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Grassmannians of spheres in Möbius and in Euclidean spaces

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Abstract. In this note we show that Euclidean and Möbius geometry of appropriate high dimension both can be expressed as incidence structures with k-spheres as points and $(k+1)$-spheres as blocks, incidence being the inclusion.

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1 Introduction

In this note we show that Möbius geometry of appropriate high dimension and Euclidean geometry both can be expressed as incidence structures with k-spheres as points and $(k+1)$-spheres as blocks, incidence being the inclusion.

There were several attempts made to base a geometry on the notion of circle (and/or a sphere). The idea originates, perhaps, in [4]. But this is merely an idea, which can be realized in various concrete ways.

To be more exact let us quote several results related to our problem. In [6] it was proved that the system consisting of the points and the circles can be used to characterize plane Euclidean geometry (and also the system consisting of the circles and the tangency can be used for this purpose). In [3] it was shown that the system of the hyperspheres and their tangency can be used to characterize Euclidean geometry and hyperbolic geometry. In [5] some systems of adjacencies of (linear) subspaces of hyperbolic geometry were studied; since hyperbolic subspaces are (via the "ends" construction) counterparts of spheres of Möbius geometry, the quoted results say also something about adjacencies of spheres in Möbius geometry. In this note an adjacency plays merely an auxiliary role, and title geometries are shown to be defined on spheres in spirit of the Grassmann geometry.

The fact that the system consisting of the k-spheres and the $(k+1)$-spheres of Möbius geometry together with the inclusion suffices to express this geometry can be derived from [5, Thm. 5.10]. Nevertheless, we give an independent proof; modifying this proof the result for Euclidean geometry follows easily.

Most of the lemmas and facts of this note are given without any proof. The proofs are easy and elementary and (we hope) can be left for the reader. We indicate only a way which leads to our two main results: Propositions 2 and 3.

Note that these propositions yield immediately that a pair of bijections, the first defined on the k-spheres, and the latter defined on the $(k+1)$-spheres which preserves the inclusion is necessarily determined by an automorphism of the underlying (Euclidean or Möbius, resp.) space (cf. a series of related famous results, e.g. [1], [2]).

2 Preliminary definitions and facts

There is a series of classical results which link Euclidean spaces and Möbius spaces.

Let $\mathfrak{E} = \langle X, \mathcal{L}, \perp \rangle$ be a finite dimensional Euclidean space defined over an ordered euclidean field $\mathfrak{F}$. The symbol X stands for the points, $\mathcal{L}$ stands for the lines, and $\perp$ is the relation of orthogonality of lines of $\mathfrak{E}$. In terms of these notions one can define other standard notions of Euclidean geometry, in particular, the equidistance relation and the class of spheres. Let $\mathcal{S}_k(\mathfrak{E})$ be the set of k-dimensional spheres in $\mathfrak{E}$. We write $\mathcal{C}(\mathfrak{E})$ for the set $\mathcal{S}_1(\mathfrak{E})$ of circles in $\mathfrak{E}$.

Let $\mathfrak{M} = \langle \mathcal{S}, \mathcal{C} \rangle$ be a Möbius space embedded into a finite dimensional projective space $\mathfrak{P}$ over $\mathfrak{F}$ i.e. let $\mathcal{S}$ be a non ruled quadric (a projective sphere) in $\mathfrak{P}$ and $\mathcal{C}$ be the family of all the nondegenerate planar sections of $\mathcal{S}$. So, $\mathcal{C}$ consists of the conics which lie on $\mathcal{S}$. In a more intuitive (and isomorphic/equivalent) way we can imagine a Möbius space as an incidence structure defined on the points of an (Euclidean) sphere $\mathcal{S}$, whose set $\mathcal{C}$ of blocks consists of the Euclidean circles which lie on $\mathcal{S}$. The "projective" representation of $\mathfrak{M}$, though less illustrative, is more convenient in further reasoning and therefore we consider it as basic in the article. In the paper we do not enter into discussion on axiomatic foundations of Möbius and of Euclidean geometry.

