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Unfolding of singularities and differential equations

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Abstract. Interrelation between Thom's catastrophes and differential equations revisited. It is shown that versal deformations of critical points for singularities of A,D,E type are described by the systems of Hamilton-Jacobi type equations. For particular nonversal unfoldings the corresponding equations are equivalent to the integrable two-component hydrodynamic type systems like classical shallow water equation, dispersionless Toda system and others. Peculiarity of such integrable systems is that the generating functions for the corresponding hierarchies, which obey Euler-Poisson-Darboux equation, contain information about normal forms of higher order and higher corank singularities.

1 Introduction

Connection between catastrophe theory (see e.g. [1, 2, 3, 4, 5]) and theory of singularities for differential equations has been studied for more than fourty years. It was R.Thom himself [1] to observe that the singularities developed by solutions of the wave equation are related to the unfolding of singularities of functions. In 1968 E. Zeeman noted [6] that the breaking waves phenomenon (gradient catastrophe) is associated with the hyperbolic umbilic catastrophe. Within the formalism of Lagrangian submanifolds the above interrelation has been studied by Guckenheimer [7], Arnold [8, 9] and other authors (see e.g. [10, 11, 12, 13, 14, 15, 16] and references therein). Some aspects of the tangent fields for singularity unfoldings in connection with deformations of complex spaces have been considered in [17] . In topological quantum field theory [18] unfoldings of singularities and corresponding Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations have appeared within the study of the deformed Ginzburg-Landau models [19, 20]. A connection of the topological field theory and Saito's approach in singularity theory [21] with Frobenius manifolds [22]and associated WDVV and Hirota equations has been discussed in [22, 23, 24, 25, 26, 27, 28]. It is well established nowadays that the base spaces of versal unfoldings of singularities may carry rich family of algebraic and geometrical structures including various types of differential equations.

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Within different studies an appearance of normal forms of singularities has been observed recently in the papers [29, 30, 31, 32, 33, 34]. In [30, 31, 32] cusp and umbilic catastrophes have arisen within the study of the behaviour of the full dispersive integrable equations near the points of gradient catastrophe. In contrast, an analysis performed in [29, 33, 34] deals directly with the dispersionless integrable systems of certain class. In the papers [33, 34, 35] it was shown that the hodograph equations for these systems are nothing else than the equations defining critical points of the functions W which obey Euler-Poisson-Darboux equations. Such functions W have several important properties: they are the generating functions for the whole integrable hierarchies, they drastically simplify an analysis of the singular sectors for these hierarchies. But, perhaps, more interesting fact is that they contain information about the normal forms of singularities of functions from the catastrophe theory. The observations made in the papers [29, 33, 34] deal with the singularities of A_n type and umbilic type.

In the present paper we will discuss one particular aspect of the relation between the singularity theory and differential equations. Let the function $F(x;t) = F_0(x) + \sum_{k=0}^{n} e_k(x)t_k$ with certain functions $e_k(x)$ and deformation parameters t_k defines the unfolding of the singularity with the normal form given by the function $F_0(x)$ of m variables $x_1, ..., x_m$. We will be interested in differential equations which govern the dependence of the critical points $u_i, i = 1, ..., m$ for the function $F(x;t)$ on deformation parameters. We will show, using an elementary technique and addressing mainly to nonexperts, that

1. For the versal deformations of the A,D,E singularities ($m = 1$ or 2) the dependence of their critical points on t_k outside the catastrophe sets is described by the systems of mn differential equations of the first order. These systems imply that $u_i = \varphi_{t_i}$ and the function φ obeys the systems of Hamilton-Jacobi type equations. For A_n singularities these equations are of the form

$$\varphi_{t_k} = (\varphi_{t_1})^k, \quad k = 2, ..., n-1$$

or

$$u_{t_k} = (u^k)_{t_1}, \quad k = 2, ..., n-1,$$

which are the Burgers-Hopf and n-3 higher Burgers-Hopf equations. For D_n, E_6, E_7, E_8 cases (m=2) the corresponding equations are

$$\varphi_{t_l} = e_l(\varphi_{t_1}, \varphi_{t_2}), \quad l = 3, ..., \mu - 1$$

or

$$u_{t_l} = \frac{\partial e_l(u,v)}{\partial t_1}, \quad v_{t_l} = \frac{\partial e_l(u,v)}{\partial t_2}, \quad l = 3, ..., \mu - 1$$

where $u = u_1, v = u_2$ and $e_l(x,y)$ are elements of a basis of the local algebra Q_{F_0} for critical points of the dimension μ. In all these cases for the versal deformations

$$F(u; dt) = d\varphi. \tag{1}$$

2. For the particular nonversal unfolding of the umbilic singularities, for which $F(x,y;t) = F_0(x,y) + t_3 F_{0x} + t_1 x$ and the constraint $(F_{xx} - \delta F_{yy})(u,v) = 0$ with $\delta = \pm 1$ is verified, the critical points $u(t)$ and $v(t)$ obey integrable systems of hydrodynamic type of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_3} = 2 \begin{pmatrix} \alpha u, & v \\ \delta v, & \alpha u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{t_1}. \tag{2}$$

For certain values of α this system represents the well-known integrable systems like the classical shallow waver equation ($\alpha = 2$) and dispersionless Toda system ($\alpha = 0$). In this case the catastrophe set defined by the equation $\Delta \div (F_{xx}F_{yy} - F_{xy}^2)(u,v) = 0$ decomposes in two components. For such integrable systems the corresponding hodograph equations coincide with the equations for critical points u and v of the functions W obeying the Euler-Poisson-Darboux equations. For this unfolding and other nonversal unfoldings of such type the relation (1) is valid too.

3. It is observed that the generating functions $W(x;t) = \sum_{k \geq 1} t_k W_k(x)$ for the integrable n-component hydrodynamic type systems contain information about normal forms of singularities of higher orders and coranks. It is shown that in the two-component case, namely, for the system (2)

$$W_3 \sim x^3 \pm 3xy^2, \quad W_4 \sim x^4 + dx^2y^2 + y^4,$$

i.e. the functions W_3 and W_4 are proportional to the normal forms of D_4 and X_9 singularities of corank two, respectively. In the three-component case

$$W_3 \sim x^3 + y^3 + z^3 + dxyz,$$

that is the normal form of the unimodular singularity P_8 (or $T_{3,3,3}$) of corank three. Dependence of the critical points for this singularity on deformation parameters is governed by the dispersionless three-component coupled KdV equation.

