12) $z_i \in z_n U_n^4\ (i \geq n)$.

For $i < k_2, i \neq k_1$, let x_i be an arbitrary element of s_i. Now let $i \in \mathbb{N}$ be such that there exists $n \geq 2$ with $k_n < i < k_{n+1}$. Because of $(7)_{n-1}$ and $i \overset{(2)}{>} l_{n-1}$, we can find $x_i \in s_i$ such that $x_i \in z_{n-1} V_{n-1}$. Altogether (cf. (2), (3)), we thus have defined a sequence $x_i \in s_i (i \in \mathbb{N})$ in X such that

(13) $x_i \in z_{n-1} V_{n-1}$ if $n \geq 2$ and $k_n \leq i < k_{n+1}$.

We complete the proof by showing that $(x_n)_{n \in \mathbb{N}}$ is an $\mathcal{L} \vee \mathcal{R}$-Cauchy sequence: Let $n \geq 2$ and $i \geq k_{n+1}$. Choose $m > n$ such that $k_m \leq i < k_{m+1}$; then we have

$x_i \overset{(13)}{\in} z_{m-1} V_{m-1} \overset{(12,2)}{\subset} z_n U_n^5$ and

$x_i \overset{(13)}{\in} z_{m-1} V_{m-1} \overset{(4)}{\subset} W_{m-1} x_{k_{m-1}} V_{m-1} \overset{(6,12,2)}{\subset} U_n^6 x_{k_n}$.

Since $(U_n)_{n \in \mathbb{N}}$ is a basis of $\mathcal{U}_e(X)$, this implies that $(x_n)_{x \in \mathbb{N}}$ is an $\mathcal{L}$- and $\mathcal{R}$-Cauchy sequence. ∎

To improve on Theorem 1 we first need some definitions.

Let X be a set and $\mathcal{V}, \mathcal{W}$ uniformities on X. We say *the pair*$(\mathcal{V}, \mathcal{W})$ *satisfies condition* (B) if

$$(B)\ \forall M \in \mathcal{V}\ \exists M_0 \in \mathcal{V} \forall a \in X\ \exists N \in \mathcal{W} : N[M_0[a]] \subset M[a].$$

(This conditions was introduced in [14, Definition 9.10] to study precompactness and completions of topological groups).

If X is a topological group and G a neutral subgroup of X, then, by [14, Lemma 9.20], the pairs $(\mathcal{L}/G, \mathcal{R}/G)$ and $(\mathcal{R}/G, \mathcal{L}/G)$ (and in particular, the pairs $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{R}, \mathcal{L})$) satisfy condition (B).

A Cauchy filter on a uniform space $(X, \mathcal{V})$ is called *minimal* if there does not exist a strictly coarser Cauchy filter on $(X, \mathcal{V})$. If $\mathcal{F}$ is a $\mathcal{V}$-Cauchy filter on X, then by [2, II,§3.2, Proposition 5] $\{N[F] : N \in \mathcal{V}, F \in \mathcal{F}\}$ is the unique minimal Cauchy filter $\mathcal{F}_0$ on X such that $\mathcal{F}_0 \subset \mathcal{F}$.

Refining the lifting technique of Cauchy sequences in the proof of Theorem 1, we can show the following theorem:

Theorem 2. *Let X be a set and let $\mathcal{V}, \mathcal{W}$ be inframetrizable uniformities on X such that $(\mathcal{V}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{V})$ satisfy condition (B). Let R be an equivalence relation on X which is compatible with $\mathcal{V}$ and $\mathcal{W}$. Then every $(\mathcal{V}/R) \vee (\mathcal{W}/R)$-Cauchy filter $\mathcal{G}$ on X/R can be lifted (w.r.t. the quotient map $q : X \to X/R$) to a $\mathcal{V} \vee \mathcal{W}$Cauchy filter $\mathcal{F}$ on X; in particular, if $(X, \mathcal{V} \vee \mathcal{W})$ is complete, then also $(X/R, (\mathcal{V}/R) \vee (\mathcal{W}4\, R))$ is complete.*

