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Automorphisms of Group Extensions

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Abstract. After a brief survey of the theory of group extensions and, in particular, of automorphisms of group extensions, we describe some recent reduction theorems for the inducibility problem for pairs of automorphisms.

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1 Background from Extension Theory

A *group extension* $\mathbf{e}$ of N by Q is a short exact sequence of groups and homomorphisms

$$\mathbf{e}: \quad N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q,$$

so that $N \simeq \text{Im } \mu = \text{Ker } \varepsilon$, $G/\text{Ker } \varepsilon \simeq Q$. Usually one writes N additively, G and Q multiplicatively.

A *morphism* of extensions is a triple (α, β, γ) of homomorphisms such that the diagram

$$\begin{array}{lccccc} \mathbf{e}_1: & N_1 & \xrightarrow{\lambda_1} & G_1 & \xrightarrow{\mu_1} & Q_1 \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \mathbf{e}_2: & N_2 & \xrightarrow{\lambda_2} & G_2 & \xrightarrow{\mu_2} & Q_2 \end{array}$$

commutes. If α and γ – and hence β – are isomorphisms, then (α, β, γ) is an *isomorphism* of *extensions*. If α, γ are identity maps, it is called an *equivalence*. Let

$$[\mathbf{e}]$$

denote the equivalence class of $\mathbf{e}$ and write

$$\mathcal{E}(Q, N) = \{[\mathbf{e}] \mid \mathbf{e} \text{ an extension of } N \text{ by } Q\}$$

for the category of equivalence classes and morphisms of extensions of N by Q. The main object of extension theory is to describe the set $\mathcal{E}(Q, N)$.

Automorphisms

An isomorphism (α, β, γ) from $\mathbf{e}$ to $\mathbf{e}$ is called an *automorphism* of $\mathbf{e}$,

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} \\ N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \end{array}$$

The pair $(\alpha, \gamma) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ is then said to be *induced* by β in $\mathbf{e}$. The automorphisms of $\mathbf{e}$ clearly form a group $\mathrm{Aut}(\mathbf{e})$ and

$$\mathrm{Aut}(\mathbf{e}) \simeq N_{\mathrm{Aut}(G)}(\mathrm{Im}\ \mu)\ \leq \mathrm{Aut}(G).$$

We would like to understand the group $\mathrm{Aut}(\mathbf{e})$ and, in particular, to determine which pairs (α, γ) are *inducible in* $\mathbf{e}$.

Couplings and factor sets

Given an extension $\mathbf{e} : N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$, choose a *transversal function*

$$\tau : Q \to G,$$

i.e., a map such that $\tau\varepsilon =$ the identity map on Q. Conjugation in $\mathrm{Im}\ \mu$ by x^τ, $(x \in Q)$, induces an automorphism x^ξ in N,

$$(a^{x^\xi})^\mu = (x^\tau)^{-1} a^\mu x^\tau,\ (a \in N),$$

so we have a function

$$\xi : Q \to \mathrm{Aut}(N).$$

Note that x^ξ depends on the choice of τ, but $x^\xi(\mathrm{Inn}(N))$ does not. Define $x^\chi = x^\xi(\mathrm{Inn}(N)) \in \mathrm{Out}(N)$. Then

$$\chi : Q \to \mathrm{Out}(N)$$

is a homomorphism which is independent of τ. This is the *coupling* of the extension $\mathbf{e}$. Equivalent extensions have the same coupling, so we can form

$$\mathcal{E}_\chi(Q, N),$$

the subcategory of extensions of N by Q with coupling χ.

The function τ is usually not a homomorphism, but

$$x^\tau y^\tau = (xy)^\tau (\varphi(x, y))^\mu$$

where $\varphi(x, y) \in N$. The associative law $(x^\tau y^\tau) z^\tau = x^\tau (y^\tau z^\tau)$ implies that

$$\varphi(x, yz) + \varphi(y, z) = \varphi(xy, z) + \varphi(x, y) \cdot z^\xi \qquad (*)$$

for $x, y, z \in Q$. Such a function $\varphi : Q \times Q \to N$ is called a *factor set*. We may assume that $1_Q^\tau = 1_G$, in which case $\varphi(1, x) = 0 = \varphi(x, 1)$ for all $x \in Q$, and φ is called a *normalized* factor set.

