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On a canonical immersion of the A-jet manifolds into a Grassmann bundle

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Abstract. For a given smooth manifold M we will consider the ideals I of $\mathcal{C}^\infty(M)$ such that $\mathcal{C}^\infty/I$ is a Weil algebra of order k; the set of these ideals is the disjoint union of several A-jets manifolds; by fixing $\dim \mathcal{C}^\infty/I$ we will immerse the above mentioned set into a Grassmann bundle of the k-th cotangent bundle of M, explicitly showing the equations of such an immersion. Finally, in a particular case, we will see how the aforesaid A-jets manifolds are placed.

Keywords: Jet, contact element, Weil bundle, Grassmann bundle.

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Introduction

The theory of Weil bundles [5], describes in an elegant and powerful way an ample class of objects of the global analysis and differential geometry, comprised such ones as the bundles of (m, r)-velocities and iterated tangent bundles (see [3, 4]); moreover, that notion recovers the old and useful idea of S. Lie of considering not only the points of a manifold themselves but also infinitesimal manifolds or 'valued points'.

On the other hand, given a Weil bundle M^A, where A is a Weil algebra, was proved in [1] that, roughly speaking, the quotient under the action of the group $\operatorname{Aut} A$ is a manifold $J^A M$ which consists of the kernels of the corresponding A-points (see below); when A is the algebra of polynomials of order $\leq k$ in m undetermined, $\mathbb{R}^k_m$, we obtain the well-known (m, k)-jet spaces of M which constitute a decisive tool when studying partial differential equations (see, for example, [3, 4] and references therein).

One can easily deduce the interest of knowing the properties of the bundles $J^A M$; in [2] some affine properties are obtained; in [1] was deduced the tangent structure and also an immersion of $J^A M$ into certain Grasmann bundle.

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Here, we are concerned with a different aspect. First, being the elements of each J^AM ideals of the ring $\mathcal{C}^\infty(M)$ we will study here the spaces of ideals (of a suitable type), obtaining the equations defining this space into the aforementioned Grasmann bundle. Second, we will study in a particular case how the several manifolds J^AM are distributed into each one of those spaces of ideals.

1 Preliminaries

A Weil algebra, A, is a finite dimensional local rational $\mathbb{R}$-algebra; let us denote by $\mathfrak{m}_A$ its maximal ideal, $m = \dim \mathfrak{m}_A/\mathfrak{m}_A^2$, and k the integer such that $\mathfrak{m}_A^{k+1} = 0$, $\mathfrak{m}_A^k \neq 0$; we will call k the order of A.

Remark 1. If the classes of $f_1, \ldots, f_m \in \mathfrak{m}_A$ generate $\mathfrak{m}_A/\mathfrak{m}_A^2$, then any element of A can be obtained as a polynomial in the f_i, that is, $A = \mathbb{R}\left[f_1, \ldots, f_m\right]$.

Examples of Weil algebras are $\mathbb{R}$, $\mathbb{R}\left[\varepsilon\right]/\varepsilon^2$ or, more in general, $\mathbb{R}^k_m \overset{def}{=} \mathbb{R}\left[\varepsilon_1, \ldots, \varepsilon_m\right] / \left(\varepsilon_1, \ldots, \varepsilon_m\right)^{k+1}$ and the tensor products $\mathbb{R}^{k_1}_{m_1} \otimes \cdots \otimes \mathbb{R}^{k_r}_{m_r}$.

Let us fix a n-dimensional smooth manifold M.

Definition 1. The set M^A of the $\mathbb{R}$-algebra morphisms

$$p^A : \mathcal{C}^\infty\left(M\right) \to A$$

is the so-called space of A-points of M associated to A; we have a map $M^A \xrightarrow{\pi} M$ which sends p^A to the point $p \in M$ corresponding to the composition $\mathcal{C}^\infty\left(M\right) \xrightarrow{p^A} A \to A/\mathfrak{m}_A = \mathbb{R}$. In fact, M^A can be endowed with a smooth structure such that π becomes a fiber bundle which is known as the Weil bundle on M associated to A. We will say that a A-point p^A is regular if it is surjective; the set of regular A-points $\check{M}^A$ is a dense open set of M^A (see [3, 4]).

