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Some stability results on nearly–integrable systems (with dissipation)

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Abstract. The stability of nearly–integrable systems can be studied over different time scales and with different techniques. In this paper we review some classical methods, like the averaging technique, the classical perturbation theory, KAM theorem and Nekhoroshev's stability for exponential times. We investigate also conformally symplectic systems, in particular nearly–integrable systems with dissipation, and we present some results about KAM and exponential stability in the dissipative context.

Keywords: Stability, perturbation theory, KAM theorem, Nekhoroshev's theorem

MSC 2000 classification: primary 70K20, secondary 70H14

1 Introduction

Many physical models are governed by nearly–integrable Hamiltonian systems. Most notably, the restricted three–body problem in Celestial Mechanics is described by a Hamiltonian function composed by an integrable part, representing the Keplerian motion, plus a perturbation depending on a small parameter measuring the primaries' mass ratio. The study of the stability of these systems is of primary importance; several techniques are nowadays available and in this work we review some of them.

The general aim is to provide a bound on the variation of the action variables (which correspond to the so–called elliptic elements in the three–body problem, precisely the semimajor axis, the eccentricity and the inclination). The stability of the actions can be ensured over different time scales. Using the averaging technique ([26]), one compares the dynamics with that of the averaged system, providing stability bounds over times of the order of the inverse of the perturbing parameter. Though having interesting applications (for example in the theory of artificial satellites), the stability time provided by the averaging technique might be too small in comparison with astronomical times, like the age of the solar system. Classical perturbation theory (see, e.g., [7]) can be implemented to obtain estimates valid on longer time scales. However, the most important results in this field are definitely represented by the Kolmogorov–Arnold–Moser (hereafter KAM, see [20], [1], [23]) theorem and by the Nekhoroshev's theorem ([24]). KAM theory applies under mild assumptions (a non–degeneracy of the unperturbed Hamiltonian and a strong non–resonance condition of the frequency); it provides the stability of invariant tori on which a quasi–periodic motion takes place. When dealing with low–dimensional systems, KAM theory provides a strong stability property in the sense of confinement in the phase space. This property does not hold for higher dimensional systems, since diffusion might take place. However, Nekhoroshev's theorem allows us to provide a bound on the action variables for exponentially long times. We review such theorems in the nearly–integrable Hamiltonian context, providing also some results for conformally symplectic systems, namely dissipative systems for which the symplectic form is transformed into a multiple of itself.

This paper is organized as follows. In Section 2 we introduce the stability problem over different time scales; in Section 3 we review the averaging technique and perturbation theory is presented in Section 4. KAM theory for conservative and dissipative systems is presented in Section 5, while stability estimates for exponential times are discussed in Section 6.

2 Stability of nearly–integrable (dissipative) systems

We consider an n–dimensional *nearly–integrable* Hamiltonian system in action–angle variables $(y, x) \in V \times \mathbb{T}^n$, where V is an open set of $\mathbb{R}^n$; we introduce the associated Hamiltonian function as

$$\mathcal{H}(y, x) = h(y) + \varepsilon f(y, x) \ , \tag{1}$$

where h and f are analytic functions to which we refer to, respectively, as the unperturbed (or integrable) Hamiltonian and the perturbing function; moreover, ε is a small parameter which measures the strength of the perturbation. In the *integrable* approximation $\varepsilon = 0$ Hamilton's equations can be solved as

$$\begin{aligned} \dot{y} &= -\frac{\partial h(y)}{\partial x} = 0 \qquad \Rightarrow \quad y(t) = y(0) \\ \dot{x} &= \frac{\partial h(y)}{\partial y} \equiv \omega(y) \qquad \Rightarrow \quad x(t) = \omega\left(y(0)\right) t + x(0) \ , \end{aligned}$$

where $(y(0), x(0))$ denotes the initial condition and where we have introduced the *frequency vector*

$$\omega(y) \equiv \frac{\partial h(y)}{\partial y} \ .$$

We see that the actions are constants, while the angle variables vary linearly with the time, so that the solution takes place on a torus with given frequency $\omega = \omega(y(0))$. For $\varepsilon \neq 0$ the equations of motion are given by

$$\begin{aligned} \dot{y} &= -\varepsilon \frac{\partial f(y,x)}{\partial x}, \\ \dot{x} &= \omega(y) + \varepsilon \frac{\partial f(y,x)}{\partial y} \ ; \end{aligned} \tag{2}$$

these equations might no longer be integrable and chaotic motions can appear.