Let us write $\mathcal{S}_k(\mathfrak{M})$ for the set of k-dimensional spheres on $\mathcal{S}$/in $\mathfrak{M}$ and $\wp_k(\mathfrak{P})$ for the class of k-dimensional subspaces of the projective space $\mathfrak{P}$. An external definition of $\mathcal{S}_k(\mathfrak{M})$ goes as follows:

$$\mathcal{S}_k(\mathfrak{M}) \quad := \quad \{\mathcal{S} \cap Z \colon Z \in \wp_{k+1}(\mathfrak{P}),\ |Z \cap \mathcal{S}| > 1\}. \tag{1}$$

The class $\mathcal{S}_k(\mathfrak{M})$ coincides with the class $\wp_k(\mathfrak{M})$ of k-dimensional subspaces of $\mathfrak{M}$, and a definition of subspaces of $\mathfrak{M}$ can be also formulated entirely in the language of $\mathfrak{M}$:

- $\wp_0(\mathfrak{M}) = \mathcal{S}$; $\quad \wp_1(\mathfrak{M}) = \mathcal{C}$;
- $\wp_{i+1}(\mathfrak{M}) =$
  $= \Big\{ \bigcup\{C \in \mathcal{C} \colon |C \cap C_0|, |C \cap T| = 2\} \colon C_0 \in \mathcal{C}, T \in \wp_i(\mathfrak{M}), |C_0 \cap T| = 2 \Big\}.$

Recall two major links between Euclidean and Möbius geometry:

Proposition 1.

(i) *Let $\mathfrak{E} = \langle X, \mathcal{L}, \perp \rangle$ be an Euclidean space and $\infty \notin X$ be an arbitrary element. Then the structure*

$$\widetilde{\mathfrak{E}} = \langle X \cup \{\infty\}, \mathcal{C}(\mathfrak{E}) \cup \{L \cup \{\infty\} \colon L \in \mathcal{L}\} \rangle$$

is a Möbius space. Moreover, $\wp_k(\widetilde{\mathfrak{E}}) = \mathcal{S}_k(\mathfrak{E}) \cup \{H \cup \{\infty\} \colon H \in \wp_k(\mathfrak{E})\}$.

(ii) *Let $\mathfrak{M} = \langle \mathcal{S}, \mathcal{C} \rangle$ be a Möbius space and $p \in \mathcal{S}$. The structure* derived at p

$$\mathfrak{M}_{(p)} = \langle \mathcal{S} \setminus \{p\}, \{C \setminus \{p\} \colon p \in C \in \mathcal{C}\} \rangle$$

is an affine space. There is a unique structure of an Euclidean space $\mathfrak{E}$ on $\mathfrak{M}_{(p)}$ such that $\mathcal{S}_k(\mathfrak{E}) = \{C \colon p \notin C \in \mathcal{S}_k(\mathfrak{M})\}$.

Let us pay attention to the Grassmann structures

$$\mathbf{G}_k(\mathfrak{M}) = \langle \mathcal{S}_k(\mathfrak{M}), \mathcal{S}_{k+1}(\mathfrak{M}), \subset \rangle \text{ and } \mathbf{G}_k(\mathfrak{E}) = \langle \mathcal{S}_k(\mathfrak{E}), \mathcal{S}_{k+1}(\mathfrak{E}), \subset \rangle.$$

In view of the above we can consider $\mathbf{G}_k(\mathfrak{E})$ as the substructure

$$\mathbf{G}_k(\mathfrak{M}_{-p}) = \langle \{S \in \mathcal{S}_k(\mathfrak{M}) \colon p \notin S\}, \{S \in \mathcal{S}_{k+1}(\mathfrak{M}) \colon p \notin S\}, \subset \rangle$$

of the Grassmannian $\mathbf{G}_k(\mathfrak{M})$ for a suitable Möbius space $\mathfrak{M} = \widetilde{\mathfrak{E}}$ and its arbitrary point p. And we can use known results related to geometry with spheres in Möbius spaces.