2 Thom's catastrophes

First, we recall briefly the basic facts about unfolding of singularities (see [1, 2, 3, 4, 5]). Let the normal form of the singularity of the corank m is given by the function $F_0(x_1, ..., x_m)$. An unfolding (deformation) $F(x; \lambda)$ of the function F_0 is a germ of a smooth function $F : R^m \times R^n \to R$ at the point (0,0) such that $F(x; 0) = F_0(x)$. The space R^n of the second argument of F is called the base space of unfolding and $\lambda_1, ..., \lambda_n$ are called the parameters of unfolding. Dimension of the base space for different deformations of the same function can be different. Critical set of $F(x; \lambda)$ is given by the solutions of the equations $F_{x_i}(u; \lambda) \div \frac{\partial F}{\partial x_i}(u; \lambda) = 0, i = 1, ..., m$ and the catastrophe set is defined by vanishing of the Hessian $|F_{x_i x_k}|(u) = 0$. The tangent space of the orbit of the germ $F_0(x)$ is its gradient (Jacobian) ideal $I_J = (F_{0x_1}, ..., F_{0x_m})$ which consists of all germs of the functions of the form $\sum_{i=1}^{i=m} h_i(x) F_{0x_i}$. Local algebra Q_{F_0} of the critical point is the quotient $R[x]/I_J$. It's dimension μ is called the Milnor number (or multiplicity) of the critical point. If the functions $e_0 = 1, e_1(x), ..., e_{\mu-1}(x)$ form a basis in Q_{F_0}, then a versal (R-versal) unfolding of $F_0(x)$ can be represented in the form

$$F(x; t) = F_0(x) + \sum_{k=0}^{\mu-1} e_k(x) t_k. \tag{3}$$

Here t_k are the deformation parameters and hence the initial velocities of deformation $V_k \div F_{t_k}(x; t)|_{t=0} = e_k(x), k = 0, 1, ..., \mu - 1$. Nonversal unfoldings of $F_0(x)$ may depend on less than μ parameters or have other forms.

Let us begin with the celebrated seven Thom's catastrophes. The corresponding functions F_0 and their versal unfoldings are (see [1, 2, 3, 4, 5]): for corank one A_n type catastrophes (fold, cusp, swallow tail, butterfly)

$$F_n^A(x; t) = x^{n+1} + \sum_{k=0}^{n-1} t_k x^k, \quad n = 2, 3, 4, 5, \tag{4}$$

for corank two hyperbolic and elliptic umbilic catastrophes ($D_4^{\pm}$)

$$F_4^{\pm}(x, y; t) = x^3 \pm 3xy^2 + t_3(x^2 \mp y^2) + t_2 y + t_1 x + t_0, \tag{5}$$

and for the parabolic umbilic catastrophe D_5

$$F_5(x, y; t) = x^2 y + y^4 + t_4 y^2 + t_3 x^2 + t_2 y + t_1 x + t_0. \tag{6}$$

We begin with the A_n type singularities. Critical points u of the functions (4) are defined by the equation

$$F_{nu}^A \div F_{nx}^A(u; t) = (n+1)u^n + \sum_{k=1}^{n-1} k t_k u^{k-1} = 0 \tag{7}$$

where F_x etc denotes the derivative w.r.t. x. Calculating the differential of (7), one gets

$$F^A_{nuu}du + \sum_{k=1}^{n-1} ku^{k-1}dt_k = \sum_{k=1}^{n-1}(F^A_{nuu}u_{t_k} + ku^{k-1})dt_k = 0. \tag{8}$$

Hence, the critical points $u(t)$ obey the system of equations

$$F^A_{nuu}u_{t_k} + ku^{k-1} = 0, \quad k = 1, 2, ..., n-1. \tag{9}$$

Outside the catastrophe sets where $F^A_{nuu} \neq 0$ one has the system

$$u_{t_k} = -\frac{ku^{k-1}}{F^A_{nuu}}, \quad k = 1, 2, ..., n-1. \tag{10}$$

On the catastrophe sets defined by the condition $F^A_{nuu} = 0$ the velocities u_{t_k} of variations of the positions of the critical point becomes unbounded which is a standard manifestation of the catastrophe.

For $n = 2$ one has a single equation while at $n \geq 3$ the system (10) is equivalent to the system

$$u_{t_1} = -\frac{1}{F^A_{nuu}}, \qquad u_{t_k} = (u^k)_{t_1}, \quad k = 2, 3, ..., n-1 \tag{11}$$

Equations (11) imply that there exists a function φ such that $u = \varphi_{t_1}$ and that the n-2 last equations (11) take the form

$$\varphi_{t_k} = (\varphi_{t_1})^k, \quad k = 2, 3, ..., n-1. \tag{12}$$

As a consequence, evaluating the infinitesimal variation of the function (4) on the critical set, one obtains

$$\delta F^A_n(u) \div F^A_n(u; t + dt) - F^A_n(u; t) = F^A_n(u; dt) = d\varphi. \tag{13}$$

Equation (11) with $k = 2$ is the well-known Burgers-Hopf (BH) equation (see e.g.[36]) while equations (11) at $k \geq 3$ represent $n-3$ higher flows commuting with them. In the form (12) this system is the family of Hamilton-Jacobi type equations. The fact that the deformations of the critical points for the first four Thom's catastrophes are governed by the BH and higher BH equations has been discussed in [29] and [26].

Umbilic type singularities can be considered simultaneously. Indeed, let us take the function

$$F(x, y; t) = F_0(x, y; t) + t_4y^2 + t_3x^2 + t_2y + t_1x + t_0. \tag{14}$$

For D_4^+ singularity $F_0 = x^3 + 3xy^2$ and $t_4 + t_3 = 0$. For D_4^- singularity $F_0 = x^3 - 3xy^2$ and $t_4 - t_3 = 0$ and in D_5 case $F_0 = x^2y + y^4$. Critical set is defined by the equations

$$F_x(u, v; t) = F_{0x}(u, v) + 2t_3x + t_1 = 0, \tag{15}$$

$$F_y(u, v; t) = F_{0y}(u, v) + 2t_4y + t_2 = 0. \tag{16}$$

Calculating differentials of these equations, taking into account that in the critical set $du = \sum_{k=1}^{k=4} u_{t_k} dt_k$, $dv = \sum_{k=1}^{k=4} v_{t_k} dt_k$ and solving systems of linear equations, one gets

$$u_{t_1} = -\frac{F_{vv}}{\Delta}, \quad v_{t_1} = \frac{F_{uv}}{\Delta}, \tag{17}$$

$$u_{t_2} = \frac{F_{uv}}{\Delta}, \quad v_{t_2} = -\frac{F_{\mathsf{uu}}}{\Delta}, \tag{18}$$

$$u_{t_3} = -2u\frac{F_{vv}}{\Delta}, \quad v_{t_3} = 2u\frac{F_{uv}}{\Delta}, \tag{19}$$

$$u_{t_4} = 2v\frac{F_{uv}}{\Delta}, \quad v_{t_4} = -2v\frac{F_{\mathsf{uu}}}{\Delta} \tag{20}$$

where $F_{\mathsf{uu}} = F_{xx}(u, v)$ etc and $\Delta = F_{\mathsf{uu}}F_{vv} - (F_{uv})^2$. These equations govern the dependence of the critical points on deformation parameters. On the catastrophe set $\Delta = 0$ derivatives of u and v become infinite and one observes a very fast (catastrophic) change of positions and number of critical points.