Our aim is to investigate stability results according to which the actions remain bounded over a given interval of time, namely there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ one gets that

$$|y(t) - y(0)| \leq \alpha(\varepsilon) \quad \text{for} \quad t \leq T(\varepsilon) \ ,$$

where $T = T(\varepsilon)$ is the stability time and $\alpha = \alpha(\varepsilon)$ is a bounded regular function depending on ε (compare with Figure 1). In particular we aim to establish the

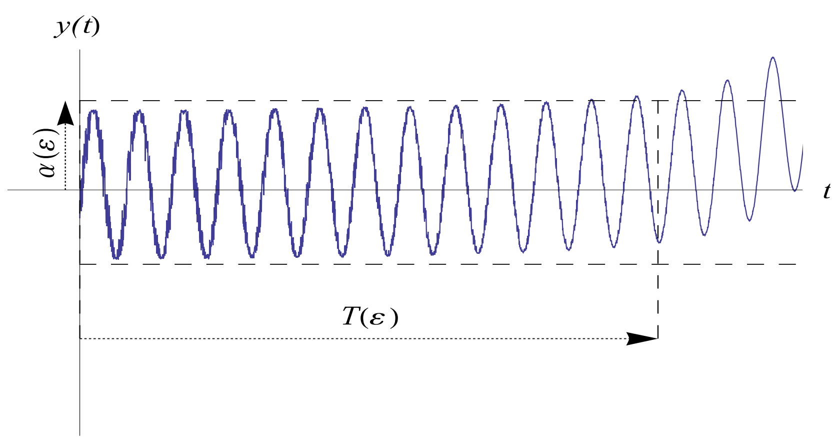

Figure 1. The stability problem: the actions are bounded by a function $\alpha = \alpha(\varepsilon)$ over a time $T(\varepsilon)$ (courtesy of C. Lhotka).

existence of a parameter interval, say $0 \leq \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$, such that one of the following results is obtained:

i) the actions are bounded as $|y(t) - y(0)| \leq \alpha(\varepsilon)$ for any $t \leq T(\varepsilon)$ with $T(\varepsilon)$ of the order of $1/\varepsilon$;

ii) the actions are bounded for any $t \leq T(\varepsilon)$ with $T(\varepsilon)$ of the order of $1/\varepsilon^2$;

iii) the actions remain bounded for an exponentially long time;

iv) there exists perpetual stability such that $|y(t) - y(0)| \leq \alpha(\varepsilon)$ for all times.

The first item is obtained through the method of averaging (see Section 3), while the second item consists in performing classical perturbation theory (see Section 4). Stability estimates for exponential times are obtained through Nekhoroshev's theorem (see Section 6), which provides a bound on the actions for times of the order of $T(\varepsilon) = T_0 \ e^{(\frac{\varepsilon_0}{\varepsilon})^a}$ for some positive real constants T_0, a, provided $\varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$. For low–dimensional systems, say $n \leq 2$, perpetual stability is obtained by KAM theory (see Section 5) by proving the persistence of invariant tori on which a quasi–periodic motion takes place. In fact, in 2–dimensional systems the phase space has dimension 4, the constant energy surfaces have dimension 3, while invariant tori are 2–dimensional. Such invariant tori provide a confinement in the constant energy phase space and one can prove perpetual stability by showing the existence of two bounding invariant tori, which confine the motion from above and below. We remark that the confinement property is no more valid for $n > 2$ due to the so–called Arnold's diffusion (see, e.g., [2]), since the motion can diffuse through invariant tori, reaching arbitrarily far regions of the phase space.

