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$*$-group identities on units of group rings

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Abstract. Analogous to $*$-polynomial identities in rings, we introduce the concept of $*$-group identities in groups. When F is an infinite field of characteristic different from 2, we classify the torsion groups with involution G so that the unit group of FG satisfies a $*$-group identity. The history and motivations will be given for such an investigation.

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1 Introduction and motivations

The motivation for the study of this topic is from two sides:

(a) Hartley's conjecture on group identities of units of group rings,

(b) Amitsur's Theorem on $*$-polynomial identities in rings.

Let F be a field and G a group. Write $\mathcal{U}(FG)$ for the unit group of the group algebra FG. We say that a subset S of $\mathcal{U}(FG)$ satisfies a *group identity* if there exists a non-trivial word $w(x_1, \ldots, x_n)$ in the free group on a countable set of generators $\langle x_1, x_2, \ldots\rangle$ such that $w(u_1, \ldots, u_n) = 1$ for all $u_1, \ldots, u_n \in S$.

Brian Hartley in the 80s conjectured that when F is infinite and G is torsion, *if $\mathcal{U}(FG)$ satisfies a group identity then FG satisfies a polynomial identity.* We recall that a subset H of an F-algebra A satisfies a *polynomial identity* if there exists a non-zero polynomial $f(x_1, \ldots, x_n)$ in the free associative algebra on non-commuting variables $x_1, x_2, \ldots$ over F, $F\{x_1, x_2, \ldots\}$, such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in H$ (in this case we shall write also that H is PI).

Hartley's conjecture was solved affirmatively by Giambruno, Jespers and Valenti [3] in the semiprime case (hence, in particular, for fields of characteristic zero) and by Giambruno, Sehgal and Valenti [7] in the general case. Its solution was at the basis of the work of Passman [18] who characterized group algebras whose units satisfy a group identity. Recall that, for any prime p, a group G is said to be p-abelian if its commutator subgroup G' is a finite p-group, and 0-abelian means abelian.

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Theorem 1. *Let F be an infinite field of characteristic $p > 0$ and G a torsion group. The following statements are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;

(iii) G has a normal p-abelian subgroup of finite index and G' is a p-group of bounded exponent.

In the characteristic zero case, when G is torsion, $\mathcal{U}(FG)$ satisfies a group identity if, and only if, G is abelian. In particular, the fact that G contains a normal p-abelian subgroup of finite index (condition (iii) of the theorem) is equivalent to saying that FG must satisfy a polynomial identity, as was established earlier by Isaacs and Passman (see Corollaries 5.3.8 and 5.3.20 of [17]). More recently, the above results have been extended to the more general context of finite fields in [15] and [16] and arbitrary groups in [9].

Along this line, a natural question of interest is to ask whether group identities satisfied by some special subset of the unit group of FG can be lifted to $\mathcal{U}(FG)$ or force FG to satisfy a polynomial identity. In this framework, the *symmetric units* have been the subject of a good deal of attention.

Assume that F has characteristic different from 2. The linear extension to FG of the map $*$ on G such that $g^* = g^{-1}$ for all $g \in G$ is an *involution* of FG, namely an antiautomorphism of order 2 of FG, called the *classical* involution. An element $\alpha \in FG$ is said to be *symmetric with respect to* $*$ if $\alpha^* = \alpha$. We write FG^+ for the set of symmetric elements, which are easily seen to be the linear combinations of the terms $g + g^{-1}$, $g \in G$. Let $\mathcal{U}^+(FG)$ denote the set of symmetric units. Giambruno, Sehgal and Valenti [8] confirmed a stronger version of Hartley's Conjecture by proving

Theorem 2. *Let FG be the group algebra of a torsion group G over an infinite field F of characteristic different from 2 endowed with the classical involution. If $\mathcal{U}^+(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

Under the same restrictions as in the above theorem, they also obtained necessary and sufficient conditions for $\mathcal{U}^+(FG)$ to satisfy a group identity. Obviously, group identities on $\mathcal{U}^+(FG)$ do not force group identities on $\mathcal{U}(FG)$. To see this it is sufficient to observe that if Q_8 is the quaternion group of order 8, for any infinite field F of characteristic $p > 2$ FQ_8^+ is commutative, hence $\mathcal{U}^+(FQ_8)$ satisfies a group identity but, according to Theorem 1, $\mathcal{U}(FQ_8)$ does not satisfy a group identity. For a complete overview of these and related results we refer to the monograph [13].