From $x^\tau y^\tau = (xy)^\tau \varphi(x, y)^\mu$ we deduce that

$$x^\xi y^\xi = (xy)^\xi \overline{\varphi(x, y)}, \; (x, y \in Q) \qquad (**)$$

where $\overline{a}$ denotes conjugation by a in N. Call ξ and φ *associated functions* for the extension $\mathbf{e}$.

Constructing extensions

Suppose we are given groups N, Q and functions $\xi : Q \to \mathrm{Aut}(N)$ and $\varphi : Q \times Q \to N$ (normalized), satisfying $(*)$ and $(**)$. Then we can construct an extension

$$\mathbf{e}(\xi, \varphi) \; : \; N \overset{\mu}{\rightarrowtail} G(\xi, \varphi) \overset{\varepsilon}{\twoheadrightarrow} Q,$$

where $G(\xi, \varphi) = Q \times N$, with group operation

$$(x, a)(y, b) = (xy, \, \varphi(x, y) + ay^\xi + b), \quad (x, y \in Q, \, a, b \in N).$$

Also $a^\mu = (1, a)$ and $(x, a)^\varepsilon = x$. Then the transversal function $x \mapsto (x, 0)$ yields associated functions ξ, φ for $\mathbf{e}(\xi, \varphi)$.

If N is abelian, it is a Q-module via the coupling $\xi = \chi : Q \to \mathrm{Out}(N) = \mathrm{Aut}(N)$ and $\varphi \in Z^2(Q, N)$ is a 2-*cocycle*, while there is a bijection

$$\mathcal{E}_\chi(Q, N) \longleftrightarrow H^2(Q, N).$$

2 The Automorphism Group of an Extension

Consider an extension

$$\mathbf{e} : N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$$

with coupling χ. Assume $\mu : N \hookrightarrow G$ is inclusion and $\varepsilon : G \to Q = G/N$ is the canonical map. If $\alpha \in \mathrm{Aut}(\mathbf{e})$, then α induces automorphisms $\alpha|_N$ in N, $\alpha|_Q$ in Q, while $\alpha \mapsto (\alpha|_N, \; \alpha|_Q)$ is a homomorphism,

$$\Psi : \mathrm{Aut}(\mathbf{e}) \to \mathrm{Aut}(N) \times \mathrm{Aut}(Q).$$

If $\alpha \in \mathrm{Ker}\ \Psi$, then α is trivial on N and G/N, so $[G, \alpha] \leq A = Z(N)$, while the map $gN \mapsto g^{-1}g^\alpha$, $(g \in G)$, is a *derivation* or 1-cocycle from Q to $Z(N) = A$. In fact $\mathrm{Ker}\ \Psi \simeq Z^1(Q, A)$ and there is an exact sequence

$$0 \to Z^1(Q, A) \to \mathrm{Aut}(\mathbf{e}) \overset{\Psi}{\to} \mathrm{Aut}(N) \times \mathrm{Aut}(Q).$$

It is more difficult to identify $\mathrm{Im}\ \Psi$. This is where the *Wells sequence* comes into play.

Theorem 1. (C. Wells [12]) *Let* $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$ *be an extension with coupling* $\chi : Q \to \mathrm{Out}(N)$ *and let* $A = Z(N)$*. Then there is an exact sequence*

$$0 \to Z^1(Q, A) \to \mathrm{Aut}(\mathbf{e}) \xrightarrow{\Psi} \mathrm{Comp}(\chi) \xrightarrow{\Lambda} H^2(Q, A)$$

where $\mathrm{Comp}(\chi)$ *is the subgroup of* χ*-compatible pairs* $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$*, i.e., pairs satisfying* $\varphi\chi = \chi\overline{\vartheta}$*, with* $\overline{\vartheta}$ *conjugation by* ϑ *in* $\mathrm{Out}(N)$*.*

To see where the compatibility condition comes from, let $\alpha \in \mathrm{Aut}(\mathbf{e})$ induce (ϑ, φ), so that $(\alpha)\Psi = (\vartheta, \varphi)$. From

$$(a^{x^\tau})^\alpha = (a^\alpha)^{(x^\tau)^\alpha}, \quad (a \in N, x \in Q),$$

we get $x^\xi\vartheta \equiv \vartheta(x^\varphi)^\xi \bmod \mathrm{Inn}(N)$. Thus $\vartheta^{-1}x^\chi\vartheta = (x^\varphi)^\chi$ in $\mathrm{Out}(N)$, i.e. $\chi\overline{\vartheta} = \varphi\chi$.