We will also consider the main results in the more general context of conformally symplectic systems. In particular, we are interested to nearly–integrable dissipative systems, which we assume to be described by a vector field of the form (2) plus a dissipation depending on the actions, say

$$\begin{aligned} \dot{y} &= -\varepsilon \frac{\partial f(y,x)}{\partial x} - \mu(g(y) - \eta), \\ \dot{x} &= \omega(y) + \varepsilon \frac{\partial f(y,x)}{\partial y} \ , \end{aligned} \tag{3}$$

where $\mu > 0$ is the *dissipative constant*, $g = g(y)$ is an analytic function and η is the so–called *drift term*. As we will see, vector fields of the form (3) find many applications in Celestial Mechanics.

3 The averaging method

The averaging method is used to get an approximation of the solution over a time scale of the order of $1/\varepsilon$. The averaging technique has been used in several fields; as an example, we quote the work of Lagrange concerning Celestial Mechanics, where averaging is introduced through the following procedure: the equations of the variations of constants are introduced, the vector field is expanded in Fourier series and the secular terms, namely the first order average, are considered. We briefly recall the averaging method, referring to [26] for an exhaustive presentation.

Consider the n–dimensional nearly–integrable Hamiltonian function (1) and let us write the corresponding Hamilton's equations as

$$\begin{aligned} \dot y &= \varepsilon F(y,x) \\ \dot x &= \omega(y) + \varepsilon G(y,x)\ , \end{aligned} \tag{4}$$

where $F(y,x) \equiv -\frac{\partial f(y,x)}{\partial x}$, $G(y,x) = \frac{\partial f(y,x)}{\partial y}$. We split F as its average plus an oscillating part, say $F(y,x) = \overline{F}(y) + \tilde F(y,x)$; then, (4) becomes

$$\begin{aligned} \dot y &= \varepsilon \overline{F}(y) + \varepsilon \tilde F(y,x) \\ \dot x &= \omega(y) + \varepsilon G(y,x)\ . \end{aligned} \tag{5}$$

After averaging (5) with respect to the angle variables, we get a differential equation in a new set of coordinates, say $z \in \mathbb{R}^n$, such that

$$\dot z = \varepsilon \overline{F}(z)\ . \tag{6}$$

Let us denote by $y_\varepsilon(t)$ the solution associated to (5) with initial condition $y_\varepsilon(0)$ and let $z_\varepsilon(t)$ be the solution of (6) with initial condition $z_\varepsilon(0) = y_\varepsilon(0)$. The aim of averaging theory is to study under which conditions the solution of the averaged equation (6) represents a good approximation of the complete solution associated to (5), so that one has:

$$\lim_{\varepsilon\to 0} |y_\varepsilon(t) - z_\varepsilon(t)| = 0 \qquad \text{for } 0 \le t \le \frac{1}{\varepsilon}\ .$$

To give an example, let us consider the following Hamiltonian function with $n > 1$ degrees of freedom:

$$\mathcal{H}(y,x) = \omega \cdot y + \varepsilon f(x)\ ,$$

where $\omega \in \mathbf{R}^n \backslash \{0\}$ and the dot denotes the scalar product. The equations of motion are

$$\begin{aligned} \dot y &= \varepsilon F(x) \\ \dot x &= \omega\ , \end{aligned} \tag{7}$$

with $F(x) \equiv -\frac{\partial f(x)}{\partial x}$. Let $\overline{F}$ be the average of $F(x)$, that we write as

$$F(x) = \overline{F} + \sum_{k\in\mathcal{K}} \hat F_k e^{ik\cdot x}$$

for a suitable sublattice $\mathcal{K}$ of $\mathbf{Z}^n \backslash \{0\}$; then, one can prove the following result ([7]), which provides an approximation of the actions over a time scale of the order of $1/\varepsilon$.