Proof. Let $\mathcal{V}_0 \subset \mathcal{V}$ (resp. $\mathcal{W}_0 \subset \mathcal{W}$) be a pseudometrizable uniformity according to the definition of the inframetrizability of $\mathcal{V}$ (resp. $\mathcal{W}$). Let $(\hat{M}_n)_{n\in\mathbb{N}}$ be a basis of $\mathcal{V}_0$ and $(P_n)_{n\in\mathbb{N}}$ a basis of $\mathcal{W}_0$. By induction, we shall define sequences $(M_n)_{n\in\mathbb{N}}$ in $\mathcal{V}$, $(\hat{N}_n)_{n\in\mathbb{N}}, (N_n)_{n\in\mathbb{N}}$ in $\mathcal{W}$, $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$, $(x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ in X such that we have for every $n \in \mathbb{N}$:

$(1)_n$ $\hat{N}_n^2 \subset \hat{N}_{n-1}$ if $n \geq 2$,
$(2)_n$ $q(x_n) \in A_n$, $N_n \subset \hat{N}_n \subset P_n$,
$(3)_n$ $x_n \in M_{n-1}[z_{n-1}]$ if $n \geq 2$,
$(4)_n$ $z_n \in N_n[x_n]$,
$(5)_n$ $\hat{N}_n^2[M_{n-1}[z_{n-1}]] \subset \hat{M}_n[z_{n-1}]$ if $n \geq 2$,
$(6)_n$ $M_n[N_n[x_n]] \subset \hat{N}_n[x_n]$,
$(7)_n$ $B_n \subset q(M_n[z_n])$,
$(8)_n$ $A_n \subset q(N_n[x_n])$,
$(9)_n$ $\forall x \in X\, \exists N \in \mathcal{W} : N[M_n[x]] \subset \hat{M}_{n+1}[x]$.

Induction. Let $n \in \mathbb{N}$ and assume that $M_1, \ldots, {}_{n-1}, \hat{N}_1, \ldots, \hat{N}_{n-1}, N_1, \ldots, N_{n-1}$, $A_1, \ldots, A_{n-1}, B_1, \ldots, B_{n-1}, x_1, \ldots, x_{n-1}, z_1, \ldots, z_{n-1}$ have already been constructed satisfyng the conditions above. If $n = 1$, define $\hat{N}_1 := P_1$. If $n \geq 2$, then, by $(9)_{n-1}$, we can choose $\hat{N}_n \in \mathcal{W}, \hat{N}_n \subset P_n$, such that $(1)_n$ and $(5)_n$ hold. Since $(\mathcal{W}, \mathcal{V})$ sastisfies condition (B), we can choose $N_n \in \mathcal{W}, N_n \subset \hat{N}_n$, such that

(10) $\forall x \in X\, \exists M \in \mathcal{V} : M[N_n[x]] \subset \hat{N}_n[x]$.

Ar R is compatible with $\mathcal{W}$, there exists $N \in \mathcal{W}$ such that

(11) $N \circ R \subset R \circ N_n$.

Since $\mathcal{G}$ is (in particular) a $\mathcal{W}/R$-Cauchy filter and since R is compatible with $\mathcal{W}$, there exists $A_n \in \mathcal{G}$ such that

(12) $A_n \subset q(N[a])\,(a \in A_n)$.

If $n = 1$, let x_1 be an arbitrary element in X such that $q(x_1) \in A_1$. If $n > 1$, we can choose by $(7)_{n-1}$ an element $x_n \in X$ such that $q(x_n) \in B_{n-1} \cap A_n$ and that $(3)_n$ is satisfied. By (10) and since $(\mathcal{V}, \mathcal{W})$ satisfies condition (B), we can choose $M_n \in \mathcal{V}, M_n \subset \hat{M}_n$, such that $(6)_n$ and $(9)_n$ are satisfied. As R is compatible with $\mathcal{V}$, there exists $M \in \mathcal{V}$ satisfyng

(13) $M \circ R \subset R \circ M_n$.

Since $\mathcal{G}$ is a $\mathcal{V}/R$-Cauchy filter and R is compatible with $\mathcal{V}$, we can find $B_n \in \mathcal{G}$ such that

(14) $B_n \subset q(M[b])\,(b \in B_n)$.