So, let us fix a point ∞ of a Möbius space $\mathfrak{M}$ and let $\mathfrak{E}$ be the respective Euclidean structure defined on $\mathfrak{M}_{(\infty)}$. In particular, we can write

$$\mathcal{S}_k(\mathfrak{E}) = \Big\{ S \in \mathcal{S}_k(\mathfrak{M}) \colon \infty \notin S \Big\}$$

for each integer k.

The symbol $\sim$ will be used for the binary joinability relation defined in an arbitrary Grassmannian by the formula $U' \sim U'' \iff \exists Y [U', U'' \subset Y]$. Specifically, we write $\approx$ for $\sim$ considered in $\mathbf{G}_k(\mathfrak{M})$ and $\simeq$ for $\sim$ defined over $\mathbf{G}_k(\mathfrak{M}_{-\infty})$ for a Möbius space $\mathfrak{M}$.

With every $S \in \mathcal{S}_k(\mathfrak{M})$ we associate the projective subspace $\Pi(S) \in \wp_{k+1}(\mathfrak{P})$ spanned by S. Clearly, $\Pi(S) \cap \mathcal{S} = S$ for each $S \in \mathcal{S}_k(\mathfrak{M})$. Note that the relation $\sim$ makes sense in the projective Grassmannian $\mathbf{G}_{k+1}(\mathfrak{P})$ as well; this one will be also denoted by $\approx$. Let $\sqcup$ be the (binary) operation which associates with any two subspaces the least subspace that contains these two (in the currently considered geometry).

Next, we denote by Γ the triangle relation:

$$\Gamma(U_1, U_2, U_3) \quad :\iff \quad U_1 \sim U_2 \sim U_3 \sim U_1 \wedge \neg(\exists Y)[U_1, U_2, U_3 \subset Y]. \tag{2}$$

We write, as above, Γ for triangle relation defined over $\mathbf{G}_k(\mathfrak{P})$ and over $\mathbf{G}_k(\mathfrak{M})$, and $\underline{\Gamma}$ for the triangle relation defined over $\mathbf{G}_k(\mathfrak{M}_{-\infty})$.

For any two spheres $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$ we say that they are *tangent* (*in a point* a) and write $S_1 \mid S_2$ ($S_1 \mid_a S_2$, resp.) when either $a \in S_1 = S_2$ or $S_1 \neq S_2$, $S_1 \approx S_2$, and $S_1 \cap S_2 = \{a\}$. The following is a folklore

Fact 1.

(i) Let $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$, $a \in \mathcal{S}$. Assume that $S_1 \mid_a S_2$ and $S_1 \neq S_2$ and write $L = \Pi(S_1) \cap \Pi(S_2)$. Then

(a) the projective k-subspace L is tangent to $\mathcal{S}$ in a (i.e. $L \cap \mathcal{S} = \{a\}$, write $L \mid_a \mathcal{S}$), and
(b) within $\Pi(S_i)$ the hypersphere S_i is tangent to L as well.

(ii) Let $L \in \wp_k(\mathfrak{P})$ be tangent to $\mathcal{S}$ in a and let $\pi \in \wp_{k+1}(\mathfrak{P})$, $L \subset \pi$. Then

(a) either $\pi \mid_a \mathcal{S}$
(b) or $S := \pi \cap \mathcal{S} \in \mathcal{S}_k(\mathfrak{M})$ and then $S \mid_a L$.

Moreover, if L as in (ii) and $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$ then $S_1, S_2 \mid_a L$ yields $S_1 \mid_a S_2$.

Let us note also an evident

Lemma 1. *Let $S, S' \in \mathcal{S}_k(\mathfrak{M})$. Then*

(i) $S \approx S' \iff \Pi(S) \approx \Pi(S')$.

(ii) $S \in \mathcal{S}_k(\mathfrak{E}) \iff \infty \notin \Pi(S)$.