Proposition 1. *Let $y_\varepsilon(t)$ and $z_\varepsilon(t)$ denote, respectively, the solutions at time t of (7) and of the averaged equations with initial data equal to $y_\varepsilon(0)$ and $z_\varepsilon(0) = y_\varepsilon(0)$. Provided the set $\mathcal{K}_0 \equiv \{k \in \mathcal{K} : \ k \cdot \omega = 0\}$ is empty, then one gets*

$$\lim_{\varepsilon\to 0} |\underline{y}_\varepsilon(t) - \underline{z}_\varepsilon(t)| = 0 \qquad \text{for any} \quad 0 \le t \le \frac{1}{\varepsilon} \ .$$

4 Perturbation theory

Classical perturbation theory aims to construct a canonical transformation, which allows to remove the perturbation to higher orders in the perturbing parameter ε. Taking into account the Hamiltonian (1), we define a canonical change of variables, say $\mathcal{C} : V \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n$ with $(y', x') = \mathcal{C}(y, x)$, such that (1) is transformed to

$$\mathcal{H}'(y', x') \ = \ h'(y') + \varepsilon^2 f'(y', x') \ , \tag{8}$$

where h' and f' denote, respectively, the new unperturbed Hamiltonian and the new perturbing function. This transformation is constructed by defining a canonical change of variables close to the identity, expanding the original Hamiltonian in Taylor series around $\varepsilon = 0$ and requiring that the transformation removes the dependence on the angle variables up to second order terms. We stress that the same technique can be performed to higher orders in ε. By expanding the perturbing function in Fourier series, one easily gets the explicit form of the canonical transformation.

More precisely, we introduce a change of variables close–to–identity as

$$\begin{aligned} y &= y' + \varepsilon \, \frac{\partial \Phi(y', x)}{\partial x} \\ x' &= x + \varepsilon \, \frac{\partial \Phi(y', x)}{\partial y'} \ , \end{aligned} \tag{9}$$

where $\Phi = \Phi(y', x)$ must be determined so that (1) takes the form (8). To this end, let us write the perturbing function as

$$f(y, x) = \overline{f}(y) + \tilde{f}(y, x) \ ,$$

where $\overline{f}(y)$ denotes the average over the angles and $\tilde{f}(y, x)$ is the oscillating part. Inserting (9) in (1) and expanding in Taylor series around $\varepsilon = 0$ up to the second order, one obtains

$$\mathcal{H}(y', x) = h(y') + \varepsilon \, \omega(y') \cdot \frac{\partial \Phi(y', x)}{\partial x} + \varepsilon \overline{f}(y') + \varepsilon \tilde{f}(y', x) + O(\varepsilon^2) \ .$$

The transformed Hamiltonian is integrable up to $O(\varepsilon^2)$ provided that the function Φ satisfies the equation

$$\omega(y') \cdot \frac{\partial \Phi(y',x)}{\partial x} + \tilde{f}(y',x) = 0 \ . \tag{10}$$

As a consequence, we immediately obtain the new unperturbed Hamiltonian as

$$h'(y') \ = \ h(y') + \varepsilon \overline{f}(y') \ . \tag{11}$$

An explicit expression of the generating function is obtained solving (10) as follows. Let us expand Φ and $\tilde{f}$ in Fourier series as

$$\begin{aligned} \Phi(y',x) \ &= \ \sum_{m \in \mathbf{Z}^n \setminus \{0\}} \hat{\Phi}_m(y') \ e^{im \cdot x} \ , \\ \tilde{f}(y',x) \ &= \ \sum_{m \in \mathcal{I}} \hat{f}_m(y') \ e^{im \cdot x} \ , \end{aligned} \tag{12}$$

where $\mathcal{I}$ denotes a suitable set of integer vectors associated to the Fourier indexes of $\tilde{f}$. Inserting (12) in (10), one obtains the Fourier coefficients $\hat{\Phi}_m(y')$, which allow to write the generating function as

$$\Phi(y',x) = i \sum_{m \in \mathcal{I}} \frac{\hat{f}_m(y')}{\omega(y') \cdot m} \ e^{im \cdot x} \ . \tag{13}$$

