Shock structure problem
in multi-temperature gaseous mixtures

Srboljub Simić[i]

Department of Mechanics, Faculty of Technical Sciences
University of Novi Sad, Serbia
`ssimic@uns.ac.rs`

Abstract. This paper discusses the shock structure problem in hyperbolic model of multi-temperature gaseous mixtures. The model is used as a paradigm for physically motivated hyperbolic dissipative systems. Existence of continuous shock profile is usually related to the properties of simpler, equilibrium subsystem. The purpose of this paper is to show that appearance of the admissible shock profile can be related to stability properties of equilibrium states. This result can be obtained by analysis of the complete hyperbolic system, i.e. without prior study of equilibrium subsystem.

1   Introduction

The shock structure problem is standard but still challenging problem in mathematical modelling of irreversible processes in physics. One way for its study was developed within the framework of continuum physics. It is relied on physical balance laws, but description of shock phenomena is affected by the kind of dissipation introduced in the model. For example, Euler equations of gas dynamics predict the appearance of the shock wave – a moving singular surface on which jump of state variables occur. On the other hand, classical Navier-Stokes-Fourier (NSF) model predicts existence of a travelling wave with narrow region in which large gradients of state variables appear. Physically, viscosity and heat conduction smooth out the shock wave predicted by the Euler model. Mathematically, hyperbolic system of conservation laws (Euler) is replaced by the hyperbolic-parabolic system (NSF) which regularizes the singularity.

Although physically motivated and mathematically efficient, parabolic (or viscous, as usually named) regularizations of hyperbolic systems are physically relevant only in processes which occur in the neighborhood of local equilibrium state. Another way of capturing non-equilibrium effects is to extend the set of state variables and to write the governing equations for them in the form of

---

balance laws. Source terms are supposed to introduce dissipation in the model and thus regularize the solutions of hyperbolic systems which comprise singular surfaces. An example of this kind is Grad's 13 moments model which firstly emerged in kinetic theory of gases [6] and later appeared in extended thermodynamics [9].

Either in parabolic, or in hyperbolic dissipative models the existence of shock structure, i.e. continuous shock profile which connects equilibrium states, is related to the structure of equilibrium subsystem obtained in the limit of vanishing dissipation (viscosity or relaxation). In other words, one always seek for a continuous solution which asymptotically connects the states (a) which are related by Rankine-Hugoniot equations and (b) satisfy some kind of admissibility criterion. The aim of this paper is to show that appearance of admissible shock profile can be indicated by the change of stability properties of equilibrium states and inequality thus obtained is parallel to shock admissibility criterion. The main feature of this approach is that it can be applied to complete hyperbolic system of balance laws directly, i.e. without prior study of an equilibrium subsystem. Due to lack of the proof of equivalence between stability condition and shock admissibility criterion, the present study of shock structure will be restricted to a hyperbolic model of multi-temperature gaseous mixtures which will serve as a paradigm for hyperbolic systems of balance laws.

The paper is organized as follows. In Section 2 we shall give a brief account on dissipative models in continuum physics and their mathematical properties which are important in the study of shock waves. Section 3 will be devoted to description of multi-temperature model for mixtures of Euler fluids. It will clearly put the model in the framework of hyperbolic dissipative models. Finally, in Section 4 we give our main result – a criterion based upon stability properties of equilibrium states which indicates the appearance of admissible shock structure.

2   Dissipative models in continuum physics

It is well known that continuum models in physics are usually expressed in the form of conservation laws

$$\partial_t \mathbf{F}^0(\mathbf{u}) + \partial_x \mathbf{F}(\mathbf{u}) = \mathbf{0} \tag{1}$$

where $\mathbf{u}(x,t) \in \mathbf{R}^n$ is the vector of state variables, $\mathbf{F}^0(\mathbf{u})$ is the vector of densities and $\mathbf{F}(\mathbf{u})$ the vector of fluxes. For the sake of simplicity we confined our study in this Section to one space dimension indicated by variable x.

When densities and fluxes are functions of state variables solely, i.e. not of their derivatives, it is expected that (1) is hyperbolic, at least in some region of

the state space. In other words, the eigenvalue problem

$$(-\lambda\mathbf{A}^0(\mathbf{u}) + \mathbf{A}(\mathbf{u}))\mathbf{r} = \mathbf{0};$$ (2)
$$\mathbf{A}^0(\mathbf{u}) = \partial\mathbf{F}^0(\mathbf{u})/\partial\mathbf{u}; \qquad \mathbf{A}(\mathbf{u}) = \partial\mathbf{F}(\mathbf{u})/\partial\mathbf{u},$$

has n real eigenvalues $\lambda_i(\mathbf{u})$ called characteristic speeds, and n linearly independent eigenvectors $\mathbf{r}_i(\mathbf{u})$, $i = 1,\dots,n$; $\mathbf{A}^0(\mathbf{u})$ is assumed to be nonsingular. This property permits modelling of wave propagation through space with finite speeds.

Hyperbolicity is the main cause for non-existence of smooth solutions – even smooth initial data can evolve into jump discontinuities in finite time interval. These discontinuities – shock waves – are located on the singular surface $\Sigma(x,t)$. If $[[\cdot]] = (\cdot)_1 - (\cdot)_0$ denotes the jump of any quantity in front (upstream) $(\cdot)_0$ and behind (downstream) $(\cdot)_1$ the surface Σ which propagates with speed s, then Rankine-Hugoniot conditions relate the jump of field variables across the surface to the shock speed

$$[[\mathbf{F}(\mathbf{u})]] = s[[\mathbf{F}^0(\mathbf{u})]].$$ (3)

Note that (3) is not affected by the interchange of the roles of upstream and downstream state.

Not all jump discontinuities which satisfy (3) are observable in reality. Along with Rankine-Hugoniot conditions, they have to satisfy additional ones which serve as selection rules for physically admissible solutions [4, 13]. One of the simplest conditions of this kind is Lax condition

$$\lambda_i(\mathbf{u}_1) \geq s \geq \lambda_i(\mathbf{u}_0).$$ (4)

It rules out the possibility for interchanging the roles of $\mathbf{u}_0$ and $\mathbf{u}_1$ and thus represents a particular form of irreversibility condition.