Examples of Weil bundles are the very $M = M^{\mathbb{R}}$, the tangent bundle $TM = M^{\mathbb{R}^1_1}$, the iterated tangent bundles $TT\overset{r}{\cdots}TM = M^{\mathbb{R}^1_1\otimes\cdots\otimes\mathbb{R}^1_1}$, the frame bundle $\mathcal{R}(M) = \check{M}^{\mathbb{R}^1_n}$, etc.

Definition 2. The kernel of a regular A-point p^A will be called the jet of p^A and we will denote it by $\mathfrak{p}^A = \mathrm{Ker}(p^A)$. The set J^AM comprised by the jets of regular A-points will be called space of A-jets of M.

Proposition 1. *The set J^AM can be endowed with an smooth manifold structure in such a way that the map* $\mathrm{Ker} : \check{M}^A \to J^AM$ *becomes a principal fiber bundle with structural group* $\mathrm{Aut}\, A$.

Proof. See [1] QED

Let $\mathfrak{p}^A$ be the jet of p^A, which projects onto $p \in M$; in particular, $\mathfrak{p}^A$ is an ideal of the ring $\mathcal{C}^\infty(M)$ containing $\mathfrak{m}_p^{k+1}$, where $\mathfrak{m}_p$ is the maximal ideal of the functions vanishing at p and k is the order of A. Therefore we have $\mathfrak{m}_p^{k+1} \subseteq \mathfrak{p}^A \subseteq \mathfrak{m}_p$.

Definition 3. An ideal $I \subset \mathcal{C}^\infty(M)$ such that $\mathfrak{m}_p^{k+1} \subseteq I \subseteq \mathfrak{m}_p$, $\mathfrak{m}_p^k \subsetneq I$, for a point $p \in M$, will be called a Weil ideal of order k at $p \in M$.

Observe that a Weil ideal of order k defines a Weil algebra of order k, $\mathcal{C}^\infty(M)/I$; also observe that such a I is completely determined by its class modulo $\mathfrak{m}_p^{k+1}$.

Let us denote $d(I) \overset{def}{=} \dim I/\mathfrak{m}_p^{k+1}$; the set of Weil ideals of order $\leq k$ at a point p with fixed $d = d(I)$ will be denoted by $I_{d,p}^k$; the same way we put $I_d^k = \coprod_{p\in M} I_{d,p}^k$.

Each ideal $I \in I_{d,p}^k$ can be identified with a d-dimensional subspace of $\mathfrak{m}_p/\mathfrak{m}_p^{k+1}$; that is, $I_{d,p}^k$ is a subset of the Grassmann manifold $Gr(d, \mathfrak{m}_p/\mathfrak{m}_p^{k+1})$. More in general, we have a natural inclusion

$$I_d^k \subseteq Gr(d, T^{*,k}M) \tag{1}$$

where $T^{*,k}M$ is the k-th cotangent fiber bundle of M (the fiber of $T^{*,k}M$ at $p \in M$ is $T_p^{*,k}M = \mathfrak{m}_p/\mathfrak{m}_p^{k+1}$).

In Section 2 we will obtain the equations of that inclusions.

On the other hand, let $\mathcal{A}$ be the set of non isomorphic Weil algebras A such that there exists at least a Weil ideal I with $A \simeq \mathcal{C}^\infty(M)/I$; then,

$$I_d^k = \coprod_{A\in\mathcal{A}} J^A M \tag{2}$$

How do the jet manifolds J^AM are distributed into I_d^k, and hence, into $Gr(d, \mathfrak{m}_p/\mathfrak{m}_p^{k+1})$? In Section 3 we will completely solve this problem in a particular situation: $\dim M = d = k = 2$; we hope the results of this example can give same light about the general situation.