The system of equations (17)-(20) can be represented in different equivalent forms. For example, it implies that

$$u_{t_2} = v_{t_1}, \tag{21}$$

and

$$u_{t_3} = (u^2)_{t_1}, \quad v_{t_3} = (u^2)_{t_2}, \tag{22}$$

$$u_{t_4} = (v^2)_{t_1}, \quad v_{t_4} = (v^2)_{t_2}. \tag{23}$$

Hence, there exists a function φ such that $u = \varphi_{t_1}, v = \varphi_{t_2}$ and equations (22) and (23) are reduced to two equations

$$\varphi_{t_3} = (\varphi_{t_1})^2, \quad \varphi_{t_4} = (\varphi_{t_2})^2. \tag{24}$$

It is easy to see that the system (17)-(20) is equivalent to the following system for the function φ

$$\varphi_{t_1t_1} = -\frac{F_{vv}}{\Delta}, \quad \varphi_{t_1t_2} = \frac{F_{uv}}{\Delta}, \quad \varphi_{t_2t_2} = -\frac{F_{\mathsf{uu}}}{\Delta}, \tag{25}$$

$$u = \varphi_{t_1}, \quad v = \varphi_{t_2}, \tag{26}$$

$$\varphi_{t_3} = (\varphi_{t_1})^2, \quad \varphi_{t_4} = (\varphi_{t_2})^2. \tag{27}$$

As the consequence of equations (26), (27) one has

$$\delta F(u,v) \div F(u,v;t+dt) - F(u,v;t) = F(u,v;dt) = \sum_{k=0}^{k=4} \varphi_{t_k} dt_k = d\varphi \qquad (28)$$

with $\varphi_{t_0} = 1$.

Equations (17)-(27) describe dependence of critical points of D_5 singularity on deformation parameters. To get the corresponding equations for $D_4^{\pm}$ singularities one can pass in equations (17)-(27) to new variables $t_{\pm} = \frac{1}{2}(t_4 \pm t_3)$ and then impose the constraints $t_+ = 0$ or $t_- = 0$ or use directly the relation (28). Indeed, the functions $F^{\pm}$ for $D_4^{\pm}$ singularities are $F^{\pm} = F(x,y;t)|_{t_4=\mp t_3}$. So,

$$d\varphi^{\pm} = \sum_{k=0}^{k=3} \varphi_{t_k}^{\pm} dt_k = d\varphi|_{t_4=\mp t_3} = ((\varphi_{t_1}^{\pm})^2 \mp (\varphi_{t_2}^{\pm})^2) dt_3 + \varphi_{t_2}^{\pm} dt_2 + \varphi_{t_1}^{\pm} dt_1 + dt_0. \quad (29)$$

Hence, for the $D_4^{\pm}$ singularities one has the equation

$$\varphi_{t_3}^{\pm} = (\varphi_{t_1}^{\pm})^2 \mp (\varphi_{t_2}^{\pm})^2. \qquad (30)$$

plus the corresponding equations (25), (26). In terms of $u = \varphi_{t_1}^{\pm}$ and $v = \varphi_{t_2}^{\pm}$ equation (30) assumes the form of the 2+1-dimensional hydrodynamical type system

$$\begin{aligned} u_{t_3} &= (u^2 \mp v^2)_{t_1}, \\ v_{t_3} &= (u^2 \mp v^2)_{t_2}. \end{aligned} \qquad (31)$$

The catastrophe sets $\Delta = 0$ for the umbilic singularities discussed above are those subsets of variables t_k for which solutions of considered systems of differential equations exhibit gradient catastrophe $u_{t_k}, v_{t_k} \to \infty$.

3 A,D,E singularities

In his seminal paper [8] Arnold proved that the list of functions with simple (without moduli) degenerate critical points consists of two infinite series $A_n (n \geq 1), D_n (n \geq 4)$ and three exceptional cases E_6, E_7, E_8. The corresponding normal forms and versal unfoldings are as follows:

$$A_n: \quad F(x;t) = x^{n+1} + t_{n-1}x^{n-1} + t_{n-2}x^{n-2} + \cdots + t_1 x + t_0, \tag{32}$$

$$D_n: \quad F(x,y;t) = x^2 y \pm y^{n-1} + t_{n-1}y^{n-2} + \cdots + t_2 y + t_1 x + t_0, \tag{33}$$

$$E_6: \quad F(x,y;t) = x^3 \pm y^4 + t_5 xy^2 + t_4 xy + t_3 y^2 + t_2 y + t_1 x + t_0, \tag{34}$$

$$E_7: \quad F(x,y;t) = x^3 + xy^3 + t_6 xy + t_5 y^4 + t_4 y^3 + t_3 y^2 + t_2 y + t_1 x + t_0, \tag{35}$$

$$E_8: \quad F(x,y;t) = x^3 + y^5 + t_7 xy^3 + t_6 xy^2 + t_5 xy + t_4 y^3 + t_3 y^2 + t_2 y + t_1 x + t_0. \tag{36}$$

Equations governing the dependence of the critical points of A_n type has been found, in fact, in the previous section. It is sufficient to extend the formulas (7)-(13) to arbitrary n. Thus, in this case (for $n \geq 3$) one has the system of equations

$$u_{t_k} = (u^k)_{t_1}, \quad k = 2, 3, ..., n-1 \tag{37}$$

or

$$\varphi_{t_k} = (\varphi_{t_1})^k, \quad k = 2, 3, ..., n-1 \tag{38}$$

and $F(u; dt) = d\varphi$. The system (37) contains the BH equation together with its $n-3$ higher flows. Thus, the system of differential equations governing versal deformations of the critical points for the entire A_n series with unbounded n represents the whole infinite BH hierarchy (see [29] and [26]).