Classical continuum physics motivates so-called viscosity approach which regularizes the system (1) by adding parabolic terms

$$\partial_t\mathbf{F}^0(\mathbf{u}) + \partial_x\mathbf{F}(\mathbf{u}) = \varepsilon\partial_x(\mathbf{B}(\mathbf{u})\partial_x\mathbf{u})$$ (5)

where $\mathbf{B}(\mathbf{u})$ is viscosity matrix and $\varepsilon > 0$ small parameter, usually related to physical viscosity, heat conductivity and diffusivity. This model predicts infinite speed of propagation of disturbances. Therefore, characteristic speeds cannot even be defined for this class of models. However, parabolic terms regularize the shock waves which correspond to genuinely nonlinear characteristic speeds of (1). In such a way (5) comprise solutions representing the shock profile travelling uniformly with shock speed s. These models are proved to be reliable as long as the state of the system is not far from the local equilibrium one.

Another way of description of dissipative mechanisms is to take into account the relaxation effects. This calls for extension of the set of state variables $\mathbf{u} \in \mathbf{R}^n$ by $\mathbf{w} \in \mathbf{R}^k$, $n + k = N$, which are governed by the additional set of balance laws. In particular, we have

$$\partial_t \hat{\mathbf{F}}^0(\mathbf{U}) + \partial_x \hat{\mathbf{F}}(\mathbf{U}) = \frac{1}{\tau} \mathbf{Q}(\mathbf{U}), \tag{6}$$

where

$$\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{U}) = \begin{pmatrix} \mathbf{0} \\ \mathbf{q}(\mathbf{u}, \mathbf{w}) \end{pmatrix}, \tag{7}$$

$$\hat{\mathbf{F}}^0(\mathbf{U}) = \begin{pmatrix} \mathbf{f}^0(\mathbf{u}, \mathbf{w}) \\ \mathbf{g}^0(\mathbf{u}, \mathbf{w}) \end{pmatrix}, \qquad \hat{\mathbf{F}}(\mathbf{U}) = \begin{pmatrix} \mathbf{f}(\mathbf{u}, \mathbf{w}) \\ \mathbf{g}(\mathbf{u}, \mathbf{w}) \end{pmatrix},$$

and $\tau > 0$ is a small parameter – relaxation time. It is assumed that $\mathbf{q}(\mathbf{u}, \mathbf{w}) = \mathbf{0}$ uniquely determines the "equilibrium manifold" $\mathbf{w}_E = \mathbf{h}(\mathbf{u})$ as $\tau \to 0$, on which the system (6) reduces to (1) with $\mathbf{F}^0(\mathbf{u}) = \mathbf{f}^0(\mathbf{u}, \mathbf{h}(\mathbf{u}))$ and $\mathbf{F}(\mathbf{u}) = \mathbf{f}(\mathbf{u}, \mathbf{h}(\mathbf{u}))$. The first n equations in (6) are conservation laws, while remaining k ones are balance laws with source terms $\mathbf{q}(\mathbf{u}, \mathbf{w})/\tau$ which describe dissipative effects off the equilibrium manifold.

It is customary to expect that system (6) is hyperbolic at least in some subset of the extended state space $\mathbf{R}^N$ which contains the equilibrium manifold. Corresponding characteristic speeds $\Lambda_j(\mathbf{U})$, $j = 1, \ldots, N$, and the set of linearly independent eigenvectors $\mathbf{R}_j(\mathbf{U})$ are determined from the eigenvalue problem for the differential part of (6)

$$(-\Lambda \hat{\mathbf{A}}^0(\mathbf{U}) + \hat{\mathbf{A}}(\mathbf{U}))\mathbf{R} = \mathbf{0}; \tag{8}$$
$$\hat{\mathbf{A}}^0(\mathbf{U}) = \partial \hat{\mathbf{F}}^0(\mathbf{U})/\partial \mathbf{U}; \qquad \hat{\mathbf{A}}(\mathbf{U}) = \partial \hat{\mathbf{F}}(\mathbf{U})/\partial \mathbf{U}.$$

Important property of characteristic speeds $\Lambda_j(\mathbf{U})$ is that they provide bounds for the characteristic speeds of (1) on equilibrium manifold through the sub-characteristic condition [1, 3]

$$\min_{1 \le j \le N} \Lambda_j(\mathbf{u}, \mathbf{h}(\mathbf{u})) \le \lambda_i(\mathbf{u}) \le \max_{1 \le j \le N} \Lambda_j(\mathbf{u}, \mathbf{h}(\mathbf{u})). \tag{9}$$

However, the spectrum $\lambda_i(\mathbf{u})$ of the system (1) does not have to be contained in the spectrum $\Lambda_j(\mathbf{u}, \mathbf{h}(\mathbf{u}))$ of the hyperbolic dissipative system (6), i.e. λ's may not coincide with Λ's on the equilibrium manifold. Moreover, Rankine-Hugoniot conditions for (6) have different form than for (1), and consequently may predict jump discontinuities which appear off the equilibrium manifold. These discrepancies between hyperbolic systems and their dissipative counterparts call for the

answer to the question how can one relate the jump discontinuities of (1) to the shock structure solutions expected to be derived from (6). First results may be found in [8] for systems of two balance laws. General existence result for shock profiles has been given in [18], under certain reasonable structural conditions.

Dissipative character of parabolic (5) and hyperbolic (6) system comes from the compatibility with entropy inequality. In former case it imposes restrictions on viscosity matrix $\mathbf{B}(\mathbf{u})$, while in the latter one determines the structure of source terms $\mathbf{q}(\mathbf{u}, \mathbf{w})$.

If one assumes the solution in the form of a travelling wave, $\mathbf{u} = \mathbf{u}(x - st)$ or $\mathbf{U} = \mathbf{U}(x - st)$, models (5) and (6) reduce to systems of ordinary differential equations. It is shown [4] that (5) yields a system which can describe the shock structure as a heteroclinic orbit which connects stationary points. These stationary points correspond to equilibrium states of (5). They are related by Rankine-Hugoniot equations which can also be recognized as stationarity conditions of the reduced system. On the other hand, reduced system of the hyperbolic model (6) does not have such desirable properties or, at least, they are not obvious. Nevertheless, it was shown that these models also smooth out the shock waves predicted by the equilibrium subsystem [17, 9].