2 The equations of the space of Weil ideals

Let V be a $\mathbb{K}$-vector space, $E \subset V$ a d-dimensional vector subspace and φ an endomorphism of V. Later we will need to obtain the conditions for $\varphi(E) \subseteq E$.

Let $\omega_E \in \bigwedge^d V$ be a representative element of E; that is, if $\{e_1, \dots, e_d\}$ is a basis of E we take the exterior product $\omega_E = e_1 \wedge \cdots \wedge e_d$. Let us consider the $\mathbb{K}$-derivation

$$D_\varphi \colon \bigwedge^d V \to \bigwedge^d V \tag{3}$$

induced by φ; in other words, if $\sigma = v_1 \wedge \cdots \wedge v_d \in \bigwedge^d V$ then

$$D_\varphi(\sigma) \overset{def}{=} \sum_i v_1 \wedge \cdots \wedge v_{i-1} \wedge \varphi(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_d$$

Proposition 2. *A vector subspace E of V is stable by an endomorphism φ (i.e. $\varphi(E) \subseteq E$) if and only if there is an scalar λ such that*

$$D_\varphi \omega_E = \lambda \omega_E \tag{4}$$

for a representative element $\omega_E \in \bigwedge^d V$ of E. In such a case, λ is the trace of φ when restricted to E.

Proof. If $\varphi(E) \subseteq E$ then trivially $D_\varphi \omega_E = \lambda \omega_E$. For the converse let us suppose that $D_\varphi \omega_E = \lambda \omega_E$, where $\omega_E = e_1 \wedge \cdots \wedge e_d$ for a given basis $\mathcal{B} = \{e_1, \dots, e_d\}$ of E. If, for example, $\varphi(e_1) = v \notin E$, we have,

$$D_\varphi(\omega_E) = v \wedge e_2 \wedge \cdots \wedge e_d + e_1 \wedge \sum_{j \geq 2} (e_2 \wedge \cdots \wedge e_{j-1} \wedge \varphi(e_j) \wedge e_{j+1} \wedge \cdots \wedge e_d)$$

then, $e_1 \wedge D_\varphi(\omega_E) \neq 0$ but $e_1 \wedge \omega_E = 0$. We deduce that $D_\varphi(\omega_E)$ cannot be proportional to ω_E. QED

Now we will apply the result above to the following problem: when does a vector subspace $\overline{E}$, with $\mathfrak{m}_p^{k+1} \subseteq \overline{E} \subseteq \mathfrak{m}_p$, is an ideal?

Lemma 1. *Let $(x_1, \dots, x_n)$ be local coordinates around $p \in M$ and $\overline{E}$ a vector subspace with $\mathfrak{m}_p^{k+1} \subseteq \overline{E} \subseteq \mathfrak{m}_p$; then, $\overline{E}$ is an ideal of $\mathcal{C}^\infty(M)$ if and only if*

$$(x_i - x_i(p)) \cdot \overline{E} \subset \overline{E}, \qquad i = 1, \dots, n$$

Proof. Let us suppose the condition $(x_i - x_i(p)) \cdot \overline{E} \subset \overline{E}$, $i = 1, \dots, n$, is satisfied. Each function $f(x) \in \mathcal{C}^\infty(M)$ can be written as $f(x) = P(x) + \overline{f}(x)$, where $P(x)$ is a polynomial in the $(x_i - x_i(p))$, and $\overline{f} \in \mathfrak{m}_p^{k+1}$; obviously $\overline{f} \cdot \overline{E} \subset \mathfrak{m}_p^{k+1} \subset \overline{E}$ and, by hypothesis, $P(x) \cdot \overline{E} \subset \overline{E}$; then $\overline{E}$ is an ideal. The converse is trivial. QED