In order to consider D and E cases let us write the functions (33)-(36) as

$$F(x,y;t) = F_0(x,y) + \sum_{k=0}^{\mu-1} e_k(x,y) t_k \tag{39}$$

with corresponding F_0, e_k and μ. Critical points are defined by the system

$$F_u = F_{0u} + \sum_{k=0}^{\mu-1} e_{ku} t_k = 0, \tag{40}$$

$$F_v = F_{0v} + \sum_{k=0}^{\mu-1} e_{kv} t_k = 0. \tag{41}$$

Differentiating these equations w.r.t. t_l, one gets the system

$$\begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix} \begin{pmatrix} u_{t_l} \\ v_{t_l} \end{pmatrix} = - \begin{pmatrix} e_{lu} \\ e_{lv} \end{pmatrix}, \quad l = 1, ..., \mu - 1 \tag{42}$$

and consequently

$$u_{t_l} = -\frac{1}{\Delta}\begin{vmatrix} e_{lu} & F_{uv} \\ e_{lv} & F_{vv} \end{vmatrix}, \quad v_{t_l} = -\frac{1}{\Delta}\begin{vmatrix} F_{uu} & e_{lu} \\ F_{uv} & e_{lv} \end{vmatrix}, \quad l = 1, ..., \mu - 1. \tag{43}$$

From equations (42) it follows that

$$\frac{\partial e_l(u,v)}{\partial t_m} - \frac{\partial e_m(u,v)}{\partial t_l} = e_{lu}u_{t_m} + e_{lv}v_{t_m} - e_{mu}u_{t_l} - e_{mv}v_{t_l} = 0. \tag{44}$$

Hence, there exists a function $\varphi(t)$ such that

$$e_l(u,v) = \varphi_{t_l}, \quad l = 1, ..., \mu - 1. \tag{45}$$

For D and E cases one has $e_0 = 1, e_1 = x, e_2 = y$. Thus, $\varphi_{t_0} = 1, u = \varphi_{t_1}, v = \varphi_{t_2}$ and equations (45) are of the form

$$\varphi_{t_l} = e_l(\varphi_{t_1}, \varphi_{t_2}), \quad l = 3, ..., \mu - 1. \tag{46}$$

In terms of u and v one has the systems

$$u_{t_l} = \frac{\partial e_l(u,v)}{\partial t_1}, \quad v_{t_l} = \frac{\partial e_l(u,v)}{\partial t_2}, \quad l = 3, ..., \mu - 1. \tag{47}$$

In virtue of (44) the systems (46),(47) describe commuting flows. From the above equations one also concludes that for all D and E cases

$$\delta F(u,v) \div F(u,v;dt) = d\varphi. \tag{48}$$

The relations of the type (48) have appeared earlier within the study of semiversal unfoldings of hypersurface singularities (see e.g. [25], Chapter 5).

The systems (46) or (47) governs motion of critical points for versal deformations of D and E singularities.

Concretely, for the D_n case one has the system

$$\varphi_{t_k} = (\varphi_{t_2})^{k-1}, \quad k = 3, 4, ..., n - 1 \tag{49}$$

and $u = \varphi_{t_1}, v = \varphi_{t_2}$. Effectively, it is again the family of the BH and higher BH equations as in A_n case.

For E_6 case one has the equations

$$\varphi_{t_3} = (\varphi_{t_2})^2, \quad \varphi_{t_4} = \varphi_{t_1}\varphi_{t_2}, \quad \varphi_{t_5} = \varphi_{t_1}(\varphi_{t_2})^2 \tag{50}$$

or three systems

$$\begin{matrix} u_{t_3} = (v^2)_{t_1,} \\ v_{t_3} = (v^2)_{t_2,} \end{matrix}; \quad \begin{matrix} u_{t_4} = (uv)_{t_1,} \\ v_{t_4} = (uv)_{t_2,} \end{matrix}; \quad \begin{matrix} u_{t_5} = (uv^2)_{t_1,} \\ v_{t_5} = (uv^2)_{t_2,} \end{matrix}. \tag{51}$$

For E_7 singularity (35) one has

$$\varphi_{t_3} = (\varphi_{t_2})^2, \quad \varphi_{t_4} = (\varphi_{t_2})^3, \quad \varphi_{t_5} = (\varphi_{t_2})^4, \quad \varphi_{t_6} = \varphi_{t_1}\varphi_{t_2} \tag{52}$$

or equivalently

$$\begin{matrix} u_{t_3} = (v^2)_{t_1,} \\ v_{t_3} = (v^2)_{t_2,} \end{matrix}; \quad \begin{matrix} u_{t_4} = (v^3)_{t_1,} \\ v_{t_4} = (v^3)_{t_2,} \end{matrix}; \quad \begin{matrix} u_{t_5} = (v^4)_{t_1,} \\ v_{t_5} = (v^4)_{t_2,} \end{matrix}; \quad \begin{matrix} u_{t_6} = (uv)_{t_1,} \\ v_{t_6} = (uv)_{t_2,} \end{matrix}. \tag{53}$$

Finally, for E_8 singularity one has the system of five equations

$$\begin{gathered} \varphi_{t_3} = (\varphi_{t_2})^2, \quad \varphi_{t_4} = (\varphi_{t_2})^3, \quad \varphi_{t_5} = \varphi_{t_1}\varphi_{t_2}, \\ \varphi_{t_6} = \varphi_{t_1}(\varphi_{t_2})^2, \quad \varphi_{t_7} = \varphi_{t_1}(\varphi_{t_2})^3 \end{gathered} \tag{54}$$

or the systems

$$\begin{gathered} \begin{matrix} u_{t_3} = (v^2)_{t_1,} \\ v_{t_3} = (v^2)_{t_2,} \end{matrix}; \ \begin{matrix} u_{t_4} = (v^3)_{t_1,} \\ v_{t_4} = (v^3)_{t_2,} \end{matrix}; \ \begin{matrix} u_{t_5} = (uv)_{t_1,} \\ v_{t_5} = (uv)_{t_2,} \end{matrix}; \\ \begin{matrix} u_{t_6} = (uv^2)_{t_1,} \\ v_{t_6} = (uv^2)_{t_2,} \end{matrix}; \ \begin{matrix} u_{t_7} = (uv^3)_{t_1,} \\ v_{t_7} = (uv^3)_{t_2,} \end{matrix}. \end{gathered} \tag{55}$$

Interrelations between the systems (50)-(55) and their 1+1-dimensional reductions are of interest. It would be also of interest to analyse a connection between apparently different systems of equations corresponding to different choices of a basis for the local algebra Q_{F_0} of the critical points.