Shock structure problem in hyperbolic dissipative models requires analysis of equilibrium subsystem. In view of existence problem [18] it is assumed that equilibrium states are admissible solutions of Rankine-Hugoniot equations for equilibrium subsystem. In view of actual calculations [17] hyperbolic models are extensions of Euler model of gas dynamics and one intuitively expects that shock structure will appear in supersonic regime. It will be shown in the sequel that shock structure analysis in hyperbolic models can be performed without prior study of equilibrium subsystem. Stability analysis of stationary points of the reduced hyperbolic system yields simple criterion for appearance of admissible shock structure. It turns out that it is naturally related to admissible solutions of Rankine-Hugoniot equations for equilibrium subsystem. Thus far it was shown that this procedure yields correct results in the case of Burgers equation and isothermal viscoelasticity [14] as well as continuum models of gas dynamics [15]. In this study a multi-temperature model of gaseous mixtures [12] will be used as a paradigmatic hyperbolic model to show that stability criterion is reliable criterion for admissible shock structures.

3   Multi-temperature mixture of Euler fluids

The mixture model we shall deal with in this account arose within the framework of rational thermodynamics [16]. Roughly speaking, it is based upon assumption that behavior of each fluid component is governed by the same balance

laws as a single fluid, but mutual interactions among them have to be taken into account through appropriate source terms. On the other hand, governing equations for the mixture ought to be reconstructed from equations for components. However, these general principles allow for different levels of accuracy depending on the number of variables chosen to describe the state of each component.

We shall briefly describe one possible model of mixtures developed within the framework of extended thermodynamics [12] which is in full accordance with the principles mentioned above. Its main feature is that to each component is ascribed its own velocity and temperature field. The main motif for its development was to overcome the difficulties which appear in extended thermodynamics of mixtures with a single temperature. The aim was to derive reliable and tractable model at the same time which will accurately describe non-equilibrium processes in which disparate masses are main driving agent of thermal non-equilibrium.

The model of the mixture is consisted of balance laws of mass, momentum and energy for each component $\alpha = 1, \ldots, n$

$$
\begin{aligned}
&\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) = \tau_\alpha, \\
&\frac{\partial}{\partial t}(\rho_\alpha \mathbf{v}_\alpha) + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \mathbf{t}_\alpha) = \mathbf{m}_\alpha, \\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha\right) + \operatorname{div}\left\{\left(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha\right)\mathbf{v}_\alpha - \mathbf{t}_\alpha \mathbf{v}_\alpha + \mathbf{q}_\alpha\right\} = e_\alpha.
\end{aligned}
\tag{10}
$$

By ρ_α, $\mathbf{v}_\alpha$, ε_α, $\mathbf{t}_\alpha$ and $\mathbf{q}_\alpha$ we denote respectively the mass density, the velocity, the internal energy density, the stress tensor and the heat flux of each component. Source terms τ_α, $\mathbf{m}_\alpha$ and e_α describe the actions of other components which cause the change of mass, momentum and energy density of the chosen one. In the sequel it will be assumed that all the components of the mixture are Euler fluids, i.e. they are neither viscous, nor heat-conducting, so that stress tensor and heat flux are assumed to be

$$
\mathbf{t}_\alpha = -p_\alpha \mathbf{I}, \qquad \mathbf{q}_\alpha = \mathbf{0},
\tag{11}
$$

where p_α is the partial pressure. In such a model, the state of each component is determined by its own mass density, velocity and temperature field, $(\rho_\alpha, \mathbf{v}_\alpha, T_\alpha)^T$, which gives the name *multi-temperature* mixture. This approach is not widespread in continuum modelling of mixtures, except in the case of plasma, but naturally arises in macroscopic equations derived from kinetic theory of gases, i.e. Boltzmann equations for mixtures. The temperatures of components can be related to translational degrees of freedom of gas molecules, and do not have any relation to vibrational or rotational ones. They were introduced

with the aim to show that, in the absence of viscosity and heat conductivity, large mass disparity is the main driving agent for thermal nonequilibrium.

To reconstruct the governing equations for the mixture we need some more assumptions and definitions. It will be assumed that source terms satisfy the following relations

$$\sum_{\alpha=1}^{n} \tau_\alpha = 0, \qquad \sum_{\alpha=1}^{n} \mathbf{m}_\alpha = \mathbf{0}, \qquad \sum_{\alpha=1}^{n} e_\alpha = 0. \tag{12}$$

Along with them the following definitions for field variables of the whole mixture will be introduced

$$\rho = \sum_{\alpha=1}^{n} \rho_\alpha, \qquad \qquad \text{mass density,}$$

$$\mathbf{v} = \frac{1}{\rho} \sum_{\alpha=1}^{n} \rho_\alpha \mathbf{v}_\alpha, \qquad \qquad \text{velocity,}$$

$$\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}, \qquad \qquad \text{diffusion velocities,}$$

$$\varepsilon_I = \frac{1}{\rho} \sum_{\alpha=1}^{n} \rho_\alpha \varepsilon_\alpha, \qquad \qquad \text{intrinsic internal energy density,}$$

$$\varepsilon = \varepsilon_I + \frac{1}{2\rho} \sum_{\alpha=1}^{n} \rho_\alpha u_\alpha^2, \qquad \qquad \text{internal energy density} \tag{13}$$

$$\mathbf{t} = \sum_{\alpha=1}^{n} \left(\mathbf{t}_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha \right), \qquad \qquad \text{stress tensor,}$$

$$\mathbf{q} = \sum_{\alpha=1}^{n} \left\{ \mathbf{q}_\alpha + \rho_\alpha \left(\varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) \mathbf{u}_\alpha - \mathbf{t}_\alpha \mathbf{u}_\alpha \right\}, \quad \text{flux of internal energy.}$$

Summing up equations (10) one obtains the conservation laws of mass, momentum and energy for the whole mixture

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) = \mathbf{0}, \tag{14}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho \varepsilon \right) + \text{div} \left\{ \left(\frac{1}{2} \rho v^2 + \rho \varepsilon \right) \mathbf{v} - \mathbf{t}\mathbf{v} + \mathbf{q} \right\} = 0.$$

Note that assumptions (11) lead to a simplified form of the stress tensor $\mathbf{t}$ and the heat flux $\mathbf{q}$. However, they are not simple componentwise sums of corre-

sponding quantities, but are influenced by the diffusion process through $\mathbf{u}_\alpha$

$$\mathbf{t} = -p\mathbf{I} - \sum_{\alpha=1}^{n} \left(\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha \right); \qquad p = \sum_{\alpha=1}^{n} p_\alpha,$$

$$\mathbf{q} = \sum_{\alpha=1}^{n} \left\{ \rho_\alpha \left(\varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) + p_\alpha \right\} \mathbf{u}_\alpha,$$

where p denotes the total pressure of the mixture. Our concern will be even restricted class of Euler fluids – perfect gases whose constitutive equations are

$$p_\alpha = \rho_\alpha \frac{k_B}{m_\alpha} T_\alpha, \qquad \varepsilon_\alpha = \frac{k_B}{m_\alpha} \frac{T_\alpha}{\gamma_\alpha - 1}. \tag{15}$$

This completes the results based upon principles of rational thermodynamics.