Because of $A_n \overset{(12)}{\subset} q(N[q(x_n)]) \overset{(11)}{\subset} q(N_n[x_n])$, condition $(8)_n$ is satisfied; therefore we can find $z_n \in X$ such that $q(z_n) \in A_n \cap B_n$ and $(4)_n$ is satisfied. Since $B_n \overset{(14)}{\subset} q(M[q(z_n)]) \overset{(13)}{\subset} q(M_n[z_n])$, inclusion $(7)_n$ is also satisfied.

For $n \in \mathbb{N}$ we have

$$\hat{N}_{n+1}^2[x_n + 1] \overset{(3,4)}{\subset} \hat{N}_{n+1}^2[M_n[N_n[x_n]]] \overset{(6)}{\subset} \hat{N}_{n+1}^2[\hat{N}_n[x_n]] \overset{(1)}{\subset} \hat{N}_n^2[x_n],$$

$$\hat{N}_{n+1}^2 \overset{(3)}{\subset} \hat{N}_{n+1}^2[M_n[z_n]] \overset{(5)}{\subset} \hat{M}_{n+1}[z_n].$$

Since $(\hat{M}_n)_{n\in\mathbb{N}}$ (resp. $(P_n)_{n\in\mathbb{N}}$) is a basis of $\mathcal{V}_0$ (resp. $\mathcal{W}$) and $\hat{N}_n \overset{(2)}{\subset} P_n (n \in \mathbb{N})$, this implies that the set $\{\hat{N}_n^2[x_n] : n \in \mathbb{N}\}$ forms a basis of a $\mathcal{V}_0 \vee \mathcal{W}_0$-Cauchy filter. For $A \in \mathcal{G}$ and $n \in \mathbb{N}$ we get, by $(8)_n$ and $N_n \overset{(2)}{\subset} \hat{N}_n^2$, that $A \cap q(\hat{N}_n^2[x_n]) \neq 0$ and therefore that $\overset{-1}{q}(A) \cap \hat{N}_n^2[x_n] \neq 0$. Thus the set $\{\overset{-1}{q}(A) \cap \hat{N}_n^2[x_n] : A \in \mathcal{G}, n \in \mathbb{N}\}$ forms a basis of a $\mathcal{V}_0 \vee \mathcal{W}_0$-Cauchy filter $\mathcal{H}$ on X such that $q(\mathcal{H}) \supset \mathcal{G}$. Let $\mathcal{F} \supset \mathcal{H}$ be an ultrafilter on X. By a remark after the definition of 'inframetrizable', $\mathcal{F}$ is a $\mathcal{V} \vee \mathcal{W}$-Cauchy filter; trivially we have $q(\mathcal{F}) \supset \mathcal{G}$. ∎

Theorem 2, applied to the special case of topological groups, yelds:

Theorem 3. *Let X be a topological group and let D and G be neutral subgroups of X such that $D \subset G$ and such that $\mathcal{L}/D$ and $\mathcal{R}/D$ are inframetrizable*[1]*. Then every $(\mathcal{L}/G) \vee$*

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[1] If $D = \{e\}$, then the inframetrizability of $\mathcal{L}/D(= \mathcal{L})$ and $\mathcal{R}/D(= \mathcal{R})$ is equivalent (as mentioned after the definition of «inframetrizable»). In [14, Example 11.10] we can find an example of a topological group and an open (and therefore neutral) subgroup G such that $\mathcal{L}/G$ is discrete, but $\mathcal{R}/G$ is not inframetrizable.

$(\mathcal{R}/G)$-Cauchy filter $\mathcal{G}$ on X/G can be lifted (w.r.t. the canonical map $p : X/D \to X/G$) to an $(\mathcal{L}/D) \vee (\mathcal{R}/D)$-Cauchy filter $\mathcal{F}$ on X/D; in particular, if $(X/D, (\mathcal{L}/D) \vee (\mathcal{R}/D))$ is complete, then also $(X/G, (\mathcal{L}/G) \vee (\mathcal{R}/G))$ is complete.