Analysis presented above can be easily extended to the singularities and their versal unfoldings of any corank m. One derives that the set of critical points $u_i (i = 1, ..., m)$ are components of the gradient $u_i = \varphi_{t_i} (i = 1, ..., m)$ and the function $\varphi(t_1, ..., t_{\mu-1})$ obeys the system of equations

$$\varphi_{t_k} = e_k(\varphi_{t_1}, ..., \varphi_{t_m}), \quad k = m+1, ..., \mu - 1 \tag{56}$$

where $e_k(x_1, ..., x_m), k = 0, 1, ..., \mu - 1$ form a basis of the local algebra Q_{F_0} for the critical point of the function $F_0(x_1, ..., x_n)$. Equivalently, one has the systems

$$\frac{\partial u_i}{\partial t_k} = \frac{\partial e_k(u_1, ..., u_m)}{\partial t_i}, \quad i = 1, ..., m; \quad k = m+1, ..., \mu - 1 \tag{57}$$

and also

$$F(u_1, ..., u_m; dt) = d\varphi. \tag{58}$$

Properties of these equations will be discussed elsewhere.

4 Nonversal unfoldings and integrable systems of hydrodynamic type.

Construction given in the previous section is applicable to nonversal unfoldings too. For nonversal unfoldings, even if the infinitesimal deformation is the form (39), the functions e_k may not form a basis of the local algebra or number of parameters of deformation may be less than μ. Nevertheless, it is not difficult to show that for such unfoldings of corank two singularities the general formulas (40)-(45) remain unaltered, only number of the variables t_k can be different.

Here we will consider a particular unfolding of the umbilic singularities given by the function

$$F(x,y;t_1,t_3) = \alpha x^3 + 3xy^2 + t_3(\beta x^2 + \gamma y^2) + t_1 x \tag{59}$$

where α, β, γ are parameters. At $\alpha = \pm 1$ and $\beta \pm \gamma = 0$ this unfolding is equivalent to the versal unfoldings of $D_4^{\pm}$ singularity with the "frozen" parameters $t_2 = 0$ and $t_0 = 0$. At $\alpha = 0$ the function (59) represents the two-dimensional part of the infinite dimensional deformation of the germ $3xy^2$ of the critical point with infinite multiplicity [8].

We will be interested in the subclass of unfoldings (59) for which $e_3 = \beta x^2 + \gamma y^2$ belongs to the tangent space of the germ $\alpha x^3 + 3xy^2$, i.e. $e_3 \sim F_{0x}$. This constraint is verified if $\alpha\gamma = \beta$. In this case the function F after trivial rescaling takes the form

$$F(x,y;t_1,t_3) = \alpha x^3 + 3xy^2 + t_3(\alpha x^2 + y^2) + t_1 x. \tag{60}$$

Repeating the calculation performed in the previous section, one gets the system

$$u_{t_1} = -\frac{F_{vv}}{\Delta}, \quad v_{t_1} = \frac{F_{uv}}{\Delta}, \tag{61}$$

$$\begin{aligned} u_{t_3} &= -2\alpha u \frac{F_{vv}}{\Delta} + 2v\frac{F_{uv}}{\Delta}, \\ v_{t_3} &= -2v\frac{F_{uu}}{\Delta} + 2\alpha u \frac{F_{uv}}{\Delta}. \end{aligned} \tag{62}$$

Using (61), one can rewrite the last system as

$$\begin{aligned} u_{t_3} &= (\alpha u^2 + v^2)_{t_1}, \\ v_{t_3} &= -2v\tfrac{F_{uu}}{\Delta} + 2\alpha u v_{t_1}. \end{aligned} \tag{63}$$

This system is not of hydrodynamical type. In absence of t_2 equations (18) we cannot express $\frac{F_{uu}}{\Delta}$ in terms of u and v and their derivatives. A way to get the hydrodynamical type system is to impose a constraint on F. It is not difficult to show that among the constraints having form of linear relation between second order derivatives of F only the constraint $F_{uu} = \delta F_{vv}$ where δ is a constant is admissible (nontrivial). Under this constraint $\frac{F_{uu}}{\Delta} = \delta\frac{F_{vv}}{\Delta} = -\delta u_{t_1}$ and the system (63) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_3} = 2\begin{pmatrix} \alpha u, & v \\ \delta v, & \alpha u \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{t_1}. \tag{64}$$

The first equation of this system, i.e. $u_{t_3} = (\alpha u^2 + v^2)_{t_1}$ implies the existence of a function $\varphi(t_1, t_3)$ such that $u = \varphi_{t_1}$ and $\alpha u^2 + v^2 = \varphi_{t_3}$. As the consequence one has

$$F(u, v; dt) = F(u, v; t + dt) - F(u, v; t) = t_3(\alpha u^2 + v^2) + t_1 u = d\varphi.$$

Among two parameters in this system only one is relevant. Indeed for nonvanishing δ the rescaling $u \to u, v^2 \to |\delta|\, v^2, t_1 \to |\delta|\, t_1, \alpha \to \frac{\alpha}{|\delta|}$ converts the system (64) into the same system with $\delta = \pm 1$ leaving only the parameter α free. For nonvanishing α one can convert the system (64) into that with $\alpha = 1$ and free δ. Since we will consider the case $\alpha = 0$ among the others the first choice is preferable.

The system (64) with $\delta = \pm 1$ and particular values of α coincides with some well-known integrable hydrodynamic type systems. Indeed, at $\alpha = 0$ one has the system

$$u_{t_3} = 2vv_{t_1}, \quad v_{t_3} = 2\delta v u_{t_1} \tag{65}$$

which is equivalent to the dispersionless Toda equation $\Phi_{t_3 t_3} = 4\delta(\exp\Phi)_{t_1 t_1}$ for $\Phi = \log v^2$. For $\alpha = 2$ and $\delta = 1$ it is the hyperbolic one-layer Benney system (classical shallow water equation) for variables u and v^2 (see e.g.[36] and [37]). At $\alpha = 2$ and $\delta = -1$ it is the elliptic one-layer Benney system which is equivalent to the dispersionless Da Rios system. For $\alpha = \delta = \pm 1$ the corresponding system $u_{t_3} = (\pm u^2 + v^2)_{t_1}, v_{t_3} = \pm 2(uv)_{t_1}$ decomposes into two BH equations $(u \pm \sqrt{\pm}v)_{t_3} = \mp((u \pm \sqrt{\pm}v)^2)_{t_1}$. For other values of α the system (64) is equivalent to the system ($t = -2\alpha t_3, x = t_1, \rho = v^{\alpha\delta}$)

$$\rho_t + (\rho u)_x = 0, \quad u_t + u_x u + \frac{p_x}{\rho} = 0, \tag{66}$$

which describes one-dimensional motions of ideal barotropic gas with the density ρ and pressure $p = \frac{1}{\alpha(\alpha\delta+2)}\rho^{\frac{\alpha\delta+2}{\alpha\delta}}$ (see e.g. [38]).