The structure of source terms in (10) is determined using the general principles of extended thermodynamics – Galilean invariance and the entropy principle. Galilean invariance [10] restricts the velocity dependence of source terms to the following form [12]:

$$\tau_b = \hat{\tau}_b, \qquad \mathbf{m}_b = \hat{\tau}_b \mathbf{v} + \hat{\mathbf{m}}_b, \qquad e_b = \hat{\tau}_b \frac{v^2}{2} + \hat{\mathbf{m}}_b \cdot \mathbf{v} + \hat{e}_b, \tag{16}$$

for $b = 1, \ldots, n-1$, where $\hat{\tau}_b$, $\hat{\mathbf{m}}_b$ and $\hat{e}_b$ are independent of the mixture velocity $\mathbf{v}$. The final form of velocity independent parts of source terms is determined by means of entropy inequality [12]

$$\hat{\tau}_b = -\sum_{c=1}^{n-1} \varphi_{bc} \left(\frac{\mu_c - \frac{1}{2} u_c^2}{T_c} - \frac{\mu_n - \frac{1}{2} u_n^2}{T_n} \right),$$

$$\hat{\mathbf{m}}_b = -\sum_{c=1}^{n-1} \psi_{bc} \left(\frac{\mathbf{u}_c}{T_c} - \frac{\mathbf{u}_n}{T_n} \right),$$

$$\hat{e}_b = -\sum_{c=1}^{n-1} \theta_{bc} \left(-\frac{1}{T_c} + \frac{1}{T_n} \right),$$

where:

$$\mu_\alpha = \varepsilon_\alpha - T_\alpha S_\alpha + \frac{p_\alpha}{\rho_\alpha}, \qquad (\alpha = 1, \ldots, n)$$

are chemical potentials of the constituents and φ_{bc}, ψ_{bc} and θ_{bc}, $(b, c = 1, \ldots, n-1)$ are phenomenological symmetric positive definite matrices. Source terms are given for $n-1$ components since the last one can be determined from restrictions (12).

One of the most important properties of the mixture model (10) is its hyperbolicity. Characteristic speeds calculated from the differential part are almost trivial – they consist of independent subsets of characteristic speeds of the components

$$\lambda_\alpha^{(1)} = v_{\alpha n} - c_{s\alpha}, \qquad \lambda_\alpha^{(2,3,4)} = v_{\alpha n}, \qquad \lambda_\alpha^{(5)} = v_{\alpha n} + c_{s\alpha},$$

$$c_{s\alpha} = \left(\gamma_\alpha \frac{k_B}{m_\alpha} T_\alpha \right)^{1/2},$$

where $v_{\alpha n} = \mathbf{v}_\alpha \cdot \mathbf{n}$ are component velocities normal to the wave front and $c_{s\alpha}$ are local sound speeds of the components.

It is crucial for further study to use the possibility to substitute the balance laws for one component from (10), say n, with conservation laws for the mixture (14)

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) = \mathbf{0},$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon \right) + \text{div}\left\{ \left(\frac{1}{2}\rho v^2 + \rho\varepsilon \right)\mathbf{v} - \mathbf{t}\mathbf{v} + \mathbf{q} \right\} = 0,$$

$$\frac{\partial \rho_b}{\partial t} + \text{div}(\rho_b \mathbf{v}_b) = \tau_b, \tag{17}$$

$$\frac{\partial}{\partial t}(\rho_b \mathbf{v}_b) + \text{div}(\rho_b \mathbf{v}_b \otimes \mathbf{v}_b - \mathbf{t}_b) = \mathbf{m}_b,$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_b v_b^2 + \rho_b\varepsilon_b \right) + \text{div}\left\{ \left(\frac{1}{2}\rho_b v_b^2 + \rho_b\varepsilon_b \right)\mathbf{v}_b - \mathbf{t}_b\mathbf{v}_b + \mathbf{q}_b \right\} = e_b.$$

These two sets of governing equations, (10) and (17), are equivalent. They are obtained by the change of state variables $(\rho_\alpha, \mathbf{v}_\alpha, T_\alpha)^T \to (\rho, \mathbf{v}, T, \rho_b, \mathbf{v}_b, T_b)^T$. As a consequence, their characteristic speeds are the same.

In this setting $\mathbf{v}_b$ (or $\mathbf{u}_b$) and T_b can be regarded as nonequilibrium variables. The hyperbolic single-temperature system appears as a principal subsystem of (17). Complete hierarchy of principal subsystems was analyzed in [12] and it was shown that in the case of non-reacting mixtures condition of equilibrium reads

$$\mathbf{v}_1 = \ldots = \mathbf{v}_n = \mathbf{v}; \qquad (\mathbf{u}_1 = \ldots = \mathbf{u}_n = \mathbf{0}), \qquad T_1 = \ldots = T_n = T, \tag{18}$$

and leads to an equilibrium subsystem (14). Corresponding characteristic speeds are

$$\lambda_E^{(1)} = v_n - c_{sE}, \qquad \lambda_E^{(2,3,4)} = v_n, \qquad \lambda_E^{(5)} = v_n + c_{sE},$$

where $v_\mathrm{n} = \mathbf{v} \cdot \mathbf{n}$ and c_sE is equilibrium sound speed of the mixture. Since (17) is endowed with convex entropy, the following subcharacteristic condition is valid

$$\min_\alpha \left(v_{\alpha \mathrm{n}} - c_{\mathrm{s}\alpha}^*\right) \leq \lambda_\mathrm{E}^{(1)}, \qquad \lambda_\mathrm{E}^{(5)} \leq \max_\alpha \left(v_{\alpha \mathrm{n}} + c_{\mathrm{s}\alpha}^*\right)$$

where $c_{\mathrm{s}\alpha}^* = (\gamma_\alpha (k_\mathrm{B}/m_\alpha)T)^{1/2}$ are the sound speeds of the full system evaluated in equilibrium. Explicit form of the equilibrium sound speed of the mixture c_sE will be left for the case of binary mixture. It is only important to note that it is evaluated when (18) is satisfied.