Proposition 3. *Let* $\overline{E}$ *be as above,* $E \overset{def}{=} \overline{E}/\mathfrak{m}_p^{k+1} \subset \mathfrak{m}_p/\mathfrak{m}_p^{k+1}$ *and denote by* $\varphi_i\colon \mathfrak{m}_p/\mathfrak{m}_p^{k+1} \to \mathfrak{m}_p/\mathfrak{m}_p^{k+1}$ *the endomorphisms defined as* $\varphi_i[f] = [(x_i - x_i(p))\cdot f]$, $i = 1,\dots,n$, *where* $f \in \mathfrak{m}_p$ *and* $[\ ]$ *means the class* $\operatorname{mod}\mathfrak{m}_p^{k+1}$. *Then,* $\overline{E}$ *is an ideal if and only if*

$$D_{\varphi_i}\omega_E = 0, \qquad i = 1,\dots,n. \tag{5}$$

Proof. By Lemma 1, $\overline{E}$ is an ideal if and only if E is stable by the φ_i. According to Proposition 2, that is equivalent to $D_{\varphi_i}\omega_E = \lambda_i\omega_E$; in this case, each $\lambda_i \in \mathbb{R}$ is the trace of φ_i when restricted to E. But, obviously, the endomorphisms φ_i are nilpotent and hence they have no trace. QED

We will use the above characterization to getting the equations of the subspace $I_{d,p}^k$ comprised by the points of $Gr(d, \mathfrak{m}_p/\mathfrak{m}_p^{k+1})$ that represent Weil ideals.

Let us fix a local chart $\{\mathcal{U}, (x_1,\dots,x_n)\}$, $p \in \mathcal{U}$, and denote $\overline{x}_i = x_i - x_i(p)$. Let us take the products $\overline{x}^\alpha \overset{def}{=} \overline{x}_1^{a_1}\cdots\overline{x}_n^{a_n}$, $\alpha = (a_1,\dots,a_n) \in \mathbb{N}^n$, $|\alpha| = a_1 + \cdots + a_n \le k$. The classes $[\overline{x}^\alpha] \equiv \overline{x}^\alpha \operatorname{mod}\mathfrak{m}_p^{k+1}$ define a basis of the vector space $V \overset{def}{=} \mathfrak{m}_p/\mathfrak{m}_p^{k+1}$.

Now we order the indexes α according to the lexicographic rule: let $\alpha = (a_1,\dots,a_n)$, $\beta = (b_1,\dots,b_n)$; then we say that $\alpha < \beta$ if and only if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $a_1 = b_1, \dots a_{i-1} = b_{i-1}, a_i > b_i$, for some i. For example, if $n = 2$, we have $(1,0) < (0,1) < (2,0) < (1,1) < (0,2) < \cdots$.

For any ordered multi-index $H = (\alpha_1,\dots,\alpha_n)$ (i.e., $\alpha_1 < \alpha_2 < \cdots$), we form the d-vector

$$e_H \overset{def}{=} [\overline{x}^{\alpha_1}] \wedge \cdots \wedge [\overline{x}^{\alpha_n}] \in \bigwedge^d V; \tag{6}$$

The collection $\{e_H\}$ provides a basis of $\bigwedge^d V$. Thus, each point $P \in Gr(d,V) \subseteq \mathbb{P}(\bigwedge^d V)$ (where $\mathbb{P}(\bigwedge^d V)$ is the projective space associated to $\bigwedge^d V$) is represented in the following way,

$$e_P = \sum_H \lambda_{H,p} e_H \in \bigwedge^d V; \tag{7}$$

where the coefficients $\lambda_{H,p} \in \mathbb{R}$ are the homogeneous coordinates of $P \in \mathbb{P}(\bigwedge^d V)$ and verify the Plücker relations.