Formulae (11) and (13) provide, respectively, a constructive way to build the new Hamiltonian function (which is just the sum of the old integrable part plus the average of the perturbing function). This algorithm can be applied provided that no zero divisors appear in (13), namely there do not exist an integer vector $m \in \mathcal{I}$, $m \neq 0$, such that

$$\omega(y') \cdot m = 0 \ . \tag{14}$$

We note that terms of the form $\omega(y') \cdot m$, though not being zero, can become arbitrarily small; for this reason they are called *small divisors*. Their existence can prevent the convergence of the generating function and therefore the implementation of perturbation theory (see [27]).

Remark 1. A frequency vector ω satisfying the relation (14) is said a *resonant* frequency. Resonant frequencies are often found in physical models. For example, in Celestial Mechanics one speaks of *mean motion resonances*, whenever the frequencies of revolution of two planets around the Sun, equivalently of two satellites around a planet, are rationally dependent. In rotational dynamics one speaks of *spin–orbit resonance*, whenever the frequency of rotation and the frequency of revolution of a satellite around a planet are rationally dependent. For example, the Moon satisfies a spin–orbit resonance, since the period of revolution around the Earth is the same as the period of rotation of the Moon about its spin–axis.

5 KAM theory

In this section we review some results concerning KAM theory in the conservative (see Section 5.1) and in the dissipative (see Section 5.2) settings.

5.1 Conservative KAM theorem

Kolmogorov–Arnold–Moser theory allows to overcome the small divisor problem arising in perturbation theory, provided that two main assumptions are satisfied. The first requirement is that the frequency $\omega = \omega(y_0) \equiv h'(y_0)$ for a given $y_0 \in V$ satisfies a strong non–resonance assumption, namely a Diophantine condition of the form

$$|\omega \cdot m| \geq \nu |m|^{-\tau} \qquad \text{for all } m \in \mathbb{Z}^n \backslash \{0\} \ , \quad \nu > 0 \ , \ \tau > 0 \ . \tag{15}$$

The second requirement concerns a non–degeneracy of the unperturbed Hamiltonian $h = h(y)$, which is assumed to satisfy

$$\det(h'') \neq 0 \ , \tag{16}$$

where h'' denotes the Hessian matrix associated to h. We note that the Diophantine condition is needed to control the small divisors, while the non–degeneracy condition is required to ensure the dependence of the frequency on the actions. Under such assumptions, KAM theory provides the persistence of an invariant torus under perturbation as stated in the following theorem.

Theorem 1 (Conservative KAM theorem)**.** *Consider the nearly-integrable Hamiltonian* (1) *and let* ω *be the associated frequency vector. Assume that* ω *satisfies the Diophantine condition* (15) *and that the unperturbed Hamiltonian* h *satisfies the non–degeneracy condition* (16)*. Then, there exists* $\varepsilon_{KAM} > 0$ *such that for any* $\varepsilon \leq \varepsilon_{KAM}$*, there exists an invariant torus on which a quasi–periodic motion with frequency* ω *takes place.*

For the proof of the KAM theorem we refer the reader to [20], [1], [23] (see also [7] and references therein). The proof is based on a super–convergent iterative method a la Nash–Moser on suitable scales of Banach spaces. While classical perturbation theory removes the perturbation linearly, KAM theory is based on a quadratic Newton's method. As it is well known, the KAM proof is constructive and it is typically performed through a computer assisted implementation of the interval arithmetic technique ([21], [17]). As we already mentioned in Section 2, for low–dimensional systems KAM theorem provides a strong stability property in the sense of confinement in phase space. Such property has been used in concrete applications to Celestial Mechanics to prove, e.g., the stability of some asteroids ([10]) or the confinement of the spin–orbit dynamics ([8], [7]).

