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Semistable genus 5 general type $\mathbb{P}^1$-curves have at least 7 singular fibres

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Abstract. We prove that if $f : X \to \mathbb{P}^1$ is a non-isotrivial, semistable, genus 5 fibration defined on a general type surface X then the number s of singular fibres is at least 7.

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MSC 2000 classification: primary 14D06, secondary 14J26

Introduction

We work on the field of complex numbers. Let $f : X \to \mathbb{P}^1$ be a non-isotrivial semistable genus g fibration defined on the general type surface X (this is the semistable curve alluded to in the title).

A classical issue (since Parshin's paper [5]), is determining a lower bound for the number s of singular fibers of f. The state of the art is as follows:

.If $g \geq 1$ then $s \geq 4$ ([2]),

.If $g \geq 2$ then $s \geq 5$ ([6]),

.If $g \geq 2$ and the Kodaira dimension of X is nonnegative, then $s \geq 6$ ([7] see also [4]),

.If X is of general type and $2 \leq g \leq 4$ then $s \geq 7$ ([7]).

.Some partial results for the case of fibrations on rational surfaces satisfying $s \geq 6$ can be found in [1].

In a preprint previous to the appearance of [7] it was conjectured by Tan and Tu that if X is of general type, then $s \geq 7$. Moreover if X is of general type, $s = 6$ and $g = 5$ then the minimal model S of X satisfies:

$$K_S^2 = 1, \quad p_g(S) = 2 \text{ and } q(S) = 0.$$

Moreover, in that case the fibration f is the pull-back of a pencil on S with 5 simple base points.

Call a pencil Λ transversal if two general elements intersect transversally (in particular a general element is non singular). In section 2 of this short note we shall prove:

Theorem 1. *Let S be a minimal surface of general type with $K_S^2 = 1$, $p_g(S) = 2$ and $q(S) = 0$. Then S does not admit a transversal pencil Λ of genus 5 curves with 5 base points.*

Previous remarks imply:

Theorem 2. *If $f : X \to \mathbb{P}^1$ is a non-isotrivial semistable fibration of genus 5 curves defined on the general type surface X, then the number s of singular fibers is at least 7.*

The proof of Theorem 1 is based on a construction by Horikawa ([3]): numerical restrictions in the hypothesis of Theorem 1 mean that S is on the "Noether's line" and thus after blowing up a point it can be realized as a double cover of $\mathbb{F}_2$. The author is indebted to Prof. M. Mendes-Lopes who pointed out this fact and suggested its use for proving Theorem 1, and to Prof. C. Ciliberto for indicating a mistake in the first version of this paper.

1 Proof of Theorem 1

Start with S minimal of general type, $K_S^2 = 1$, $q = 0$ and $p_g = 2$. Assume that a transversal pencil Λ of smooth genus 5 curves and with general curve F and $F^2 = 5$ exists on S.

After blowing up the base locus of $|K_S|$ consider the ramified double covering:

$$f_2 : \bar{S} \to \mathbb{F}_2.$$

The map f_2 is described as follows: the bi-canonical map of S determines a double cover on the quadric cone in $\mathbb{P}^3$, f_2 is the induced map on $\bar{S}$ after considering the desingularization $\mathbb{F}_2$ of the cone. The branch locus of f_2 is a curve B of class $6\Delta_0 + 10\Gamma$, Δ_0 and Γ denoting respectively the class of the (-2)-section and the class of the fiber in $\mathbb{F}_2$ ([3], Theorem 2.1).

Denote by $|\bar{F}|$ the induced pencil in $\bar{S}$. Note that $\bar{F}^2 = 4$ or 5 depending on whether the base point of $|K_S|$ is a base point of $|F|$ or not. Let G be the image of $\bar{F}$ under f_2. Note that if we denote $G = a\Delta_0 + b\Gamma$, then we have:

i) $G.B = 6b - 2a$,

ii) $G^2 = 2a(b-a)$,

iii) $2p_G - 2 = G^2 + G.(-2\Delta_0 - 4\Gamma) = 2a(b-a) - 2b$, with p_G denoting the arithmetic genus of G.

We distinguish two cases:

Case 1: f_2 restricted to $\bar{F}$ is $2:1$.

Denote by $f_2 : \bar{F} \to G$ the restriction. If $G \equiv a\Delta_0 + b\Gamma$, then

$$2G^2 = (f_2^* G)^2 = \bar{F}^2 = 4 \text{ or } 5.$$

Thus, $G^2 = 2 = 2a(b-a)$. This forces $a = 1$ and $b = 2$.

By iii):

$$2p_G - 2 = 2 - 2b = -2.$$

Thus, being G irreducible and of arithmetic genus 0 it must be a non-singular rational curve.

Finally, the degree of the ramification divisor of f_2 restricted to $\bar{F}$ can be computed into two different ways, namely, using Riemann-Hurwitz or intersecting G with B. Using Riemann-Hurwitz we obtain:

$$8 = 2g_{\bar{F}} - 2 = 4(g_G - 1) + \mathcal{B},$$

and therefore $\mathcal{B} = 12$. On the other hand, by i):

$$\mathcal{B} = G.B = 6b - 2a = 10.$$

This contradiction proves that Case 1 is impossible.

Case 2: f_2 restricted to F is $1:1$.

In this case we use not only the branch locus B but also the ramification divisor R on $\bar{S}$. Denote by $\pi : \bar{S} \to S$ the blowing up. The divisor R is given by $R = 5D + 6E$, with E the exceptional divisor and $D \equiv \pi^* K_S - E$, B and R are related by $f_2^* B = 2R$ ([3], page 129).

Let $f_2^* G = \bar{F} + \tilde{F}$. Note that since the ramifications of f_2 occurring on $\bar{F}$ are given by intersections of $\bar{F}$ and $\tilde{F}$, the equality $R.\bar{F} = R.\tilde{F}$ holds. Thus, we have:

$$2(\bar{F}).2R = (\bar{F} + \tilde{F}).2R = f_2^* G.f_2^* B = 2G.B. \tag{1}$$

Assume $\bar{F}^2 = 4$. First, we compute the intersection $\bar{F}.R$. Note that $\bar{F}^2 = 4$ means that the center of the blowing up is a base point of $|F|$. Thus, $\bar{F} \equiv \pi^* F - E$, $\bar{F}.E = 1$ and:

$$\bar{F}.R = (\pi^* F - E).(5\pi^* K_S - 5E + 6E) = 5\pi^* F.\pi^* K_S + 1 = 16,$$

because $g_F = 5$ and $F^2 = 5$ imply $K_S.F = 3$.

Then, by 1 and i) :

$$32 = 6b - 2a, \text{ i.e. } a = 3b - 16.$$

On the other hand,

$$0 \leq G^2 = a(b-a) = (3b-16)(-2b+16).$$

It follows that: $b \geq 16/3$ and $b \leq 8$. Thus $b = 6, 7$ or 8 and correspondingly $a = 2, 5$ or 8. But, being f_2 restricted to $\bar{F}$ a degree 1 map, the arithmetic genus p_G of G must be at least 5 and thus

$$8 \leq 2(p_G - 1) = 2a(b-a) - 2b.$$

None of the possible combinations of a and b listed before satisfy this inequality.

The case $\bar{F}^2 = 5$ follows by similar considerations. In this case $\bar{F} = \pi^* F$, $\bar{F}.E = 0$ and $\bar{F}.R = 15$. The computations are quite analogous. This prove the Theorem.

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