(iii) *Assume that $S, S' \in \mathcal{S}_k(\mathfrak{E})$. Then*

$$\begin{aligned} S \simeq S' &\iff S \approx S' \text{ and } \infty \notin S \sqcup S' \\ &\iff \Pi(S) \approx \Pi(S') \text{ and } \infty \notin \Pi(S) \sqcup \Pi(S'). \end{aligned}$$

3 Grassmannian of spheres in Möbius geometry

First, we briefly recall known results concerning Möbius geometry. Let $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$ and $S_1 \approx S_2$, $S_1 \neq S_2$. Then $L := \Pi(S_1) \cap \Pi(S_2)$ is a k-subspace of $\mathfrak{P}$, so S_1, S_2 determine a *projective star* of $(k+1)$-subspaces

$$\mathbf{S}_{\mathfrak{P}}(L) = \left\{U \in \wp_{k+1}(\mathfrak{P})\colon L \subset U\right\}. \tag{3}$$

In turn, this star determines the *star of spheres*

$$\begin{aligned} \mathbf{S}_{\mathfrak{M}}(L) &= \left\{S \in \mathcal{S}_k(\mathfrak{M})\colon L \subset \Pi(S)\right\} \\ &= \left\{U \cap \mathcal{S}\colon U \in \mathbf{S}_{\mathfrak{P}}(L),\ |U \cap \mathcal{S}| \geq 2\right\}; \end{aligned} \tag{4}$$

the subspace L is referred to as the *vertex* of $\mathbf{S}_{\mathfrak{M}}(L)$. We write, for short, $\mathbf{S}(S_1, S_2) = \mathbf{S}_{\mathfrak{M}}(\Pi(S_1) \cap \Pi(S_2))$. A *pencil of subspaces* (*of spheres*, resp) is any nondegenerate family of the form $\{X \in \wp_k\colon X \in w,\ X \subset Y\}$ where w is a star and $Y \in \wp_{k+1}$. That way a pencil is defined both in projective and in Möbius geometry.
Clearly, each pair S_1, S_2 of distinct adjacent spheres determines the pencil $\mathbf{p}(S_1, S_2)$ which contains them.

Recall one of basic properties of projective Grassmannians:

Fact 2. Let $\Gamma(U_1, U_2, U_3)$ with $U_1, U_2, U_3 \in \wp_{k+1}(\mathfrak{P})$. Then there is $L \in \wp_k(\mathfrak{P})$ such that $L \subset U_1, U_2, U_3$.

Consequently, from Lemma 1 for distinct adjacent $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$ we have

$$\{S\colon \Gamma(S_1, S_2, S)\} = \mathbf{S}(S_1, S_2) \setminus \mathbf{p}(S_1, S_2),$$

which enables us to show the following

Lemma 2. *Assume* $\dim(\mathfrak{M}) \geq k + 2$. *Then*

$$\begin{aligned} \mathbf{S}(S_1, S_2) = \left\{S\colon \Gamma(S_1, S_2, S)\right\} \cup \\ \left\{S\colon (\exists S_3)[\Gamma(S_1, S_2, S_3) \wedge \Gamma(S, S_2, S_3)]\right\} \cup \left\{S_2\right\} \end{aligned} \tag{5}$$

for each pair S_1, S_2 of distinct k-spheres in $\mathfrak{M}$.

Let $\mathfrak{S}(S_1, S_2) = \left\langle \mathbf{S}(S_1, S_2), \text{the pencils on } \mathbf{S}(S_1, S_2)\right\rangle$ for distinct $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$ and let L be the vertex of $\mathbf{S}(S_1, S_2)$. Then

- if L is tangent to $\mathcal{S}$ then $\mathfrak{S}(S_1, S_2)$ is an affine space;
- if L misses $\mathcal{S}$ then $\mathfrak{S}(S_1, S_2)$ is a hyperbolic space;
- if $|L \cap \mathcal{S}| \geq 2$ then $\mathfrak{S}(S_1, S_2)$ is a projective space.