Interesting applications of KAM theory concern also discrete systems, like the celebrated standard–map ([16]), for which KAM applications provide results in very good agreement with the experimental values ([22], [9]).

5.2 Conformally symplectic KAM theorem

In this section we extend the results presented in Section 5.1 to some dissipative systems. Precisely, we consider *conformally symplectic* flows and maps, which are defined as follows.

An n–dimensional vector field X is said to be a conformally symplectic flow, if one can determine a function $\mu : \mathbb{R}^{2n} \to \mathbb{R}$ such that, denoting by Ω the symplectic form, we have:

$$L_X\Omega = \mu\Omega \ , \tag{17}$$

where L_X stands for the Lie derivative. Denoting by Φ_t the flow associated to X, if μ is constant, then by the Lie derivative theorem one has that

$$(\Phi_t)^*\Omega = e^{\mu t}\Omega \ ,$$

where the star denotes the pull–back. Similarly, a diffeomorphism f defined on a manifold $\mathcal{M}$ is said to be conformally symplectic, if there exists a function $\mu : \mathcal{M} \to \mathbb{R}$ such that

$$f^*\Omega = \mu\Omega \ .$$

where f^* denotes the pull–back of f. In the following we will consider a family of vector fields X_η (or diffeomoriphisms f_η) depending on a parameter $\eta \in \mathbb{R}$.

An example of a conformally symplectic system is provided by a vector field of type (3) that we extend to the non–autonomous case with a dissipation proportional to the velocity, say

$$\begin{aligned} \dot{y} &= -\varepsilon\frac{\partial f(x,t)}{\partial x} - \mu(y-\eta), \\ \dot{x} &= \omega(y) \ . \end{aligned} \tag{18}$$

A physical model which is described by equations (18) is the spin–orbit problem in Celestial Mechanics, modeling the rotation of a satellite, whose center of mass revolves around a planet on a Keplerian orbit with the spin–axis perpendicular to the orbital plane. In this case $\omega(y) = y$ denotes the spin frequency, ε represents the equatorial oblateness and $f = f(x,t)$ describes the potential associated to the rotational motion on the Keplerian orbit; as for the dissipative part, μ represents the dissipation constant, depending on the physical properties of the satellite, while η is a drift function which varies according to the orbital eccentricity of the Keplerian ellipse.

We immediately remark that for $\varepsilon = 0$ and $\mu \neq 0$, equations (18) admit the invariant torus $\mathcal{T}_0 \equiv \{y = \eta\} \times \{(\theta, t) \in \mathbb{T}^2\}$, which is a global attractor with frequency η. In fact, one can readily see that the solution of (18) for $\varepsilon = 0$ is given by

$$x(t) = x(0) + \eta t + \frac{1 - e^{-\mu t}}{\mu} \left(\dot{x}(0) - \eta\right) ,$$

where $(x(0), \dot{x}(0))$ denotes the initial position and velocity. Then the question is whether there exists an invariant attractor with frequency ω for the perturbed system with $\varepsilon \neq 0$. The answer is given by a KAM theorem developed for conformally symplectic systems ([6], see also [5]), provided two assumptions (extending those of the conservative case) are satisfied, namely a non–resonance condition on the frequency and a non–degeneracy of the vector field. We remark that in the dissipative case the non–degeneracy condition is more complicated than in the conservative case, since it amounts to require a non–degeneracy in the parameters, beside the condition on the Hessian of the unperturbed Hamiltonian associated to the conservative vector field (see [6] for details).

Theorem 2. (Conformally symplectic KAM theorem). *Let us consider a conformally symplectic non–degenerate vector field satisfying* (17) *and let ω be diophantine; then, for a suitable drift η there exists $\varepsilon_{KAM} > 0$ such that for any $\varepsilon \leq \varepsilon_{KAM}$ and $0 \leq \mu < 1$, there exists a quasi–periodic solution with frequency ω.*

In the perturbative case there exists an explicit relation between the frequency ω and the drift η; such compatibility condition shows that one cannot choose independently these quantities. For a vector field of the type (18) which is associated to the spin–orbit problem, such compatibility condition amounts to require that ω and η are related as $\eta = \omega\ (1 + O(\varepsilon^2))$. Since η is a function of the eccentricity, it means that the frequency of the invariant attractor is directly linked to the orbital eccentricity (see [11]).