Let us express the equations of Proposition 3 in terms of the coordinates $\lambda_{H,p}$. Recall that $\varphi_i[f] = [\overline{x}_i f]$; in particular, $\varphi_i[\overline{x}^\alpha] = [\overline{x}_1^{\alpha_1}\cdots\overline{x}_i^{\alpha_i+1}\cdots\overline{x}_n^{\alpha_n}] = [\overline{x}^{\alpha+\epsilon_i}]$, where $\epsilon_i = (0,\dots,1^i,\dots,0)$. Therefore,

$$D_{\varphi_i} e_H = \sum_j [\overline{x}^{\alpha_1}] \wedge \cdots \wedge [\overline{x}^{\alpha_i+1}] \wedge \cdots \wedge [\overline{x}^{\alpha_d}]. \tag{8}$$

If we denote by $H+\epsilon_i^j$ the ordered multi-index obtained from $(\alpha_1,\dots,\alpha_j+\epsilon_i,\dots,\alpha_d)$ by means of a suitable number $\sigma(H,\epsilon_i^j)$ of permutations, we get

$$D_{\varphi_i}e_H=\sum_j(-1)^{\sigma(H,\epsilon_i^j)}e_{H+\epsilon_i^j}.$$

Finally, the equations determining $I_{d,p}^k$ into $Gr(d,\mathfrak{m}_p/\mathfrak{m}_p^{k+1})$ are

$$\sum_{H+\epsilon_i^j=K}(-1)^{\sigma(H,\epsilon_i^j)}\lambda_{H,p}=0,\quad |K|=d+1;\ i=1,\dots,n. \tag{9}$$

From the local chart $\{\mathcal{U},(x_1,\dots,x_n)\}$; $\mathcal{U}\subseteq M$, we define homogeneous fiber coordinates $\{\lambda_H\}$ on the bundle $\mathbb{P}(\bigwedge^d T^{*,k}M)=\bigcup_{p\in M}\mathbb{P}(\bigwedge^d\mathfrak{m}_p/\mathfrak{m}_p^{k+1})\to M$, by the rule

$$\lambda_H(P)=\lambda_{H,p}(P)$$

where P projects onto $p\in\mathcal{U}\subseteq M$ and $\lambda_{H,p}$ is defined by (7).

Proposition 4. *With the above notation, the local equations of the space of ideals I_d^k into $Gr(d,T^{*,k}M)\subseteq\mathbb{P}(\bigwedge^d T^{*,k}M)$, are*

$$\sum_{H+\epsilon_i^j=K}(-1)^{\sigma(H,\epsilon_i^j)}\lambda_H=0,\quad |K|=d+1;\ i=1,\dots,n.$$

3 The structure of I_2^2M, $\dim M=2$.

In that follows we will fix a 2-dimensional manifold M.

Consider a local chart $\{\mathcal{U},(x=x_1,y=x_2)\}$. For each $p\in\mathcal{U}$ we obtain a basis $\{e_1,e_2,e_3,e_4,e_5\}$ of $\mathfrak{m}_p/\mathfrak{m}_p^3$ defined as follows: $e_1=[\overline{x}]$, $e_2=[\overline{y}]$, $e_3=[\overline{x}^2]$, $e_4=[\overline{xy}]$, $e_5=[\overline{y}^2]$, where, $\overline{x}=x-x(p)$ and $\overline{y}=y-y(p)$ (in this case we have simplified the notation by removing multi-indexes).

From relations

$$\begin{array}{lll}\overline{x}e_1=e_3 & \overline{x}e_2=e_4 & \overline{x}e_3=\overline{x}e_4=\overline{x}e_5=0\\ \overline{y}e_1=e_4 & \overline{y}e_2=e_5 & \overline{y}e_3=\overline{y}e_4=\overline{y}e_5=0\end{array}$$