Assume that the three geometries on stars listed above do not degenerate to the geometry of a line. Clearly, there is a single elementary formula ψ_{af} such that the substructure $\mathfrak{S}(S_1, S_2)$ of $\mathbf{G}_k(\mathfrak{M})$ satisfies ψ_{af} iff it is an affine space. Consequently, we get

Lemma 3. *Assume that* $\dim(\mathfrak{M}) \geq k+2$. *There is a formula* $\psi(S_1, S_2)$ *in the language of* $\mathbf{G}_k(\mathfrak{M})$ *which holds iff* $S_1 \mid S_2$ *and* $S_1 \neq S_2$.

Roughly speaking, *tangent stars* are determined as those which carry affine geometry.

A tangent star uniquely determines a point on $\mathcal{S}$: if $w = \mathbf{S}_{\mathfrak{M}}(L)$ is a tangent star then L is tangent to $\mathcal{S}$ in a point p and each two $S_1, S_2 \in w$ are tangent at p. However, $p \in \mathcal{S}$ does not determine any star through p. To recover the incidence an additional reasoning is needed.

3.1 Grassmannian of hyperspheres

Let $k = \dim(\mathfrak{M}) - 1$. Clearly, $\mathbf{G}_k(\mathfrak{M})$ is a trivial structure, and the classes of stars and of pencils coincide. Let $\mathcal{P}$ be the class of tangent pencils of (hyper)spheres; consider the structure $\mathbf{P}_k(\mathfrak{M}) = \langle \mathcal{S}_k(\mathfrak{M}), \mathcal{P}, \in, \mid \rangle$ (note: $\mid$ is simply the binary collinearity of $\langle \mathcal{S}_k, \mathcal{P} \rangle$). It was (actually) proved in [3] that $\mathcal{P}$ is definable in $\langle \mathcal{S}_k, \mid \rangle$. One more observation of [3], generalizing a proposition of [6] is needed:

Lemma 4. *Let* $S_1, S_2, S_3 \in \mathcal{S}_k(\mathfrak{M})$, $S_1 \neq S_2$, $w = \mathbf{S}(S_1, S_2)$, *and* $S_1 \mid_a S_2$. *The formula*

$$S_3 \in w \ \vee \ \neg(\exists S')[S' \in w \wedge S_3 \mid S'] \tag{6}$$

is valid iff $a \in S_3$.

This means also that the structure $\langle \mathcal{S}, \mathcal{S}_k(\mathfrak{M}) \rangle$ is definable in $\mathbf{P}_k(\mathfrak{M})$, and, clearly, $\mathfrak{M}$ is definable in $\langle \mathcal{S}, \mathcal{S}_k(\mathfrak{M}) \rangle$.

3.2 General case: $\dim(\mathfrak{M}) \geq k+2$

Assume $\dim(\mathfrak{M}) \geq k+2$ and define the relation $\mid \subset \mathcal{S}_k(\mathfrak{M}) \times (\mathcal{S}_k(\mathfrak{M}) \times \mathcal{S}_k(\mathfrak{M}))$ as follows:

$$\begin{aligned} S_3 \mid (S_1, S_2) \ :\Longleftrightarrow & \big(S_1 \mid S_2 \wedge S_1 \neq S_2 \wedge S_3 \in \mathbf{S}(S_1, S_2)\big) \vee \\ & (\exists Y \in \mathcal{S}_{k+1})(\exists S_1', S_2')\big[S_1', S_2', S_3 \subset Y \wedge S_1' \neq S_2' \wedge S_1', S_2' \in \mathbf{S}(S_1, S_2) \wedge \\ & \qquad S_1' \mid S_2' \wedge \neg(\exists S_3')[S_3' \mid S_3 \wedge S_3' \subset Y \wedge S_3' \in \mathbf{S}(S_1, S_2)]\big]. \end{aligned} \tag{7}$$

In general case the formula (6) does not characterize the incidence. Instead, the following "approximating" property, an analogue and a consequence of Lemma 4, is valid.