In terms of Riemann invariants $\beta_1 = u + \frac{1}{\sqrt{\delta}}v, \beta_2 = u - \frac{1}{\sqrt{\delta}}v$ the system (64) is of the form

$$\beta_{1t_3} = |\delta|\,(\varepsilon(\beta_1 + \beta_2) + \beta_1)\beta_{1t_1}, \quad \beta_{2t_3} = |\delta|\,(\varepsilon(\beta_1 + \beta_2) + \beta_2)\beta_{2t_1}, \tag{67}$$

where $\varepsilon = \frac{1}{2}(\frac{\alpha}{|\delta|} - 1)$. This form of the system (64) clearly shows that only the ratio $\frac{\alpha}{|\delta|}$ is relevant. We will put $\delta = 1$ in what follows. The hodograph equations are of the form $F_{\beta_1} = 0, F_{\beta_2} = 0$ while the constraint $F_{uu} = F_{vv}$ becomes $F_{\beta_1\beta_2} = 0$. Under this condition $\Delta = F_{\beta_1\beta_1}F_{\beta_2\beta_2}$ and hence the catastrophe set where the solutions of (64) or (67) exhibit gradient catastrophe decomposes into two pieces $F_{\beta_1\beta_1} = 0$ and $F_{\beta_2\beta_2} = 0$.

The system (67) represents a two-component case of the so-called ε-systems discussed in [39] . In a different manner the systems (64)-(66) arisen within the study of hydrodynamic type systems associated with the two dimensional Frobenius manifolds [22].

5 Hodograph equations for hydrodynamic type systems as equations for critical points and Euler-Poisson-Darboux equation

Critical points for unfoldings of A and D singularities as the functions of deformation parameters represent very particular solutions of the BH equation and the system (64). What about the other solutions of these integrable equations? Do they describe certain unfoldings of critical points for other singularities?

To address this question we begin with the BH equation $u_{t_2} = (u^2)_{t_1}$. A standard hodograph equation for it (see e.g. [36, 38])

$$t_1 + 2t_2u + f(u) = 0 \tag{68}$$

is the equation defining critical point $u(t)$ of the function $W(x;t) = F_0(x) + t_2x^2 + t_1x$ where $F_{0x}(x) = f(x)$ and $f(u)$ is the function inverse to the initial data function $u_0(t_1) = u(t_1, t_2 = 0)$. Let us consider a family of BH solutions which corresponds to the family of initial data with $F_0(x) = x^{n+1} + t_nx^n + t_{n-1}x^{n-1} + \cdots + t_3x^3$ where $t_3, ..., t_n$ are parameters and n is arbitrary. These solutions $u(t_1, t_2, t_3, ..., t_n)$ of BH equation clearly are associated with unfoldings (versal at $t_n = 0$) of A_n singularities considered in the previous section. Thus, the class of solutions of BH equation which corresponds to the family of initial data of this type with unbounded n describes unfoldings of critical points for all A_n singularities. Alternatively, the parameters $t_3, ..., t_n$ can be viewed as times for higher BH equations within the infinite BH hierarchy $(n \to \infty)$ [29].

An extension of this construction to the system (64) is almost straightforward. We already saw that the particular solution of this system describes motion of the critical points for the function (60). Thus, we will look for the functions $F(x, y; t) = F_0(x, y) + t_3(\alpha x^2 + y^2) + t_1x$ with F_0 different from $\alpha x^3 + 3xy^2$. It is convenient to pass to the variables $x_1 = x + y$, $x_2 = x - y$. In these variables $F = F_0(x_1, x_2) + \frac{t_2}{4}((\varepsilon+1)x_1^2 + 2\varepsilon x_1 x_2 + (\varepsilon+1)x_2^2) + \frac{t_1}{2}(x_1 + x_2)$ where $\varepsilon = \frac{1}{2}(\alpha - 1)$ and $t_2 = 2t_3$. Equations defining the critical points $\beta_1$and β_2 are

$$F_{\beta_1} = 0, \quad F_{\beta_2} = 0. \tag{69}$$

Lemma 1. *Solution of equations (69), with the function F of the form given above and any function F_0 such that $F_{\beta_1\beta_2} = 0$ and $F_{\beta_1\beta_1}F_{\beta_2\beta_2} \neq 0$, is a solution of the system (67).*

Proof. Differentiating equation (69) w.r.t. t_1 and t_2 and assuming that $F_{\beta_1\beta_2} = 0$, one gets

$$\beta_{1t_1} = -\frac{1}{2F_{\beta_1\beta_1}}, \quad \beta_{2t_1} = -\frac{1}{2F_{\beta_2\beta_2}} \tag{70}$$

and

$$\beta_{1t_2} = -\frac{((\varepsilon+1)\beta_1 + \varepsilon\beta_2)}{2F_{\beta_1\beta_1}}, \quad \beta_{2t_2} = -\frac{(\varepsilon\beta_1 + (\varepsilon+1)\beta_2)}{2F_{\beta_2\beta_2}}. \tag{71}$$

Eliminating $F_{\beta_1\beta_1}$ and $F_{\beta_2\beta_2}$ from equations (71), one obtains the system (67). $\boxed{QED}$

The function F_0 due to the relations

$$\beta_{1t_1}(t_1, t_2 = 0) = -\frac{1}{2F_{0\beta_1\beta_1}|_{t_2=0}}, \quad \beta_{2t_1}(t_1, t_2 = 0) = -\frac{1}{2F_{0\beta_2\beta_2}|_{t_2=0}}$$

is defined implicitly by the initial data for β_1 and β_2. The functions F and aF where a is an arbitrary constant, obviously, give rise to the same equation.

Equations (67) imply that

$$\frac{1}{2}(\beta_1 + \beta_2)_{t_2} = \frac{1}{4}\left((\varepsilon+1)\beta_1^2 + 2\varepsilon\beta_1\beta_2 + (\varepsilon+1)\beta_2^2\right)_{t_1}.$$

So, $\frac{1}{2}(\beta_1 + \beta_2) = \varphi_{t_1}, \frac{1}{4}\left((\varepsilon+1)\beta_1^2 + 2\varepsilon\beta_1\beta_2 + (\varepsilon+1)\beta_2^2\right) = \varphi_{t_2}$ where φ is a function of t_1 and t_2 and hence

$$F(\beta_1, \beta_2; dt) = d\varphi. \tag{72}$$

There is a special subclass of functions F of particular interest. It is given by

$$W(x_1, x_2; t) = \frac{1}{2\varepsilon}\oint\limits_{\gamma} \frac{d\lambda}{2\pi i}\left(\sum_{k\geq 1}\lambda^{k-1}t_k\right)\left((1 - \frac{x_1}{\lambda})(1 - \frac{x_2}{\lambda})\right)^{-\varepsilon} \tag{73}$$

where γ denotes a large positively oriented circle on the λ plane. Written explicitly the function W is the series

$$\begin{aligned}W = \frac{1}{2}t_1(x_1+x_2) + t_2\frac{1}{4}\left[(\varepsilon+1)(x_1^2+x_2^2)+2\varepsilon x_1x_2\right]+\\ + t_3\frac{1}{12}(\varepsilon+1)\left[(\varepsilon+2)(x_1^3+x_2^3)+3\varepsilon(x_1x_2^2+x_2x_1^2)\right]+\\ + t_4\frac{1}{48}(\varepsilon+1)\left[(\varepsilon+2)(\varepsilon+3)(x_1^4+x_2^4)\right.\\ \left.+4\varepsilon(\varepsilon+2)(x_1x_2^3+x_2x_1^3)+6\varepsilon(\varepsilon+1)x_1^2x_2^2\right]+\cdots\end{aligned} \quad (74)$$