4 Shock structure problem in binary mixture

Analysis of the shock structure in mixtures, modelled by the hyperbolic system (17), will be restricted to a binary mixture of perfect gases without chemical reactions ($\tau_\alpha = 0$). It was shown [11] that governing equations (17) could be written in the following form in the case of binary mixture

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \mathrm{div}\left(\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I} + \frac{1}{\rho c(1-c)}\mathbf{J} \otimes \mathbf{J}\right) = \mathbf{0},$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \varepsilon\right) + \mathrm{div}\left\{\left(\frac{1}{2}\rho v^2 + \rho \varepsilon + p\right)\mathbf{v} + \left(\frac{\mathbf{v} \cdot \mathbf{J}}{\rho c(1-c)} + \frac{1}{\beta}\right)\mathbf{J}\right\} = 0,$$

$$\frac{\partial}{\partial t}(\rho c) + \mathrm{div}\,(\rho c\mathbf{v} + \mathbf{J}) = 0, \tag{19}$$

$$\frac{\partial}{\partial t}(\rho c\mathbf{v} + \mathbf{J}) + \mathrm{div}\left\{\rho c\mathbf{v} \otimes \mathbf{v} + \frac{1}{\rho c}\mathbf{J} \otimes \mathbf{J} + \mathbf{v} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{v} + \nu \mathbf{I}\right\} = \mathbf{m}_1,$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho c\left(\mathbf{v} + \frac{\mathbf{J}}{\rho c}\right)^2 + \rho c e\right)$$

$$+ \mathrm{div}\left\{\left(\frac{1}{2}\rho c\left(\mathbf{v} + \frac{\mathbf{J}}{\rho c}\right)^2 + \rho c e + \nu\right)\left(\mathbf{v} + \frac{\mathbf{J}}{\rho c}\right)\right\} = e_1,$$

provided we introduce concentration variable c and diffusion flux vector $\mathbf{J}$

$$c = \frac{\rho_1}{\rho}, \qquad \mathbf{J} = \rho_1 \mathbf{u}_1 = -\rho_2 \mathbf{u}_2.$$

Thermal inertia β reads

$$\beta = \frac{1}{g_1 - g_2}, \qquad g_\alpha = \varepsilon_\alpha + \frac{p_\alpha}{\rho_\alpha} + \frac{u_\alpha^2}{2},$$

and $\nu = p_1$, $e = \varepsilon_1$.

Thermal and caloric equation of state for the mixture could be expressed in the form of equations for a single fluid

$$p = \rho \frac{k_B}{m} T, \qquad \varepsilon_I = \frac{k_B}{m} \frac{T}{\gamma - 1}, \tag{20}$$

provided we introduce average atomic mass m, average temperature T and average ratio of specific heats γ in the following way [12]

$$\frac{1}{m} = \frac{c}{m_1} + \frac{1-c}{m_2}, \qquad T = c \frac{m}{m_1} T_1 + (1-c) \frac{m}{m_2} T_2,$$

$$\frac{1}{\gamma - 1} = \frac{c}{\gamma_1 - 1} \frac{m}{m_1} \frac{T_1}{T} + \frac{1-c}{\gamma_2 - 1} \frac{m}{m_2} \frac{T_2}{T}. \tag{21}$$

Note that $\gamma_1 = \gamma_2$ implies $\gamma = \gamma_1 = \gamma_2$. It will be useful for further analysis to use the temperature difference $\Theta = T_2 - T_1$ as a nonequilibrium variable, so that component temperatures can be expressed in terms of average temperature T and Θ

$$T_1 = T - \frac{m}{m_2}(1 - c)\Theta, \qquad T_2 = T + \frac{m}{m_1} c\Theta. \tag{22}$$

Using these relations ν, e and β can be expressed in terms of $\mathbf{u} = (\rho, \mathbf{v}, T, c, \mathbf{J}, \Theta)^T$, a new set of state variables. Using these new quantities one may determine the sound speed for equilibrium subsystem

$$c_{\mathrm{sE}} = \left(\gamma \frac{k_B}{m} T\right)^{1/2}. \tag{23}$$

Explicit form of source terms $\mathbf{m}_1$ and e_1 reads as follows

$$\hat{\mathbf{m}}_1 = -\psi_{11}\left(\frac{\mathbf{u}_1}{T_1} - \frac{\mathbf{u}_2}{T_2}\right) = -\psi_{11}\left(\frac{1}{\rho c T_1} + \frac{1}{\rho(1-c)T_2}\right)\mathbf{J},$$

$$\hat{e}_1 = -\theta_{11}\left(-\frac{1}{T_1} + \frac{1}{T_2}\right) = \theta_{11}\frac{\Theta}{T_1 T_2},$$

where T_1 and T_2 are given by (22). However, source terms cannot be completely described within the framework of extended thermodynamics since coefficients ψ_{11} and θ_{11} remain undetermined. They can be related either to experimental data, or to some other theory. We shall recourse to the results of kinetic theory of gases [2] which relate them to volumetric collision frequency Γ'_{12} in equilibrium

$$\psi_{11E} = 2T_0 \frac{m_1 m_2}{m_1 + m_2}\Gamma'_{12}, \qquad \theta_{11E} = 3k_B T_0^2 \frac{m_1 m_2}{(m_1 + m_2)^2}\Gamma'_{12}.$$

On the other hand, by linearization of source terms in the neighborhood of equilibrium they could be related to relaxation times τ_D and τ_T for diffusion and temperature in equilibrium

$$\psi_{11E} = \frac{\rho_0 c_0 (1 - c_0) T_0}{\tau_D}, \qquad \theta_{11E} = \frac{k_B}{m_1 (\gamma_1 - 1)} \frac{\rho_0 c_0 T_0^2}{\tau_T}. \tag{24}$$

Comparison of the last two equations yields the relation between relaxation times

$$\tau_T = \frac{2}{3(\gamma_1 - 1)} \frac{m_1 + m_2}{(1 - c_0) m_1} \tau_D$$

and it is frank that for real gases we have an estimate $\tau_T > \tau_D$. This fact gives further support to the multi-temperature assumption since thermal nonequilibrium needs more time for attenuation than mechanical (diffusion) one. In the sequel we shall adopt equilibrium values of coefficients (24): since we are dealing with states which are close to equilibrium ones, they will provide sufficiently accurate estimate.