Lemma 5. *Let $S_1, S_2, S_3 \in \mathcal{S}_k(\mathfrak{M})$, $S_1 \neq S_2$, $w = \mathbf{S}(S_1, S_2)$, $L = \Pi(S_1) \cap \Pi(S_2)$ be the vertex of w, and $S_1 \mid_a S_2$. Then $S_3 \mid (S_1, S_2)$ holds iff $a \in S_3$ and* $\dim(\Pi(S_3) \cap L) \geq k-1$.

We define, inductively

$$\mathcal{Z}^1_{S_1,S_2} := \{S \colon S \mid (S_1, S_2)\}; \tag{8}$$

$$\mathcal{Z}^{i+1}_{S_1,S_2} := \bigcup\{\mathcal{Z}^1_{S_1',S_2'} \colon S_1' \mid S_2',\ S_1', S_2' \in \mathcal{Z}^i_{S_1,S_2}\}. \tag{9}$$

As a simple consequence of Lemma 5 and connectedness of the projective Grassmannian $\mathbf{G}_k(\mathfrak{P})$ we obtain

Proposition 2. *Let $\dim(\mathfrak{M}) \geq k+2$. Under the assumptions of Lemma 5, $\mathcal{Z}^k_{S_1,S_2} = \{S \in \mathcal{S}_k(\mathfrak{M}) \colon a \in S\}$. Consequently, the structure $\langle \mathcal{S}, \mathcal{S}_k(\mathfrak{M})\rangle$ and thus also the underlying Möbius space $\mathfrak{M}$ both are definable in $\mathbf{G}_k(\mathfrak{M})$.*

Proof. Write $\mathcal{S}_k(a) = \{S \in \mathcal{S}_k(\mathfrak{M}) \colon a \in S\}$. The set $\mathcal{S}_k(a)$ can be identified with a point a.

Clearly, $\mathcal{Z}^i_{S_1,S_2} \subset \mathcal{S}_k(a)$ for each integer i. Let $S \in \mathcal{S}_k(a)$ and $\delta = \dim(L \cap \Pi(S))$. Let K be the k-subspace of $\mathfrak{P}$ contained in $\Pi(S)$ and tangent to S at a. We show that $S \in \mathcal{Z}^\delta_{S_1,S_2}$ and, in view of Lemma 5, to this aim it suffices to consider a sequence $L_1, \ldots, L_\delta$ of k-subspaces of $\mathfrak{P}$ such that $L_1 = L$, $L_\delta = K$, and L_i is tangent to $\mathcal{S}$ at a, $\dim(L_i \cap L_{i+1}) = k-1$ for $i = 1, \ldots, \delta-1$. This yields, finally, the desired inclusion $\mathcal{S}_k(a) \subset \mathcal{Z}^k_{S_1,S_2}$.

Since a is arbitrary, we infer that $\langle \mathcal{S}, \mathcal{S}_k(\mathfrak{M})\rangle$ is interpretable in $\mathbf{G}_k(\mathfrak{M})$. To interpret $\mathfrak{M}$ in $\langle \mathcal{S}, \mathcal{S}_k(\mathfrak{M})\rangle$ it suffices to note that given any three pairwise distinct points a, b, c the set $\bigcap\{S \in \mathcal{S}_k(\mathfrak{M}) \colon a, b, c \in S\}$ is the circle in $\mathcal{C}$ through a, b, c. $\boxed{QED}$