we obtain

$$\begin{array}{ll}
D_{\overline{x}}(e_1\wedge e_2)=-e_2\wedge e_3+e_1\wedge e_4 & D_{\overline{y}}(e_1\wedge e_2)=-e_2\wedge e_4+e_1\wedge e_5\\
D_{\overline{x}}(e_1\wedge e_3)=0 & D_{\overline{y}}(e_1\wedge e_3)=-e_3\wedge e_4\\
D_{\overline{x}}(e_1\wedge e_4)=e_3\wedge e_4 & D_{\overline{y}}(e_1\wedge e_4)=0\\
D_{\overline{x}}(e_1\wedge e_5)=e_1\wedge e_5 & D_{\overline{y}}(e_1\wedge e_5)=0\\
D_{\overline{x}}(e_2\wedge e_3)=-e_3\wedge e_4 & D_{\overline{y}}(e_2\wedge e_3)=-e_3\wedge e_5\\
D_{\overline{x}}(e_2\wedge e_4)=0 & D_{\overline{y}}(e_2\wedge e_4)=-e_4\wedge e_5\\
D_{\overline{x}}(e_2\wedge e_5)=e_4\wedge e_5 & D_{\overline{y}}(e_2\wedge e_5)=0\\
D_{\overline{x}}(e_i\wedge e_j)=0,\ i,j\geq 3 & D_{\overline{y}}(e_i\wedge e_j)=0,\ i,j\geq 3
\end{array} \tag{10}$$

where $D_{\overline{x}} = D_{\varphi_1}$ and $D_{\overline{y}} = D_{\varphi_2}$ (see the notation in Proposition 3).

Let $P \in Gr(d, T^{*,k}M)$ which projects to $p \in M$ and is represented by the 2-vector

$$e_P = \sum_{1 \leq i < j \leq 5} \lambda_{ij} e_i \wedge e_j$$

By applying (10) we see that the equations of Proposition 3 are, in this case,

$$\begin{aligned} 0 = D_{\overline{x}} e_P = & -\lambda_{12} e_2 \wedge e_3 + \lambda_{12} e_1 \wedge e_5 + \lambda_{14} e_3 \wedge e_4 \\ & + \lambda_{15} e_3 \wedge e_5 - \lambda_{23} e_3 \wedge e_4 + \lambda_{25} e_4 \wedge e_5 \\ 0 = D_{\overline{y}} e_P = & -\lambda_{12} e_2 \wedge e_4 + \lambda_{12} e_1 \wedge e_5 - \lambda_{13} e_3 \wedge e_4 \\ & + \lambda_{15} e_4 \wedge e_5 - \lambda_{23} e_3 \wedge e_5 - \lambda_{24} e_4 \wedge e_5 \end{aligned}$$

From which we get: $\lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{23} = \lambda_{24} = \lambda_{25} = 0$ and so

$$e_P = \lambda_{34} e_3 \wedge e_4 + \lambda_{35} e_3 \wedge e_5 + \lambda_{45} e_4 \wedge e_5; \tag{11}$$

in particular, the Plücker relations are automatically satisfied by such a e_P (because $e_P \in \bigwedge^2 \langle e_3, e_4, e_5 \rangle$).

For simplicity, let us denote

$$a = \lambda_{34}, \qquad b = \lambda_{35}, \qquad c = \lambda_{45}; \tag{12}$$

this way, the vector subspace (and also ideal, as we know) associated to e_P is

$$I_P = \{c_3 e_3 + c_4 e_4 + c_5 e_5 \ / \ c c_3 - b c_4 + a c_5 = 0, \ c_i \in \mathbb{R}\} \subset \mathfrak{m}_p \tag{13}$$

Now, we want to describe the possible structures of the Weil algebra $A = \mathcal{C}^\infty(M)/I_P \simeq \mathbb{R}[\overline{x}, \overline{y}]/I_P$. If $\mathfrak{m}_A$ denotes the maximal ideal of A, we have $\mathfrak{m}_A^3 = 0$ and $\dim A = \dim(\mathbb{R}[\overline{x}, \overline{y}]/\mathfrak{m}_p^3) - \dim(I_A/\mathfrak{m}_p^3) = 6 - 2 = 4$. Besides, $\dim(\mathfrak{m}_A/\mathfrak{m}_A^2) = 2$; in fact, that dimension must be lower or equal than 2, if $\dim(\mathfrak{m}_A/\mathfrak{m}_A^2) = 1$, then there exist an $f \in \mathfrak{m}_A$ such that $A = \mathbb{R}[f]$ and hence $\dim A \leq 3$, which is contradictory.