Equilibrium state is characterized by vanishing of the entropy production. In the case of binary mixture it is equivalent to vanishing of the source terms, $\mathbf{m}_1 = \mathbf{0}$ and $e_1 = 0$ (see [12]). Condition of equilibrium leads to a state without diffusion where constituents have common temperature

$$\mathbf{J} = \mathbf{J}_0 = \mathbf{0}, \qquad \Theta = \Theta_0 = 0.$$

These conditions are in agreement with (18). We shall further assume that other variables have uniform distributions in equilibrium state

$$\rho = \rho_0 = \text{const.}, \qquad \mathbf{v} = \mathbf{v}_0 = \text{const.}, \qquad T = T_0 = \text{const.}, \qquad c = c_0 = \text{const.}$$

The shock structure is assumed to be a continuous travelling wave-like solution which asymptotically connects two such equilibrium states.

The shock structure problem will be analyzed for a plane wave which propagates in x-direction with speed s. Therefore, we shall seek for a travelling wave solution in the form $\mathbf{u}(\xi) = \mathbf{u}(x - st)$ with a single independent variable $\xi = x - st$. This assumption transforms the system of governing equations (19) to the following set of ODE's which determine the shock structure

$$\frac{d}{d\xi} (\rho u) = 0,$$

$$\frac{d}{d\xi} \left(\rho u^2 + p + \frac{J^2}{\rho c (1 - c)} \right) = 0,$$

$$\frac{d}{d\xi} \left\{ \left(\frac{1}{2} \rho u^2 + \rho \varepsilon + p \right) u + \left(\frac{uJ}{\rho c (1 - c)} + \frac{1}{\beta} \right) J \right\} = 0,$$

$$\frac{d}{d\xi}\left(\rho c u + J\right) = 0, \tag{25}$$

$$\frac{d}{d\xi}\left\{\rho c u^2 + \frac{J^2}{\rho c} + 2uJ + \nu\right\} = \hat{m}_1,$$

$$\frac{d}{d\xi}\left\{\left(\frac{1}{2}\rho c\left(u + \frac{J}{\rho c}\right)^2 + \rho c e + \nu\right)\left(u + \frac{J}{\rho c}\right)\right\} = \hat{m}_1 u + \hat{e}_1,$$

where $u = v - s$ is x-component of the relative mixture velocity with respect to the shock wave and J and $\hat{m}_1$ are x-components of diffusion flux and source term. Due to change of variables we shall have a standing shock profile with upstream (unperturbed) state $\mathbf{u}_0 = \lim_{\xi \to -\infty} \mathbf{u}(\xi)$ and downstream (perturbed) state $\mathbf{u}_1 = \lim_{\xi \to \infty} \mathbf{u}(\xi)$. In view of equilibrium conditions discussed above, we adjoin the system (25) with following boundary conditions

	upstream		downstream	
$\rho = \rho_0,$	$c = c_0;$	$\rho = \rho_1,$	$c = c_1;$
$u = u_0,$	$J = 0;$	$u = u_1,$	$J = 0;$
$T = T_0,$	$\Theta = 0;$	$T = T_1,$	$\Theta = 0.$

$$(26)$$

Downstream boundary conditions will be determined in the sequel.

Further analysis will be performed in dimensionless form using the following dimensionless variables

$$\hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{u} = \frac{u}{c_{sE}}, \quad \hat{T} = \frac{T}{T_0},$$

$$\hat{J} = \frac{J}{\rho_0 c_0 c_{sE}}, \quad \hat{\Theta} = \frac{\Theta}{T_0}, \quad \hat{\xi} = \frac{\xi}{\tau_D c_{sE}}, \quad M_0 = \frac{u_0}{c_{sE}}.$$

We shall also assume that both components have the same ratio of specific heats, equal to the average one γ. Upstream boundary data thus read

$$\hat{\rho}_0 = 1, \quad \hat{u}_0 = M_0, \quad \hat{T}_0 = 1, \quad \hat{J}_0 = 0, \quad \hat{\Theta}_0 = 0, \tag{27}$$

while c_0 is already a dimensionless quantity. For convenience hats will be dropped in the sequel.

To reduce the order of the system, as well as computational effort, conservation laws for mass and momentum of the mixture $(25)_{1,2}$ and conservation law for mass of the component $(25)_4$ will be used to express ρ, J and T in terms of c and u

$$\rho = \frac{M_0}{u}, \qquad J = M_0\left(1 - \frac{c}{c_0}\right), \tag{28}$$

$$T = \frac{m}{m_0}\frac{u}{M_0}\left\{1 + \gamma M_0^2\left[1 - \frac{u}{M_0}\left(1 + \frac{(c_0 - c)^2}{c(1-c)}\right)\right]\right\},$$

where m_0 denotes the average atomic mass evaluated in upstream equilibrium state.

There will remain three equations which will finally determine the shock structure. Although there still remains one conservation law among them $(25)_3$ further elimination of state variables will cause appearance of singularities which prevent efficient numerical calculations in the sequel. Thus, inserting (28) into $(25)_{3,5,6}$ written in dimensionless form we formally obtain the following set of ODE's

$$\frac{du}{d\xi} = F(u, c, \Theta, M_0), \quad \frac{dc}{d\xi} = G(u, c, \Theta, M_0), \quad \frac{d\Theta}{d\xi} = H(u, c, \Theta, M_0), \quad (29)$$

The actual form of the ODE system is a bit different

$$\mathbf{B}(\mathbf{u}, M_0)\frac{d\mathbf{u}}{d\xi} = \mathbf{f}(\mathbf{u}, M_0) \tag{30}$$

for $\mathbf{u} = (u, c, \Theta)^T$. Under regularity assumption $\det \mathbf{B}(\mathbf{u}, M_0) \neq 0$ equations (29) can be reconstructed as $d\mathbf{u}/d\xi = \mathbf{F}(\mathbf{u}, M_0) = \mathbf{B}^{-1}(\mathbf{u}, M_0)\mathbf{f}(\mathbf{u}, M_0)$. However, the form (30) is much more convenient and will be used in stability analysis.