An important property of the function W is that it obeys the Euler-Poisson-Darboux equation $E(\varepsilon,\varepsilon)$, i.e.

$$(x_1-x_2)W_{x_1x_2} = \varepsilon(W_{x_1}-W_{x_2}). \quad (75)$$

This equation and representation of its solutions in the form (73) are known for more than a century (see [40]). In the papers [33, 34, 35] it was observed that the hodograph equations for the one-layer Benney hierarchy ($\varepsilon=\frac{1}{2}$) and dispersionless dToda hierarchy ($\varepsilon=-\frac{1}{2}$) are nothing but that the equations for critical points of the function $2\varepsilon W$.

For arbitrary $\varepsilon\neq 0$ equations

$$W_{\beta_1}=0, \quad W_{\beta_2}=0 \quad (76)$$

for the critical points are the hodograph equations for the system (67). Due to equation (75) the function W at $\beta_1\neq\beta_2$ automatically verifies the condition $W_{\beta_1\beta_2}=0$. For the function F of the form (73) the function F_0 is a special one. The variables $t_3, t_4, ...$ can be viewed as the variables parametrizing a family of initial data for the system (67). Alternatively, one can treat them as the higher "times" for the commuting systems

$$\beta_{1t_k}=\theta_{1k}(\beta_1,\beta_2)\beta_{1t_1}, \quad \beta_{2t_k}=\theta_{2k}(\beta_1,\beta_2)\beta_{2t_1}, \quad k=3,4,... \quad (77)$$

with the characteristic velocities $\theta_{1k}(\beta_1,\beta_2)=\frac{\partial}{\partial x_1}(\frac{\partial W}{\partial t_k})|_{x=\beta}$, $\theta_{2k}(\beta_1,\beta_2)=\frac{\partial}{\partial x_2}(\frac{\partial W}{\partial t_k})|_{x=\beta}$. The totality of these systems is an infinite hierarchy of systems associated with the system (67) and the function W plays the role of the generating function for this hierarchy.

In the singular case $\varepsilon=0$ the generating function $W_{\varepsilon=0}$ is given by

$$\begin{aligned}W_{\varepsilon=0} = \oint_\gamma \frac{d\lambda}{2\pi i}\left(\sum_{k\geq 1}\lambda^{k-1}t_k\right)\log\left((1-\frac{x_1}{\lambda})(1-\frac{x_2}{\lambda})\right)\\ = -\sum_{n\geq 1}\frac{1}{n}t_n(x_1^n+x_2^n).\end{aligned} \quad (78)$$

which gives rise to two independent BH hierarchies for β_1 and β_2.

The observation that the hodograph solutions of the two-component hydrodynamic systems of ε-type describe critical points of function (73) is extendable to the multi-component case. It was shown in [33] that the critical points of the function

$$W(x_1, x_2, ..., x_n; t) = \frac{1}{2\varepsilon} \oint_\gamma \frac{d\lambda}{2\pi i} \left(\sum_{k\geq 1} \lambda^{k-1} t_k \right) \left((1 - \frac{x_1}{\lambda})(1 - \frac{x_2}{\lambda}) \cdots (1 - \frac{x_n}{\lambda}) \right)^{-\varepsilon} \quad (79)$$

at $\varepsilon = \frac{1}{2}$ are described by hodograph solutions of the dispersionless n-component coupled KdV equation.

For arbitrary $\varepsilon \neq 0$ the function (79) obeys the Euler-Poisson-Darboux system

$$(x_i - x_k) W_{x_i x_k} = \varepsilon (W_{x_i} - W_{x_k}), \quad i \neq k; i, k, = 1, ..., n. \quad (80)$$

From the equations $F_{\beta_i} = 0, i = 1, ..., n$ for critical points $\beta_1, ..., \beta_n$ of the function (79) one finds $\beta_{it_k} = -\frac{W_{k\beta_i}}{W_{\beta_i \beta_i}}, i = 1, ..., n; k = 1, 2, ...$ where W_k are defined by the expansion $W(x; t) = \sum_{k\geq 1} t_k W_k(x)$. Since $W_{1\beta_i} = 1$ one gets the following hierarchy of hydrodynamic type systems governing the motion of the critical points

$$\beta_{it_k} = W_{k\beta_i}(\beta) \beta_{it_1}, \quad i = 1, ..., n; \quad k = 2, 3, ... \quad (81)$$

The first member of this hierarchy is given by the ε-system $\beta_{it_2} = (\varepsilon(\sum_{m=1}^{m=n} \beta_m) + \beta_i)\beta_{it_1}, \quad i = 1, ..., n$ considered in [39]. The system (81) implies that $W_{kt_l} - W_{lt_k} = 0, \quad k, l = 1, 2, ...$ So, $W_k = \varphi_{t_k}$ and hence

$$W(\beta; dt) = d\varphi.$$

It was observed in [34] (see also [39]) that the systems (81) have an interesting property: densities P of their conserved quantities satisfy the Euler-Poisson-Darboux equations dual to (80) (i.e. with opposite sign of ε). An infinite family of such densities is given by

$$P_n(\beta) = \oint_\gamma \frac{d\lambda}{2\pi i} \lambda^n \left((1 - \frac{\beta_1}{\lambda})(1 - \frac{\beta_2}{\lambda}) \cdots (1 - \frac{\beta_n}{\lambda}) \right)^{\varepsilon}, \quad n = 0, 1, 2, ...$$

A class of the hydrodynamic type systems for which hodograph equations coincide with the equations for critical points of certain functions is, in fact, larger. It consists of all semihamiltonian diagonal systems for which characteristic velocities θ_l are components of the gradient of a function, i.e.