The Wells map Λ

Let $(\vartheta, \varphi) \in \mathrm{Comp}(\Lambda)$. In order to understand where $(\vartheta, \varphi)\Lambda \in H^2(Q, A)$ comes from, we take note of two actions on the set $\mathcal{E}_\chi(Q, N)$.

(i) $H^2(Q, A)$ acts regularly on $\mathcal{E}_\chi(Q, N)$ by adding a fixed 2-cocycle to each factor set.

(ii) $\mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ acts in the natural way on $\mathcal{E}_\chi(Q, N)$.
Hence, given $(\vartheta, \varphi) \in \mathrm{Comp}(\chi)$ and $[\mathbf{e}] \in \mathcal{E}_\chi(Q, N)$, by regularity there is a unique $h \in H^2(Q, A)$ such that $[\mathbf{e}] = ([\mathbf{e}] \cdot (\vartheta, \varphi)) \cdot h$. Define

$$(\vartheta, \varphi)\Lambda = h,$$

so that

$$[\mathbf{e}] = ([\mathbf{e}] \cdot (\vartheta, \varphi)) \cdot (\vartheta, \varphi)\Lambda.$$

Properties of the Wells map

(i) Im $\Psi = $ Ker Λ. (This is a routine calculation.)

For a long time it was believed that Λ, which is clearly not a homomorphism, was merely a set map. Then in 2010 Jin and Liu [4] discovered two very interesting facts about Λ.

(ii) $\Lambda : \mathrm{Comp}(\chi) \to H^2(Q, A)$ is a derivation, so that $\Lambda \in Z^1(\mathrm{Comp}(\chi),\, H^2(Q, A))$ and

$$(UV)\Lambda = (U)\Lambda \cdot V + (V)\Lambda, \quad (U, V \in \mathrm{Comp}(\chi)).$$

(iii) The cohomology class

$$[\Lambda] \in H^1(\mathrm{Comp}(\chi),\, H^2(Q, A))$$

depends on $[\mathbf{e}]$ *only through its coupling* χ, i.e., extensions with the same coupling have cohomologous Wells maps Λ.

Applications of the Wells Sequence

For a given extension $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$ with coupling χ, the *inducibility problem* is to determine when a given pair $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ is induced by some automorphism of $\mathbf{e}$. This happens if and only if $(\vartheta, \varphi) \in \mathrm{Comp}(\chi)$ and $(\vartheta, \varphi)\Lambda = 0$.

We will describe theorems which reduce the inducibility problem to certain subgroups of Q.

Reduction to Sylow subgroups

Consider an extension $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q = G/N$ with coupling χ where Q is finite. Let $\pi(Q) = \{p_1, \ldots, p_k\}$ and choose $P_i \in \mathrm{Syl}_{p_i}(Q)$, say $P_i = R_i/N$. Then we have subextensions

$$\mathbf{e}_i : N \rightarrowtail R_i \twoheadrightarrow P_i$$

with couplings $\chi_i = \chi|_{P_i}$. Let $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$. Then $P_i^{\varphi} \in \mathrm{Syl}_{p_i}(Q)$, so $P_i^{\varphi} = P_i^{g_i^{-1}}$ for some $g_i \in G$. Then $P_i^{\varphi\overline{g}_i} = P_i$, so $\varphi\overline{g}_i|_{P_i} \in \mathrm{Aut}(P_i)$.

Theorem 2. *With the above notation, the pair* (ϑ, φ) *is inducible in* $\mathbf{e}$ *if and only if* $(\vartheta\overline{g}_i,\, \varphi\overline{g}_i|_{P_i})$ *is inducible in* $\mathbf{e}_i$ *for* $i = 1, 2, \ldots, k$.