4 Grassmannians of spheres in Euclidean geometry

As it was stated in Section 2, the underlying Euclidean space $\mathfrak{E}$ is the (uniquely determined) Euclidean structure on $\mathfrak{M}_{(\infty)}$, where ∞ is a point of a Möbius space $\mathfrak{M}$. Dealing with the Grassmannian $\mathbf{G}_k(\mathfrak{E}) = \mathbf{G}_k(\mathfrak{M}_{-\infty})$ essentially the same methods as those of Section 3 can be used. One important exception appears, though. Let $S_1, S_2 \in \mathcal{S}_k(\mathfrak{E})$, assume that $S_1 \simeq S_2$. Then $w := \mathbf{S}_{\mathfrak{M}}(S_1, S_2)$ contains exactly one sphere $w^\infty = ((\Pi(S_1) \cap \Pi(S_2)) \sqcup \{\infty\}) \cap \mathcal{S}$ that is not a sphere of $\mathfrak{E}$; one can write

$$\mathbf{S}_{\mathfrak{E}}(S_1, S_2) = \mathbf{S}_{\mathfrak{M}}(S_1, S_2) \setminus \left\{(\mathbf{S}_{\mathfrak{M}}(S_1, S_2))^\infty\right\}.$$

This causes some (not really serious) troubles. Namely, if S_1, S_2 are as above and $S_0 = (\mathbf{S}_{\mathfrak{M}}(S_1, S_2))^\infty$ then

$$\{S\colon \underline{\Gamma}(S_1,S_2,S)\} = \mathbf{S}_{\mathfrak{E}}(S_1,S_2) \setminus \big(\mathbf{p}_{\mathfrak{E}}(S_1,S_2) \cup \mathbf{p}_{\mathfrak{M}}(S_1,S_0) \cup \mathbf{p}_{\mathfrak{M}}(S_2,S_0)\big).$$

Then definition of a star in terms of the triangle relation (and thus, finally, in terms of geometry of $\mathbf{G}_k(\mathfrak{M}_{-\infty})$) becomes slightly more complex:

Lemma 6. *Assume that* $\dim(\mathfrak{M}) \geq k+2$. *Let* $S_1, S_2 \in \mathcal{S}_k(\mathfrak{M})$, $\infty \notin S_1, S_2$. *Then*

$$\mathbf{S}_{\mathfrak{E}}(S_1,S_2) = \big\{S\colon \underline{\Gamma}(S_1,S_2,S)\big\} \cup \big\{S\colon (\exists S_3)[(\underline{\Gamma}(S_1,S_2,S_3) \wedge \underline{\Gamma}(S,S_2,S_3)) \vee (\underline{\Gamma}(S_1,S_2,S_3) \wedge \underline{\Gamma}(S,S_1,S_3))]\big\} \cup \big\{S_1,S_2\big\}. \quad (10)$$

In this section we write $\mathbf{S}$ for $\mathbf{S}_{\mathfrak{E}}$. As usually, a pencil of k-spheres in $\mathfrak{E}$ is a (nonvoid) intersection of a star of k-spheres and a family of spheres contained in a $(k+1)$-sphere.

Let $S_1, S_2 \in \mathcal{S}_k(\mathfrak{E})$, $S_1 \neq S_2$, $S_1 \simeq S_2$, and L be the vertex of $\mathbf{S}(S_1,S_2)$. Write, as in Section 3, $\mathfrak{S}(S_1,S_2) = \big\langle \mathbf{S}(S_1,S_2), \text{the pencils on } \mathbf{S}(S_1,S_2)\big\rangle$. Then

- if L is tangent to $\mathcal{S}$ then $\mathfrak{S}(S_1,S_2)$ is a punctured affine space, i.e. an affine space with one point and all the lines through this point deleted;
- if L misses $\mathcal{S}$ then $\mathfrak{S}(S_1,S_2)$ is a punctured hyperbolic space;
- if $|L \cap \mathcal{S}| \geq 2$ then $\mathfrak{S}(S_1,S_2)$ is a punctured projective space.

The three above geometries on stars are elementarily distinguishable, provided they do not degenerate to lines. This leads us to the conclusion analogous to Lemma 3:

Lemma 7. *Assume that* $\dim(\mathfrak{E}) \geq k+2$. *The relation of tangency on* $\mathcal{S}_k(\mathfrak{E})$ *is definable in terms of geometry of* $\mathbf{G}_k(\mathfrak{E})$.