Recently, there has been a considerable amount of work on involutions of FG obtained as F-linear extensions of arbitrary group involutions on G (namely antiautomorphisms of order 2 of G) other than the classical one. The final outcome has been the complete classification of the torsion groups G such that the units of FG which are symmetric under the given involution satisfy a group identity (see [5]).

Here we discuss a more general problem, that of *$*$-group identities* on $\mathcal{U}(FG)$. We can define an involution on the free group $\langle x_1, x_2, \ldots \rangle$ via $x_{2i-1}^* := x_{2i}$ for all $i \geq 1$. Renumbering, we obtain the free group with involution $\mathcal{F} := \langle x_1, x_1^*, x_2, x_2^*, \ldots \rangle$. We say the unit group $\mathcal{U}(FG)$ satisfies a $*$-group identity if there exists a non-trivial word $w(x_1, x_1^*, \ldots, x_n, x_n^*) \in \mathcal{F}$ such that

$$w(u_1, u_1^*, \ldots, u_n, u_n^*) = 1$$

for all $u_1, \ldots, u_n \in \mathcal{U}(FG)$. Obviously, if $\mathcal{U}^+(FG)$ satisfies the group identity $v(x_1, \ldots, x_r)$, then $\mathcal{U}(FG)$ satisfies the $*$-group identity $v(x_1x_1^*, \ldots, x_rx_r^*)$. It seems of interest to understand the behaviour of the symmetric units when the group of units satisfies a $*$-group identity. The main motivation for this investigation dates back to the classical result of Amitsur on $*$-polynomial identities satisfied by an algebra with involution. Let A be an F-algebra having an involution $*$. We can define an involution on the free algebra $F\{x_1, x_2, \ldots\}$ via $x_{2i-1}^* := x_{2i}$ for all $i \geq 1$. As in the free group case, renumbering we obtain the free algebra with involution $F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$. We say that A satisfies a $*$-polynomial identity (or A is $*$-PI) if there exists a non-zero element $f(x_1, x_1^*, \ldots, x_n, x_n^*) \in F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$ such that $f(a_1, a_1^*, \ldots, a_n, a_n^*) = 0$ for all $a_1, \ldots, a_n \in A$. It is obvious that if the symmetric elements of A satisfy the polynomial identity $g(x_1, \ldots, x_r)$ then A satisfies the $*$-polynomial identity $g(x_1+x_1^*, \ldots, x_r+x_r^*)$. It is more difficult to see that if A satisfies a $*$-polynomial identity, then A^+ satisfies a polynomial identity. The deep result of Amitsur [2] shows that this is the case, by proving that if A satisfies a $*$-polynomial identity, then A satisfies a polynomial identity.

The surprising result we obtain is just a group-theoretical analogue of Amitsur's theorem for the unit groups of torsion group rings endowed with the linear extension of an arbitrary group involution. The original results were established in [6]. Recently a long and detailed survey on the subject by Lee [14] has appeared.

2 $*$-group identities on units of torsion group algebras

Let $\langle X \rangle$ be the free group of countable rank on a set $X := \{x_1, x_2, \ldots\}$. We can regard it as a group with involution by setting, for every $i \geq 1$, $x_{2i-1}^* = x_{2i}$ and extending $*$ to an involution of $\langle X \rangle$ in the obvious way. Write $X_1 := \{x_{2i-1} \,|\, i \geq 1\}$ and $X_2 := \{x_{2i} \,|\, i \geq 1\}$. The group above, we call $\mathcal{F}$, has the following universal property: if H is a group with involution, any map $X_1 \longrightarrow H$ can be uniquely extended to a group homomorphism $f : \mathcal{F} \longrightarrow H$ commuting with the involution.