Lemma 2. *Let B be a Weil algebra of dimension 4 and $\dim(\mathfrak{m}_B/\mathfrak{m}_B^2) = 2$. Let us denote s the maximum number of linearly independent (modulo $\mathfrak{m}_B^2$) solutions of the equation $f^2 = 0$, $f \in \mathfrak{m}_B$. The following isomorphisms holds:*

(1) If $s = 0$, then $B \simeq \mathbb{R}[t, \tau]/(t^2 - \tau^2, t\tau)$

(2) If $s = 1$, then $B \simeq \mathbb{R}[t, \tau]/(t^2, t\tau, \mathfrak{m}^3)$

(3) If $s = 2$, then $B \simeq \mathbb{R}[t, \tau]/(t^2, \tau^2)$

where t, τ are undetermined and $\mathfrak{m}$ denotes the maximal ideal that they generate.

Proof. Let $f, g \in \mathfrak{m}_B$ be such that their classes generate $\mathfrak{m}_B/\mathfrak{m}_B^2$; in particular, $B = \mathbb{R}[f, g]$.

Case 1) $s = 0$. If the functions f^2, fg, g^2 generate (over $\mathbb{R}$) a vector subspace of dimension greater than one, then $\dim B > 5$; so, two of them are proportional to the third one; subcase 1.1) there exist $\lambda, \mu \in \mathbb{R}$ such that $fg = \lambda f^2$, $g^2 = \mu f^2$; we deduce $f(g - \lambda f) = 0$; hence, we can suppose $\lambda = 0$; on the other hand, if $\mu \leq 0$ we have $0 = g^2 - \mu f^2 = (g - \sqrt{-\mu} f)^2$ and then $s \neq 0$; therefore $\mu > 0$ and we can take $\sqrt{\mu} f$ as a new f; that is, we can suppose that the relations are $fg = 0$ and $f^2 - g^2 = 0$; subcase 1.2) there exist $\lambda, \mu \in \mathbb{R}$ such that $f^2 = \lambda fg$, $g^2 = \mu fg$; necessarily, $\lambda, \mu \neq 0$ because $s = 0$; then have $fg = \frac{1}{\lambda} f^2$, $g^2 = \frac{\mu}{\lambda} f^2$ which correspond to 1.1; subcase 1.3) there exist $\lambda, \mu \in \mathbb{R}$ such that $fg = \lambda g^2$, $f^2 = \mu g^2$; changing the rules of f and g we are once again in the situation 1.1. Then, we can define the surjective morphism $\mathbb{R}[t, \tau]/(t^2 - \tau^2, t\tau) \to B = \mathbb{R}[f, g]$ sending $t \mapsto f$, $\tau \mapsto g$, taking into account the respective dimensions we deduce that this map is an isomorphism.

Case 2) $s = 1$. We can suppose that f is the unique independent solution of $f^2 = 0$. Because $\dim B = 4$, vectors 1, f, g, g^2, fg cannot be linearly independents; thus, there exist a non trivial relation