Unhappily, Lemma 5 is not valid in $\mathbf{G}_k(\mathfrak{E})$. A modification of this lemma valid in Grassmannians of spheres of an Euclidean space suitable for our purposes is the following. The relation $\mathsf{I} \subset \mathcal{S}_k(\mathfrak{E}) \times (\mathcal{S}_k(\mathfrak{E}) \times \mathcal{S}_k(\mathfrak{E}))$ is defined by the formula (7).

Lemma 8. *Let* $S_1, S_2, S_3 \in \mathcal{S}_k(\mathfrak{E})$, $S_1 \neq S_2$, $w = \mathbf{S}(S_1,S_2)$, $L = \Pi(S_1) \cap \Pi(S_2)$ *be the vertex of* w, *and* $S_1 \mathsf{I}_a S_2$. *Then* $S_3 \mathsf{I} (S_1,S_2)$ *holds iff* $a \in S_3$, $\dim(\Pi(S_3) \cap L) \geq k-1$, *and* $\infty \notin L \sqcup \Pi(S_3)$.

Let the sets $\mathcal{Z}^i_{S_1,S_2}$ be defined over $\mathbf{G}_k(\mathfrak{E})$ by the formulas analogous to (8), (9). Then as a consequence of Lemmas 6, 7, and 8 we conclude with the following analogue of Proposition 2.

Proposition 3. *Let* $\dim(\mathfrak{E}) \geq k+2$. *Under the assumptions of Lemma 8,* $\mathcal{Z}^{k+1}_{S_1,S_2} = \{S \in \mathcal{S}_k(\mathfrak{E})\colon a \in S\}$. *Consequently, the structure* $\langle \mathcal{S} \setminus \{\infty\}, \mathcal{S}_k(\mathfrak{E})\rangle$ *and thus also the underlying Euclidean space* $\mathfrak{E}$ *are both definable in* $\mathbf{G}_k(\mathfrak{E})$.

Proof. As in the case of Proposition 2, the point is to prove the inclusion $\mathcal{S}_k(a) := \{S \in \mathcal{S}_k(\mathfrak{E}) \colon a \in S\} \subset \mathcal{Z}^{k+1}_{S_1,S_2}$. Let $S \in \mathcal{S}_k(a)$. Adopt notation of the proof of Proposition 2 and consider respective sequence $L_1, \ldots, L_\delta$ of k-subspaces of $\mathfrak{P}$ through a. From assumptions, S_1, S_2 determine the star $\mathbf{S}(L)$. For each $i = 2, \ldots, \delta$ one can find in $\mathbf{S}_{\mathfrak{E}}(L_i)$ at least two k-spheres S'_i, S''_i such that $\infty \notin (L_{i-1} \sqcup \Pi(S'_i)), (L_{i-1} \sqcup \Pi(S''_i))$. Then $\mathbf{S}(S'_i, S''_i) = \mathbf{S}_{\mathfrak{E}}(L_i)$ and, in view of Lemma 8, $S'_i, S''_i \mid (S'_{i-1}, S''_{i-1})$ for any pair of distinct spheres in $\mathbf{S}_{\mathfrak{E}}(L_{i-1})$ and $S'_i, S''_i \in \mathcal{Z}^i_{S_1,S_2}$ So, finally, one gets $S'_\delta, S''_\delta \in \mathbf{S}_{\mathfrak{E}}(K)$ with $S'_\delta, S''_\delta \in \mathcal{Z}^\delta_{S_1,S_2}$. This gives $S \in \mathcal{Z}^{\delta+1}_{S_1,S_2}$. Finally we get the desired inclusion.

To complete the reasoning recall that the structure $\mathfrak{E}$ can be recovered in terms of the structure $\left\langle \text{the points of } \mathfrak{E}, \mathcal{C}(\mathfrak{E}) \right\rangle$. $\boxed{QED}$

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