It is of utmost importance to notice that stationary points of (29), or (30), which satisfy

$$\mathbf{f}(\mathbf{u}_0, M_0) = \mathbf{f}(\mathbf{u}_1, M_0) = \mathbf{0}$$

are determined by relations

$$\mathbf{u}_0 = (u_0, c_0, \Theta_0) = (M_0, c_0, 0), \qquad \mathbf{u}_1 = (u_1, c_1, \Theta_1) = \left(\frac{3 + M_0^2}{4M_0}, c_0, 0\right). \tag{31}$$

Actually, they satisfy the Rankine-Hugoniot equations on singular surface for the equilibrium subsystem (14). This fact is far from being obvious, but it is in a full accordance with the results obtained thus far for other hyperbolic dissipative systems (isothermal viscoelasticity and gas dynamics).

Our main concern will be the stability analysis of stationary points (31). We want to show that there is a critical value for the upstream Mach number M_0 for which stationary points change their stability properties. Moreover, we would like to show that we can rely on a simple criterion, related to eigenvalues, which provides us a selection rule for admissible shock structures. To that end we shall introduce perturbation of upstream equilibrium state $\Delta\mathbf{u} = (u - u_0, c - c_0, \Theta - \Theta_0)^T = \mathbf{u} - \mathbf{u}_0$ and write the system of linearized variational equations in the form

$$\mathbf{B}(\mathbf{u}_0, M_0)\frac{d\Delta\mathbf{u}}{d\xi} = \frac{\partial\mathbf{f}}{\partial\mathbf{u}}(\mathbf{u}_0, M_0)\Delta\mathbf{u}.$$

Eigenvalue problem which corresponds to this system reads

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, M_0) - \lambda \mathbf{B}(\mathbf{u}_0, M_0) \right) \mathbf{r} = \mathbf{0}.$$

Characteristic equation has generalized form

$$\det \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, M_0) - \lambda \mathbf{B}(\mathbf{u}_0, M_0) \right) = 0$$

and gives three eigenvalues

$$\lambda_{01} \equiv 0, \qquad \lambda_{02} = \lambda_{02}(M_0), \qquad \lambda_{03} = \lambda_{03}(M_0).$$

Note that one eigenvalue is identically zero, which is a consequence of the fact that among the system of shock structure equations (30) there is one conservation law. Other two eigenvalues can be calculated in a closed form, which is rather cumbersome. More important than its closed form is the fact that one of them changes the sign in the neighborhood of $M_0 = 1$

$$\lambda_{02}(1) = 0, \qquad \frac{d\lambda_{02}(1)}{dM_0} = 1.046; \qquad \lambda_{03}(1) = -0.3118, \tag{32}$$

the conclusion which can be drawn by continuity argument. By similar calculation one may determine the eigenvalues in downstream equilibrium state $\mathbf{u}_1$

$$\lambda_{11} \equiv 0, \qquad \lambda_{12} = \lambda_{12}(M_0), \qquad \lambda_{13} = \lambda_{13}(M_0).$$

Again, one of the eigenvalues is identically zero, and among other two there is one which changes the sign in the neighborhood of the critical value of shock speed $M_0 = 1$

$$\lambda_{12}(1) = 0, \qquad \frac{d\lambda_{12}(1)}{dM_0} = -1.046; \qquad \lambda_{13}(1) = -0.3118, \tag{33}$$

Figure 4 shows the graphs of nontrivial eigenvalues calculated in the mixture of Helium and Xenon, whose ratio of atomic masses is $m_{\mathrm{Xe}}/m_{\mathrm{He}} = 32.80$, and with equilibrium concentration of Helium $c_0 = 0.3$.

Relation between the results in (32) and (33) to admissibility conditions is based upon inequality form of both of them. Namely, admissibility of the shock wave in equilibrium subsystem reads

$$\lambda_{\mathrm{E}}^{(5)}(\mathbf{u}_0) < s < \lambda_{\mathrm{E}}^{(5)}(\mathbf{u}_1).$$
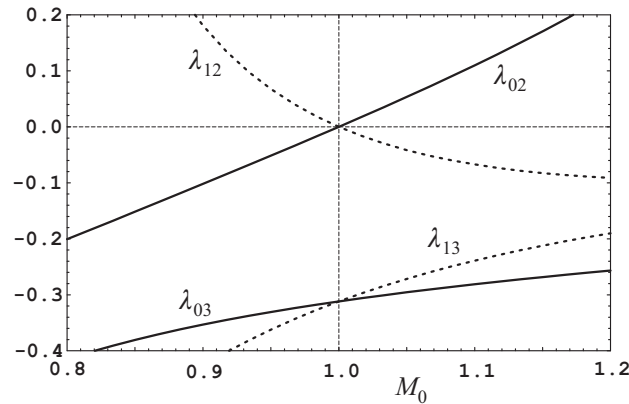
Figure 1. Eigenvalues in upstream $\mathbf{u}_0$ and downstream $\mathbf{u}_1$ stationary points in He-Xe mixture shock structure.

Left inequality corresponds to $M_0 > 1$. For $s < \lambda_{\mathrm{E}}^{(5)}(\mathbf{u}_0)$, or $M_0 < 1$, shocks are not admissible. Moreover, $\lambda_{02}(M_0) < 0$ determines the stable direction (eigenvector) in state space when $M_0 < 1$, while $\lambda_{02}(M_0) > 0$ determines unstable one for $M_0 > 1$. The opposite conclusions are valid for $\lambda_{12}(M_0)$. Therefore, we can conclude that admissible shock structure appears when

$$\lambda_{12}(M_0) < 0 < \lambda_{02}(M_0), \tag{34}$$

while $\lambda_{02}(M_0) = 0$, or $\lambda_{12}(M_0) = 0$, determine the critical value of the parameter (shock speed). Selection rule (34) is in direct relation to the properties of equilibrium subsystem, but we do not have to study its properties before the analysis of shock structure of the full dissipative system. Equilibrium states (stationary points $\mathbf{u}_0$ and $\mathbf{u}_1$), the critical value of the shock speed and stability condition (34) are determined from the shock structure equations (30) (or equivalently (25)) without prior solution of Rankine-Hugoniot equations and usage of Lax condition. Moreover, inequality form of admissibility criterion (34) strengthens its irreversible nature.