$$\beta_{lt} = \Phi_{\beta_l} \beta_{lx}, \quad l = 1, ..., n. \tag{82}$$

where $\Phi_{\beta_l} \div \frac{\partial \Phi}{\partial \beta_l}$. Indeed, according to [41] the generalized hodograph equations for semihamiltonian diagonal system $\beta_{lt} = \theta_l(\beta)\beta_{lx}, l = 1, ..., n$ are given by the system

$$x + \theta_l(\beta)t + \omega_l(\beta) = 0, \quad l = 1, ..., n \tag{83}$$

where the functions $\omega_l(\beta)$ obeys the equations

$$\frac{\omega_{i\beta_k}}{\omega_k - \omega_i} = \frac{\theta_{i\beta_k}}{\theta_k - \theta_i}, \quad i \neq k \tag{84}$$

If $\theta_i = \Phi_{\beta_i}$then the l.h.s. of (84) is skew symmetric and hence $\omega_{i\beta_k} = \omega_{k\beta_i}$. So there exists a function Φ_ω such that $\omega_i(\beta) = \Phi_{\omega\beta_i}, i = 1, .., n$. As a result, the hodograph equations (83) take the form

$$x + \Phi_{\beta_l} t + \Phi_{\omega\beta_l} = 0, \quad l = 1, ..., n$$

that coincides with the equations $F_{\beta_l} = 0$ for critical points of the function

$$W = x(x_1 + \cdots + x_n) + t\Phi(x) + \Phi_\omega(x). \tag{85}$$

From (84) it follows that this function obeys the system of equations

$$W_{x_i x_k} = \frac{\Phi_{x_i x_k}}{\Phi_{x_i} - \Phi_{x_k}} (W_{x_i} - W_{x_k}), \quad i \neq k; \quad i, k, = 1, ..., n. \tag{86}$$

as well as the function Φ_ω. Densities P of the conserved quantities for the system (82) satisfy the equations [41]

$$P_{\beta_i \beta_k} = -\frac{\Phi_{\beta_i \beta_k}}{\Phi_{\beta_i} - \Phi_{\beta_k}} (P_{\beta_i} - P_{\beta_k}), \quad i \neq k; \quad i, k, = 1, ..., n. \tag{87}$$

Thus, the hydrodynamic type system (82) describes critical points of the function W (85) which obeys equations (86) and this equations (as well as the equations for the generating function Φ_ω of symmetries) and the equations for conserved densities P of (82) are dual to each other.

6 Hierarchies of integrable systems and normal forms of singularities

Now let us discuss the hierarchies of two- and three-component integrable systems considered in the previous section and the corresponding functions W from the singularity theory viewpoint. The functions $W=\sum_{k\geq 1} t_k W_k$ provide us with an infinite families of symmetric homogeneous functions $W_k(x_1,...,x_n)$ of degrees k. Is there any relation between these functions and normal forms of germs in singularity theory? We will present here two observations which indicate that such a connection exists.

In the two-component case the function W (73) in the variables x and y is given by

$$W(x,y;t)=\frac{1}{2\varepsilon}\oint\limits_{\gamma}\frac{d\lambda}{2\pi i}\left(\sum_{k\geq 1}\lambda^{k-1}t_k\right)\left((1-2x\frac{1}{\lambda}+(x^2-y^2)\frac{1}{\lambda^2}\right)^{-\varepsilon} \tag{88}$$

or

$$\begin{aligned} W=t_1x+\frac{1}{2}t_2(\alpha x^2+y^2)+\frac{1}{6}(\alpha+1)t_3(\alpha x^3+3xy^2)\\ +\frac{1}{384}(\alpha+1)t_4(ax^4+bx^2y^2+cy^4)+\cdots \end{aligned} \tag{89}$$

where $a=15\alpha^2+24\alpha-15,\quad b=\alpha^2+8\alpha-33,\quad c=6\alpha^2-48\alpha-102.$

Third term in (89) generates the first higher commuting flow for the system (64). At the same time, it provides us with the normal form of simple D_4 singularity after the trivial rescaling of x. Fourth term gives rise to the next commuting flow for (64). Does it correspond to some standard normal form? It is easy to see that, rescaling x and y, one can convert it into the form $x^4+dx^2y^2+y^4$ where d is certain function of α . For generic α the parameter $d\neq\pm 2$. Thus one has the normal form of the unimodular X_9 singularity (see [42, 4]). Higher W_k in the expansion $W=\sum_{k\geq 1} t_k W_k$ which generate higher flows are of the form $W_k=\sum_{l=0}^{l=[\frac{k}{2}]} a_l x^{k-2l}y^{2l}$ where $a_l(\alpha)$ are certain polynomials in α. So, they describe unimodular singularities of the order k with the germs symmetric with respect to the reflection $y\to -y$.

Second example is given by the system (81) with $n=3$ and $\varepsilon=\frac{1}{2}$, i.e. by the dispersionless three-component coupled KdV system. The function W is

$$\begin{aligned} W=\frac{1}{2}t_1(x_1+x_2+x_3)+\frac{1}{8}t_2(3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3+2x_2x_3)+\\ +\frac{1}{16}t_3(5x_1^3+5x_2^3+5x_3^3+3x_1^2x_2+3x_1^2x_3+ \end{aligned}$$

$$3x_2^2x_1 + 3x_2^2x_3 + 3x_3^2x_1 + 3x_3^2x_2 + 2x_1x_2x_3) + \cdots$$

In terms of variables x,y,z defined by

$$x_1 = ax + y + z, \quad x_2 = ax + qy + q^2z, \quad x_3 = ax + q^2y + qz, \tag{90}$$

where q=exp($\frac{2\pi i}{3}$) and $a = \frac{2}{\sqrt[3]{35}}$, one has

$$W = \frac{3}{2}at_1x + \frac{3}{8}t_2(5a^2x^2 + 4yz) + \frac{1}{2^5 3^3}t_3(x^3 + y^3 + z^3 + \frac{21}{2}axyz) + \cdots \tag{91}$$

The transformation (90) is, in fact, the well-known relation between roots x_i of a cubic equation and its Lagrange resolvents ℓ_i modulo the identification $ax = \frac{1}{3}\ell_1, y = \frac{1}{3}\ell_2, z = \frac{1}{3}\ell_3$ (see e.g.[43]).

The W_3 term in (91) represents the normal form of the unimodular parabolic singularity P_8 (or $T_{3,3,3}$) of corank three (see [42, 4]). At $t_k = 0, k = 4, 5, ...$ the function (91) gives the unfolding of P_8 singularity. The dependence of the critical points u,v,w for this function on deformation parameters t_1 and t_2 is described by the three-component dispersionless coupled KdV system (81) or equivalently by the system

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} \frac{5}{2}au, & \frac{1}{a}w, & \frac{1}{a}v \\ av, & \frac{5}{2}u, & w \\ aw, & v, & \frac{5}{2}u \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1}. \tag{92}$$

The functions W_k with $k = 4, 5, ...$ represent higher order singularities of corank three and the function (91) give their unfoldings.

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