Let $1 \neq w(x_1, x_1^*, \ldots, x_n, x_n^*) \in \mathcal{F}$ and let H be a group with involution $*$. The word w is said to be a *$*$-group identity* (or $*$-GI) of H if w is equal to 1 for any evaluation $\varphi(x_i) = u_i \in H$, $\varphi(x_i^*) = u_i^* \in H$ with $1 \leq i \leq n$. Clearly a group identity is a $*$-GI. Moreover, since for any $x \in X$ xx^* is symmetric, a group identity on symmetric elements of H yields a $*$-group identity of H. We focus our attention on the *converse* problem, namely the possibility of a $*$-group identity of H to force a group identity on the symmetric elements of H itself when H is the unit group of a group algebra.

One of the key ingredients is the following result dealing with finite-dimensional semisimple algebras with involution over an infinite field.

Lemma 1. *Let A be a finite-dimensional semisimple algebra with involution over an infinite field of characteristic different from 2. If its unit group $\mathcal{U}(A)$ satisfies a $*$-GI, then A is a direct sum of finitely many simple algebras of dimension at most 4 over their centre. Moreover A^+ is central in A.*

Proof. See Lemma 5 of [6]. $\boxed{QED}$

The conclusions of the above lemma are not a novelty in the setting of algebras with involution. For instance the same happens when one considers finite-dimensional semisimple algebras with involution whose symmetric elements are Lie nilpotent (see [4]).

In the framework of group algebras, this gives crucial information on the structure of the basis group. In fact, assume that F is an infinite field of characteristic $p > 2$ and G a finite group with an involution $*$ and let FG have the induced involution. Write $P := \{x \,|\, x \in G,\ x \text{ is a } p\text{-element}\}$. Suppose that $\mathcal{U}(FG)$ satisfies a $*$-group identity w. The Jacobson radical J of the group algebra FG is nilpotent and $*$-invariant. This is sufficient to conclude that $\mathcal{U}(FG/J)$ also satisfies w. But FG/J is finite-dimensional and semisimple. By applying Lemma 1, the simple components of its Wedderburn decomposition are all of dimension at most 4 over their centres. But Lemma 2.6 of [4] or Lemma 3 of [12] show that this forces P to be a (normal and $*$-invariant) subgroup of G.

We can summarize all these deductions in the following

Lemma 2. *Let F be an infinite field of characteristic $p > 2$ and G a finite group with involution and let FG have the induced involution. If $\mathcal{U}(FG)$ satisfies a $*$-group identity, then the p-elements of G form a subgroup.*

It is trivial to see that the conclusion holds for locally finite groups G as well.

Now, let F and G be as in the lemma. We know that if $\mathcal{U}(FG)$ satisfies a $*$-GI, then P is a subgroup, $F(G/P)$ has an induced involution and $\mathcal{U}(F(G/P))$ still satisfies a $*$-GI. By Lemma 1 $F(G/P)^+$ is central in $F(G/P)$. In particular, $F(G/P)^+$ must be commutative. Therefore it is of interest to classify group algebras with linear extensions of arbitrary group involutions whose symmetric elements commute. In order to state this, a definition is required.

We recall that a group G is said to be an LC-group (that is, it has the "lack of commutativity" property) if it is not abelian, but if $g, h \in G$, and $gh = hg$, then at least one of g, h and gh must be central. These groups were introduced by Goodaire. By Proposition III.3.6 of [10], a group G is an LC-group with a unique non-identity commutator (which must, obviously, have order 2) if and only if $G/\zeta(G) \cong C_2 \times C_2$. Here, $\zeta(G)$ denotes the centre of G.

Definition 1. A group G endowed with an involution $*$ is said to be a special LC-group, or SLC-group, if it is an LC-group, it has a unique non-identity commutator z, and for all $g \in G$, we have $g^* = g$ if $g \in \zeta(G)$, and otherwise, $g^* = zg$.

The SLC-groups arise naturally in the following result proved by Jespers and Ruiz Marin [11] for an arbitrary involution on G.