Apart from giving intrinsic characterization of admissible shock structure, stability analysis has another practical importance. Namely, for particular value of Mach number M_0 one may determine eigenvectors which correspond to critical eigenvalues $\lambda_{02}(M_0)$ and $\lambda_{12}(M_0)$. They determine directions in state space which correspond to asymptotic behavior of heteroclinic orbit in the neighborhood of stationary point. These directions can be used to evaluate the initial values of state variables in the neighborhood of stationary point and solve the shock structure problem as an initial value problem. In our case integration has
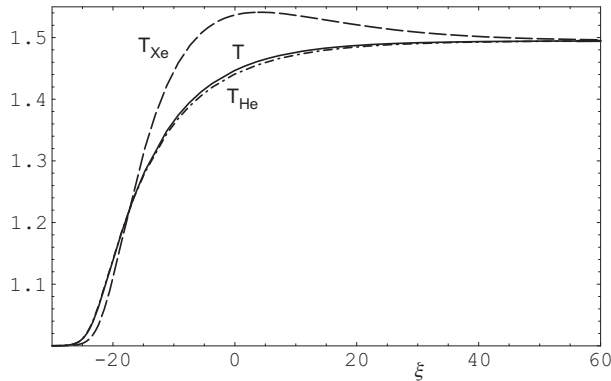
Figure 2. Temperature profiles for a shock structure in He-Xe binary mixture for $M_0 = 1.5$, $c_0 = 0.3$.

to be started from the neighborhood of upstream equilibrium $\mathbf{u}_0$, which is unstable, and finished in the neighborhood of stable downstream equilibrium $\mathbf{u}_1$. Results of numerical calculation are shown for temperature profiles in Helium-Xenon mixture for $M_0 = 1.5$ and $c_0 = 0.3$. Basic feature of multi-temperature profiles, existence of temperature overshoot, is captured by the model. However, it is very important to stress that computational effort is much lower than in the case of simulations based upon Boltzmann equation.

5   Conclusions

This paper gives a review of recent analysis of the shock structure problem in hyperbolic systems of balance laws. Since our main concern are physically motivated hyperbolic systems which have dissipative character, we put into focus a hyperbolic multi-temperature model of gaseous mixtures developed within the framework of extended thermodynamics. The aim was to show that admissible shock structure can be determined from the study of complete hyperbolic system, i.e. without prior analysis of the equilibrium subsystem. In particular, it was shown that stationary points of the system of shock structure equations carry information about equilibrium states, which are in turn related through Rankine-Hugoniot equations of an equilibrium subsystem. Moreover, exchange of stability properties of stationary points yield an inequality which can serve as a simple indicator for appearance of physically admissible continuous shock structure. Although this result is parallel to Lax admissibility condition for hyperbolic systems of conservation laws, it was obtained by direct stability analysis

of the full system of balance laws.

These results open the possibility to study shock structure problem directly, even in situations in which one cannot rely on physical intuition. Moreover, it was shown that this method provides valuable information which can be efficiently exploited for numerical analysis of the problem. Exhaustive study of these possibilities is a matter of ongoing research.

References

[1] G. Boillat, T. Ruggeri: *Hyperbolic principal subsystems: entropy convexity and sub-characteristic conditions*, Archive for Rational Mechanics and Analysis **137**, 305–320, (1997).

[2] T.K. Bose: High Temperature Gas Dynamics, Springer-Verlag, Berlin (2004).

[3] G.-Q. Chen, C.D. Levermore, T.-P. Liu: *Hyperbolic Conservation Laws with Stiff Relaxation Terms and Entropy*, Communications on Pure and Applied Mathematics, **47**, 787–830, (1994).

[4] C.M. Dafermos: Hyperbolic Conservation Las in Continuum Physiscs, Springer-Verlag, Berlin (2000).

[5] D. Gilbarg, D. Paolucci: *The structure of shock waves in the continuum theory of fluids*, J. Ration. Mech. Anal., **2**, 617–642, (1953).

[6] H. Grad: *On the kinetic theory of rarefied gases*, Communications on Pure and Applied Mathematics, **2**, 331–407, (1949).

[7] J. Guckenheimer, P. Holmes: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer-Verlag, New York (1986).

[8] T.-P. Liu: *Hyperbolic conservation laws with relaxation*, Communications in Mathematical Physics, **108**, 153–175, (1987).

[9] I. Müller, T. Ruggeri: Rational Extended Thermodynamics, Springer-Verlag, New York, (1998).

[10] T. Ruggeri: *Galilean Invariance and Entropy Principle for Systems of Balance Laws. The Structure of the Extended Thermodynamics*, Continuum Mechanics and Thermodynamics, **1**, 3–20, (1989).

[11] T. Ruggeri: *The binary mixtures of Euler fluids: a unified theory of second sound phenomena*, In: Continuum Mechanics and Applications in Geophysics and the Environment, Straughan, B.; Greve, R.; Ehrentraut, H. & Wang, Y., (Eds.), 79–91, Springer, Berlin. (2001).

[12] T. Ruggeri, S. Simić: *On the hyperbolic system of a mixture of Eulerian fluids: A comparison between single- and multi-temperature models*, Mathematical Methods in the Applied Sciences, **30**, 827–849, (2007).

[13] D. Serre: Systems of Conservation Laws 1: Hyperbolicity, Entropies, Shock Waves, Cambridge University Press, Cambridge, (1999).

[14] S.S. Simić: *A note on shock profiles in dissipative hyperbolic and parabolic models*, Publication de l'Institut Mathématique, Nouvelle série **84 (98)**, 97–107, (2008).

[15] S.S. SIMIĆ: *Shock structure in continuum models of gas dynamics: stability and bifurcation analysis*, Nonlinearity, **22**, 1337–1366, (2009).

[16] C. TRUESDELL: Rational Thermodynamics, Springer-Verlag, New York, (1969).

[17] W. WEISS: *Continuous shock structure in extended thermodynamics*, Physical Review E, **52**, R5760–R5763, (1995).

[18] W.-A. YONG, K. ZUMBRUN: *Existence of relaxation shock profiles for hyperbolic conservation laws*, SIAM Journal on Applied Mathematics, **60**, 1565–1575, (2000).