Proof. Necessity is routine. Assume the condition holds, i.e. $(\vartheta\overline{g}_i,\, \varphi\overline{g}_i|_{P_i})$ is inducible for $i = 1, 2, \ldots, k$. Let $A = Z(N)$.

(i) (ϑ, φ) *is* χ*-compatible.* This is a straightforward calculation.

(ii) (ϑ, φ) *is inducible in* $\mathbf{e}$. To see this, form a subsequence of the Wells sequence for $\mathbf{e}$ by restricting to automorphisms that leave R_i invariant.

$$0 \to Z^1(Q, A) \to N_{\mathrm{Aut}(\mathbf{e})}(R_i) \to C_i \to H^2(Q, A)$$

where $C_i = \{(\lambda, \mu) \in \mathrm{Comp}(\chi) \mid P_i^\mu = P_i\}$. Now apply the restriction map for P_i to get the commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{\Lambda} & H^2(Q, A) \\ \Big\downarrow \mathrm{res}_{P_i} & & \Big\downarrow \mathrm{res}_{P_i} \\ \mathrm{Comp}(\chi_i) & \xrightarrow{\Lambda_i} & H^2(P_i, A) \end{array}$$

Since (ϑ, φ) and $(\overline{g}_i, \overline{g}_i)$ are χ-compatible, $(\vartheta\overline{g}_i, \varphi\overline{g}_i) \in \mathrm{Comp}(\chi)$. Also

$$(\vartheta\overline{g}_i, \varphi\overline{g}_i)\mathrm{res}_{P_i} \circ \Lambda_i = (\vartheta\overline{g}_i, \varphi\overline{g}_i|_{P_i})\Lambda_i = 0,$$

and $\Lambda \circ \mathrm{res}_{P_i}$ maps $(\vartheta\overline{g}_i, \varphi\overline{g}_i)$ to 0. Since Λ is a derivation,

$$(\vartheta\overline{g}_i, \varphi\overline{g}_i)\Lambda = ((\vartheta, \varphi)(\overline{g}_i, \overline{g}_i))\Lambda = (\vartheta, \varphi)\Lambda \cdot (\overline{g}_i, \overline{g}_i) + (\overline{g}_i, \overline{g}_i)\Lambda = (\vartheta, \varphi)\Lambda.$$

This is because $(\overline{g}_i, \overline{g}_i)$ is obviously inducible and it acts trivially on $H^2(Q, A)$. Thus $((\vartheta, \varphi)\Lambda)\mathrm{res}_{P_i} = 0$ for $i = 1, \ldots, k$.

Apply the corestriction map for P_i, noting that $(\mathrm{res}_{P_i}) \circ (\mathrm{cor}_{P_i})$ is multiplication by $|Q : P_i|$. Also $|Q| \cdot |H^2(Q, A)| = 0$ and $(\vartheta, \varphi)\Lambda$ has order a p_i'-number for all i. Hence $(\vartheta, \varphi)\Lambda = 0$, and (ϑ, φ) is inducible in $\mathbf{e}$. *QED*

Special cases of Theorem 1 have appeared in [3] and [8].

Reduction to finite subgroups

Next consider an extension $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$ with coupling χ where Q is a *locally finite* group. Choose a *local system* of finite subgroups in Q

$$\{Q_i\}_{i \in I},$$

i.e., every finite subset of Q is contained in some Q_i. Let I be ordered by inclusion, i.e., $i \leq j$ if and only if $Q_i \leq Q_j$. Then $\{Q_i\}$ is a direct system and $Q = \varinjlim \{Q_i\}$. By restricting to Q_i, we form the corresponding subextension

$$\mathbf{e}_i : N \rightarrowtail G_i \twoheadrightarrow Q_i = G_i/N, \quad (i \in I),$$

with coupling $\chi_i = \chi|_{Q_i}$.

Suppose that $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ is given such that $Q_i^\varphi = Q_i$ for all i. (If φ has finite order, such a system $\{Q_i\}$ will always exist). Assume that $(\vartheta, \varphi|_{Q_i})$ is inducible in $\mathbf{e}_i$ for all $i \in I$.

Question: does this imply that (ϑ, φ) is inducible in $\mathbf{e}$?