Proof. Notice that $(\mathcal{L}/D, \mathcal{R}/D)$ and $(\mathcal{R}/D, \mathcal{L}/D)$ satisfy condition (B), and that the equivalence relation $R := \{(s,t) \in (X/D) \times (X/D) : t \subset sG\}$ on X/D is compatible with $\mathcal{L}/D$ and $\mathcal{R}/D$ (by [14, Remark (2) after 4.10]),and that $((X/D)/R, (\mathcal{L}/D)/R)$ (resp. $((X/D)/R, (\mathcal{R}/D)/R)$) is uniformly equivalent to $(X/G, \mathcal{L}/G)$ (resp. $(X/G, \mathcal{R}/G)$).

■

Remarks. (a) By [14, Remark after 11.21], the completeness of the quotient $(X/G, (\mathcal{L}/G) \vee (\mathcal{R}/G))$ (where G is an arbitrary subgroup of a topological group X) implies the completeness of $(X/G, (\mathcal{L} \vee \mathcal{R})/G)$.

(b) The following example (from W. Roelcke) shows that Theorem 1 and Theorem 3 does not hold for arbitrary (not necessarily neutral) subgroups of X:

Let X be a metrizable Raikov-complete topological group such that the infimum $\mathcal{L} \vee \mathcal{R}$ is not complete (cf. [14, Example 8.14(b)]). Then $Y := X \times X$ is also a metrizable Raikov-complete topological group. Let $\Delta := \{(x,x) \in Y : x \in X\}$; then the quptient $(Y/\Delta, (\mathcal{L} \vee \mathcal{R})/\Delta)$ (and therefore by (a) the quotient $(Y/\Delta, (\mathcal{L}/\Delta) \vee (\mathcal{R}/\Delta)))$ is not complete, because it is, by an unpublished complement to [14, Lemma 5.24(c)] by W. Roelcke (see appendix), uniformly equivalent to $(X, \mathcal{L} \wedge \mathcal{R})$.

The following lemma treats the relationship between the lifting of sequences and the lifting of filters. It permits the deduction of lifting properties of sequences from lifting properties of filters.

Lemma 4. *Let $(X, \mathcal{V})$ be an uniform space, R an equivalence relation on X and $q : X \to X/R$ the quotient map.*

(a) If R is compatible with $\mathcal{V}$, then for every $\mathcal{V}/R$ Cauchy filter $\mathcal{G}$ on X/R which can be lifted to a $\mathcal{V}$-Cauchy filter $\mathcal{F}$ on X there exists a $\mathcal{V}$-Cauchy filter $\mathcal{H} \subset \mathcal{F}$ on X with $q(\mathcal{H}) = \mathcal{G}$.

(b) If $\mathcal{V}$ is pseudometrizable and if $(s_n)_{n \in \mathbb{N}}$ is a sequence in X/R such that the filter $\mathcal{G}$ canonically generated by this sequence can be lifted to a $\mathcal{V}$-Cauchy filter $\mathcal{F}$ on X with $q(\mathcal{F}) = \mathcal{G}$, then the sequence $(s_n)_{n \in \mathbb{N}}$ can be lifted to a $\mathcal{V}$ Cauchy sequence $(x_n)_{n \in \mathbb{N}}$. (In addition, it can be achieved that the filter generated by $(x_n)_{x \in \mathbb{N}}$ is finer than the minimal $\mathcal{V}$-Cauchy filter belonging to $\mathcal{F}$).

Proof. (a) Because of $q(\mathcal{F}) \supset \mathcal{G}$, the set $\{M[A] \cap \overset{-1}{q}(B) : M \in \mathcal{V}, A \in \mathcal{F}, B \in \mathcal{G}\}$ forms a basis of a filter $\mathcal{H}$ on X, which is coarser than $\mathcal{F}$. $\mathcal{H}$ is a $\mathcal{V}$-Cauchy filter since it is finer than the minimal $\mathcal{V}$-Cauchy filter belonging to $\mathcal{F}$. Obviously we have $q(\mathcal{H}) \supset \mathcal{G}$. To show

'$\subset$' let $M \in \mathcal{V}, A \in \mathcal{F}$ and $B \in \mathcal{G}$. Since Gc is a $\mathcal{V}/R$-Cauchy filter and R is compatible with $\mathcal{V}$, there exists $C \in \mathcal{G}$ such that $q(M[x]) \in \mathcal{G}$ for all $x \in \overset{-1}{q}(C)$. Let $a \in \overset{-1}{q}(C) \cap A$; then we get $q(M[A] \cap \overset{-1}{q}(B)) = q(M[A]) \cap B \supset q(M[a]) \cap B \in \mathcal{G}$.