$$\lambda_1 g^2 + \lambda_2 fg + \lambda_3 f + \lambda_4 g + \lambda_5 1 = 0;$$

first observe that $\lambda_5 = 0$ (if not, $1 \in \mathfrak{m}_B$); moreover $\lambda_3 f + \lambda_4 g \equiv 0 \mod \mathfrak{m}_B^2$, which is impossible if λ_3, λ_4 are not identically vanishing; thus, the above relation reduces to $\lambda_1 g^2 + \lambda_2 fg = 0$; if $\lambda_1 \neq 0$ we can suppose $\lambda_1 = 1$ and then $g^2 + \lambda_2 fg = (g + \frac{\lambda_2}{2} f)^2 = 0$, which contradicts the assumption $s = 1$. As a consequence $\lambda_1 = 0$ and $\lambda_2 fg = 0$, where $\lambda_2 \neq 0$; that is, $fg = 0$. Now we define the surjective morphism $\mathbb{R}[t, \tau]/(t^2, t\tau, \mathfrak{m}^3) \to B = \mathbb{R}[f, g]$ sending $t \mapsto f$, $\tau \mapsto g$; by computing dimensions we conclude.

Case 3) $s = 2$. In this situation we can suppose that two independent solutions are f, g; that is, $f^2 = g^2 = 0$; we finish the proof as in the previous cases. QED

Let us denote the three Weil algebras appearing in Lemma 2 by B_s, $s = 0, 1, 2$. We will apply this result to classify the algebra $\mathcal{C}^\infty(M)/I_P$, depending of the parameters a, b, c. Recall that

$$I_P = \left\{ c_3 \overline{x}^2 + c_4 \overline{xy} + c_5 \overline{y}^2 \;/\; cc_3 - bc_4 + ac_5 = 0,\; c_i \in \mathbb{R} \right\}$$

If, for example, $b \neq 1$, I_P will be generated by vectors $\overline{x}^2 + \frac{c}{b}\overline{xy}$ and $\overline{y}^2 + \frac{a}{b}\overline{xy}$. Now we search for the number of solutions of $f^2 = 0$, with $f = \lambda[\overline{x}] + \mu[\overline{y}] \in \mathcal{C}^\infty(M)/I_P$ (here, symbol $[\ ]$ means the class mod I_P). Taking into account relations $\overline{x}^2 + \frac{c}{b}\overline{xy}$, $\overline{y}^2 + \frac{a}{b}\overline{xy} \equiv 0 \mod I_P$, we have $f^2 = \lambda^2[\overline{x}^2] + \lambda\mu[\overline{xy}] + \mu^2[\overline{y}^2] =$

$-\frac{c}{b}\lambda^2[\overline{xy}] + 2\lambda\mu[\overline{xy}] - \frac{a}{b}\mu^2[\overline{xy}] = (-\frac{c}{b}\lambda^2 + 2\lambda\mu - \frac{a}{b}\mu^2)[\overline{xy}]$; then $f^2 = 0$ if and only if $c\lambda^2 - 2b\lambda\mu - a\mu^2 = 0$.

The number of independent solutions of the last equation is 0, 1 or 2 if $\triangle < 0$, $\triangle = 0$ or $\triangle > 0$, respectively, where $\triangle \overset{def}{=} b^2 - ac$. The same conclusion is easily obtained if we suppose instead $a \neq 0$ or $c \neq 0$.

Therefore, by applying Lemma 2 we have finally,

Theorem 1. *If* $\dim M = 2$, *then*

$$I_2^2 M = J^{B_0}M \amalg J^{B_1}M \amalg J^{B_2}M \subset Gr(2, T^{*,2}M)$$

Moreover, with the above notation,

$$\begin{aligned} J^{B_0}M &= \left\{ \lambda_{35}^2 - \lambda_{34}\lambda_{45} < 0;\ \lambda_{ij} = 0, i \leq 2 \right\} \\ J^{B_1}M &= \left\{ \lambda_{35}^2 - \lambda_{34}\lambda_{45} = 0;\ \lambda_{ij} = 0, i \leq 2 \right\} \\ J^{B_2}M &= \left\{ \lambda_{35}^2 - \lambda_{34}\lambda_{45} > 0;\ \lambda_{ij} = 0, i \leq 2 \right\} \end{aligned}$$

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