By restriction form the commutative diagram

$$\begin{array}{ccc} \mathrm{Comp}(\chi) & \xrightarrow{\Lambda} & H^2(Q,A) \\ \Big\downarrow{\scriptstyle \mathrm{res}_{Q_i}} & & \Big\downarrow{\scriptstyle \mathrm{res}_{Q_i}} \\ \mathrm{Comp}(\chi_i) & \xrightarrow{\Lambda_i} & H^2(Q_i,A) \end{array}$$

where $A = Z(N)$. Since $(\vartheta, \varphi|_{Q_i})\Lambda_i = 0$, we have $(\vartheta, \varphi)\Lambda \in \mathrm{Ker}(\mathrm{res}_{Q_i})$ for all $i \in I$, and $(\vartheta, \varphi)\Lambda$ belongs to

$$K = \mathrm{Ker}(H^2(Q,A) \to \varprojlim H^2(Q_i, A)) :$$

note here that $\left\{H^2(Q_i, A)\right\}$ is *an inverse system* of abelian groups with restriction maps.

A spectral sequence for $H^n(\varinjlim, -)$

In general cohomology does not interact well with direct limits. However, there is a spectral sequence converging to $H^n(\varinjlim \{Q_i\}, A) = H^n(Q, A)$, namely

$$E_2^{pq} \stackrel{p+q=n}{\Longrightarrow} H^n(Q, A)$$

where

$$E_2^{pq} = \varprojlim{}^{(p)} \{H^q(Q_i, A)\}$$

and $\varprojlim{}^{(p)}$ is the pth derived functor of $\varprojlim$. (This may be deduced from the Grothendieck spectral sequence – see [6], [9]). Hence when $n = 2$ we obtain a series

$$0 = L_0 \leq L_1 \leq L_2 \leq L_3 = H^2(Q, A)$$

where $L_1 \simeq E_\infty^{20}$, $L_2/L_1 \simeq E_\infty^{11}$ and $L_3/L_2 \simeq E_\infty^{02}$. Thus $L_2 = K$ and in our situation $(\vartheta, \varphi)\Lambda \in L_2$. To prove that $(\vartheta, \varphi)\Lambda = 0$ it suffices to show that

$$E_2^{11} = 0 = E_2^{20}.$$

For this to be true additional conditions must be imposed: for example,

$$\sum_p r_p(A) \; < \; \infty,$$

the sum being for $p = 0$ or a prime, i.e., A has *finite total rank*. In fact this condition implies that

$$\varprojlim{}^{(1)} \left\{H^1(N, A)\right\} = 0 = \varprojlim{}^{(2)} \left\{A^N\right\},$$

(see [2]). Hence $(\vartheta, \varphi)\Lambda = 0$ and (ϑ, φ) is inducible in **e**.

Theorem 3. *With the above notation, assume that $Z(N)$ has finite total rank. Then (ϑ, φ) is inducible in $\mathbf{e}$ if and only if $(\vartheta, \varphi|_{Q_i})$ is inducible in $\mathbf{e}_i$ for all $i \in I$.*

By combining Theorems 1 and 2 we reduce the inducibility problem for Q locally finite to the case of a finite p-group.

Counterexamples

Theorem 3 does not hold without some conditions on $A = Z(N)$. Consider a non-split extension

$$\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$$

where G is locally finite, $\pi(N) \cap \pi(Q) = \emptyset,\ 2 \notin \pi(N)$ and N is abelian. In fact there are many such extensions – see for example [5], [11]. Let $Q_i \leq Q$ be finite. Then $H^n(Q_i, N) = 0$ for all $n \geq 1$ by Schur's theorem, so that $\mathbf{e}_i : N \rightarrowtail G_i \twoheadrightarrow Q_i = G_i/N$ splits. Let $\vartheta \in \mathrm{Aut}(N)$ be the inversion automorphism. Then $(\vartheta, 1)$ is inducible in $\mathbf{e}_i$ for every i since $\mathbf{e}_i$ is a split extension. However, $(\vartheta, 1)$ is *not* inducible in $\mathbf{e}$: for if it were, the cohomology class Δ of e would satisfy $\Delta = \Delta\vartheta_* = -\Delta$ and hence $\Delta = 0$ since $H^2(Q, N)$ has no elements of order 2. This is a contradiction.

Remark. Full details of the proofs may be found in [10].

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