(b) Let $(M_n)_{n\in\mathbb{N}}$ be a basis of $\mathcal{V}$. The sets $E(n) := \{s_i : i \geq n\}(n \in \mathbb{N})$ form a basis of $\mathcal{G}$. It is easy to see that there exists a decreasing sequence $A_m \in \mathcal{F}(m \in \mathbb{N})$ and a strictly increasing sequence $k_m \in \mathbb{N}(m \in \mathbb{N})$ such that

(1) $q(A_m) = E(k_m)(m \in \mathbb{N})$ and

(2) $A_m \times A_m \subset M_m(m \in \mathbb{N})$.

We shall define now the sequence $(x_n)_{n\in\mathbb{N}}$: if $n < k_1$, let x_n be an arbitrary element of s_n. If $n \geq k_1$, then there exists $m \in \mathbb{N}$ with $k_m \leq n < k_{m+1}$; by (1) one can choose $x_n \in A_m$ such that $q(x_n) = s_n$. Obviously the filter $\mathcal{H}$ canonically generated by yhe sequence $(x_n)_{n\in\mathbb{N}}$ is finer than the filter $\mathcal{I}$ generated by the sets $A_m(m \in \mathbb{N})$; by (2), this implies that $(x_n)_{n\in\mathbb{N}}$ is a $\mathcal{V}$-Cauchy sequence. Since $\mathcal{I}$ is a $\mathcal{V}$-Cauchy filter which is coarser than $\mathcal{F}$, the sequence $(x_n)_{n\in\mathbb{N}}$ has the required property. ■

Remark. Using Lemma 4, we can get Theorem 1 as a corollary of Theorem 3:

Let $(s_n)_{n\in\mathbb{N}}$ be an $(\mathcal{L}/G) \vee (\mathcal{R}/G)$-Cauchy sequence. By Theorem 3, the $(\mathcal{L}/G) \vee (\mathcal{R}/G)$-Cauchy filter $\mathcal{G}$ belonging to $(s_n)_{n\in\mathbb{N}}$ can be lifted to an $\mathcal{L} \vee \mathcal{R}$-Cauchy filter $\mathcal{F}$ on X. Because of Lemma 4(a) (with $\mathcal{V} = \mathcal{R}$), there exists an $\mathbf{R}$-Cauchy filter $\mathcal{H} \subset \mathcal{F}$ such that $q(\mathcal{H}) = \mathcal{G}$. Since $\mathcal{H}$ is an $\mathcal{R}$-Cauchy filter which is coarser than an $\mathcal{L} \vee \mathcal{R}$-Cauchy filter, $\mathcal{H}$ is even an $\mathcal{L} \vee \mathcal{R}$Cauchy filter (cf. [14, Lemma 10.5(a) in conjunction with Lemma 9.10]). Since, by assumption, the underlyng topological group X is pseudometrizable, Lemma 4(b) implies that $(s_n)_{n\in\mathbb{N}}$ can be lifted to an $\mathcal{L} \vee \mathcal{R}$-Cauchy sequence. ■

Kholodovskii proved in [6, Theorem 7] the following generalisation of Theorem (*) of L.G. Brown (cf. the beginning of the paper).