Theorem 3. *Let R be a commutative ring of characteristic different from 2, G a non-abelian group with an involution $*$ which is extended linearly to RG. The following statements are equivalent:*

(i) RG^+ is commutative;

(ii) RG^+ is the centre of RG;

(iii) G is an SLC-group.

We recall that in [1] Amitsur proved that *if R is a ring with involution and R^+ is PI, then R is PI.* Later the same arguments were used by him to prove that *if R is $*$-PI, then R is PI.* In particular, if R is $*$-PI then R^+ is PI. The developments for us were similar. In fact, by using exactly the same arguments as in [5] (Section 3 for the semiprime case and Sections 4 and 5 for the general case) we provide the following result which is the core of [6].

Theorem 4. *Let F be an infinite field of characteristic $p \neq 2$, G a torsion group with an involution $*$ which is extended linearly to FG. The following statements are equivalent:*

(i) the symmetric units of FG satisfy a group identity;

(ii) the units of FG satisfy a $*$-group identity;

(iii) one of the following conditions holds:

(a) FG is semiprime and G is abelian or an SLC-group;

(b) FG is not semiprime, the p-elements of G form a (normal) subgroup P, G has a p-abelian normal subgroup of finite index, and either

(1) G' is a p-group of bounded exponent, or

(2) G/P is an SLC-group and G contains a normal $*$-invariant p-subgroup B of bounded exponent, such that P/B is central in G/B and the induced involution acts as the identity on P/B.

References

[1] S.A. Amitsur, *Rings with involution*, Israel J. Math. **6** (1968), 99–106.

[2] S.A. Amitsur, *Identities in rings with involution*, Israel J. Math. **7** (1969), 63–68.

[3] A. Giambruno, E. Jespers, A. Valenti, *Group identities on units of rings*, Arch. Math. **63** (1994), 291–296.

[4] A. Giambruno, C. Polcino Milies, S.K. Sehgal, *Lie properties of symmetric elements in group rings*, J. Algebra **321** (2009), 890–902.

[5] A. Giambruno, C. Polcino Milies, S.K. Sehgal, *Group identities on symmetric units*, J. Algebra **322** (2009), 2801–2815.

[6] A. Giambruno, C. Polcino Milies, S.K. Sehgal, *Star-group identities and group of units*, Arch. Math. **95** (2010), 501–508.

[7] A. Giambruno, S.K. Sehgal, A. Valenti, *Group algebras whose units satisfy a group identity*, Proc. Amer. Math. Soc. **125** (1997), 629–634.

[8] A. Giambruno, S.K. Sehgal, A. Valenti, *Symmetric units and group identities*, Manuscripta Math. **96** (1998), 443–461.

[9] A. Giambruno, S.K. Sehgal, A. Valenti, *Group identities on units of group algebras*, J. Algebra **226** (2000), 488–504.

[10] E.G. Goodaire, E. Jespers, C. Polcino Milies, *Alternative loop rings*, North-Holland, Amsterdam, 1996.

[11] E. Jespers, M. Ruiz Marin, *On symmetric elements and symmetric units in group rings*, Comm. Algebra **34** (2006), 727–736.

[12] G.T. Lee, *Groups whose irriducible representations have degree at most* 2, J. Pure Appl. Algebra **199** (2005), 183–195.

[13] G.T. Lee, *Group identities on units and symmetric units of group rings*, Springer, London, 2010.

[14] G.T. Lee, *A survey on $*$-group identities on units of group rings*, Comm. Algebra **40** (2012), 4540–4567.

[15] C.H. Liu, *Group algebras with units satisfying a group identity*, Proc. Amer. Math. Soc. **127** (1999), 327–336.

[16] C.H. Liu, D.S. Passman *Group algebras with units satisfying a group identity II*, Proc. Amer. Math. Soc. **127** (1999), 337–341.

[17] D.S. Passman, *The algebraic structure of group rings*, Wiley, New York, 1977.

[18] D.S. Passman, *Group algebras whose units satisfy a group identity II*, Proc. Amer. Math. Soc. **125** (1997), 657–662.