(***) *Quotients of Hausdorff Raikov-complete topological groups modulo a Čech-complete normal subgroup are Raikov-complete again.*

We do *not* know if the following generalisation of both Theorem (***) and Theorem 3 is True:

Conjecture. *If X is a Raikov-complete topological group and G is a Čech-complete neutral subgroup of X, then the quotient $(X/G, (\mathcal{L}/G) \vee (\mathcal{R}/G))$ is complete.*

APPENDIX

Complement by W. Roelcke to [14, Proposition 5.31(b)]:

Let X be a topological group and G a subgroup of X such that $\mathcal{L} \vee \mathcal{R}$ is compatible with the equivalence relation $R_G := \{(x,y) \in X \times X : x^{-1}y \in G\}$. Then G is neutral.

Proof. The sets $U_{\mathcal{L}\vee\mathcal{R}} := \{(x,y) \in X \times X : y \in xU \cap Ux\}$ with $U \in \mathcal{U}_e(X)$ form a basis of $\mathcal{L} \vee \mathcal{R}$. Therefore, by [14, Lemma 4.10(ii)(c)], the hypothesis of the complement means that

$$\forall U \in \mathcal{U}_e(X)\, \exists V \in \mathcal{U}_e(X) : R_G \circ V_{\mathcal{L}\vee\mathcal{R}} \subset U_{\mathcal{L}\vee\mathcal{R}} \circ R_G, i.e.,$$

$$\forall U \in \mathcal{U}_e(X)\, \exists V \in \mathcal{U}_e(X) \forall x \in X : (xV \cap Vx)G \subset \bigcup\{xgU \cap Uxg : g \in G\}.$$

For $x = e$ this implies $VG \subset GU$, as desired. ∎

Complement by W. Roelcke to [14, Lemma 5.24(c)]:

Let Y be a group, $\Delta := \{(x,y) \in Y \times Y : y \in Y\}$, and let $\mathcal{T}_1, \mathcal{T}_2$ be group topologies on Y. Put $\mathbf{R}_i := \mathbf{R}_{(Y,\mathcal{T}_i)}$ and $\mathcal{L}_i := \mathcal{L}_{(Y,\mathcal{Y}_i)}$ for $i = 1,2$. Then $(Y \times Y, (\mathcal{R}_1 \vee \mathcal{L}_1) \times (\mathcal{R}_2 \vee \mathcal{L}_2))/\Delta$ is uniformly equivalent to $(Y, \mathcal{R}_1 \vee \mathcal{L}_2)$ with respect to the bijection $(y,z)\Delta \mapsto yz^{-1} (y, z \in Y)$.

Proof. Let W be the final uniformity on Y with respect to the surjection

(*) $h : (Y \times Y, (\mathcal{R}_1 \vee \mathcal{L}_1) \times (\mathcal{R}_2 \vee \mathcal{L}_2)) \to Y, (y,z) \mapsto yz^{-1}$.

The surjection h induces the uniform equivalence $h : (Y \times Y, (\mathcal{R}_1 \vee \mathcal{L}_1) \times (\mathcal{R}_2 \vee \mathcal{L}_2))/\Delta \to (Y, \mathcal{W})$. Trivially, $\mathcal{W}$ is finer than the final uniformity $\mathcal{V}$ on Y with respect to $h : (Y \times Y, \mathcal{R}_1 \times \mathcal{R}_2) \to Y$. By [14, Lemma 5.24(c)], $\mathcal{V}$ is equal to $\mathcal{R}_1 \wedge \mathcal{L}_2$. Therefore it remains to show that $\mathcal{R}_1 \wedge \mathcal{L}_2$ is finer than $\mathcal{W}$. Let $M \in \mathcal{W}$. Since h in (*) is uniformly continuous if its codomain Y is provided with $\mathcal{W}$, there are $U_i \in \mathcal{U}_e(\mathcal{T}_i) (i = 1,2)$, such that $(U_1 y \cap yU_1)(U_2 z \cap zU_2)^{-1} \subset M[yz^{-1}]$ for all $y, z \in Y$. This implies that $U_1 z^{-1} \subset M[z^{-1}]$ and $yU_2^{-1} \subset M[y]$ for all $y, z \in Y$. So $\mathcal{R}_1 \supset \mathcal{W}$ and $\mathcal{L}_2 \supset \mathcal{W}$, which means that $\mathcal{R}_1 \wedge \mathcal{L}_2 \supset \mathcal{W}$. ∎

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