6 Stability estimates for exponential times

As anticipated in Section 2, Nekhoroshev's theorem provides a powerful tool to give a bound on the action variables for exponentially long times. In this section we first provide results in the conservative setting (see Section 6.1) and then we analyze the dissipative context (see Section 6.2).

6.1 Nekhoroshev's theorem

The original version of Nekhoroshev's theorem was formulated under the so–called *steepness* condition ([24]), later relaxed to a convex or quasi–convex assumption on the unperturbed Hamiltonian (compare with [25]). More precisely, with reference to (1) we provide the following definitions and results.

Definition 1. The function $h(y)$ is said m–convex for some $m > 0$, if for any $y \in V$ the following inequality hold for any $v \in \mathbb{R}^n$:

$$h''(y)v \cdot v \geq m|v|^2 \ ,$$

where h'' denotes the Hessian matrix associated to h.

Definition 2. Given m, $\ell > 0$, $h(y)$ is said m, ℓ–quasi–convex, if for any $y \in V$ one of the following inequalities holds for any $v \in \mathbb{R}^n$:

$$|\omega(y) \cdot v| > \ell |v| \ , \qquad h''(y)v \cdot v \geq m|v|^2 \ .$$

Theorem 3. *(Nekhoroshev) Given a quasi–convex Hamiltonian function of the type* (1)*, there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \leq \varepsilon_0$ *one has*

$$|y(t) - y(0)| \leq C(\frac{\varepsilon}{\varepsilon_0})^a \qquad for \quad |t| \leq T_0 \ e^{(\frac{\varepsilon_0}{\varepsilon})^a}$$

for some positive constants C*,* T_0*,* a*.*

The proof of the theorem is based on three main ingredients, namely the construction of a normal form which removes the perturbation up to a suitable order, the use of the convexity or quasi–convexity assumption, a careful analysis of the geography of the resonances in order to cover the whole phase space.

The classical example to show that the quasi–convexity condition is necessary is provided by the 2–dimensional Hamiltonian function

$$\mathcal{H}(y_1, y_2, x_1, x_2) = \frac{y_1^2}{2} - \frac{y_2^2}{2} - \varepsilon \sin(x_1 + x_2) \ ;$$

the associated Hamilton's equations admit the solution

$$x_1(t) = -x_2(t) = x_0 + y_0 t + \frac{1}{2}\varepsilon t^2 \ , \qquad y_1(t) = y_2(t) = y_0 + \varepsilon t \ ,$$

for some initial conditions $(x_0, -x_0, y_0, y_0)$; this solution shows that the motion is not bounded and that Nekhoroshev's exponential estimates do not hold.

The proof of Nekhoroshev's theorem is straightforward in the case of a non–resonant frequency. More precisely, assume to start with an initial condition y_0 such that $\omega(y_0)$ satisfies a non–resonant condition up to some order $K > 0$:

$$|\omega(y_0) \cdot k| > 0 \qquad \text{for all } |k| \leq K \ .$$

Using classical perturbation theory, one constructs a normal form to some order N through a transformation from the original variables (y, x) to new variables

(y', x'); the change of coordinates is defined in terms of some functions $\Delta^{(N)}$, $\Phi^{(N)}$, say

$$\begin{aligned} y' &= y + \varepsilon \Delta^{(N)}(y, x) \\ x' &= x + \varepsilon \Phi^{(N)}(y, x) \ , \end{aligned}$$

such that the new Hamiltonian becomes

$$\mathcal{H}^{(N)}(y', x') = h^{(N)}(y') + \varepsilon^{N+1} f^{(N)}(y', x') \ . \tag{19}$$

From Hamilton's equations associated to (19) one obtains

$$\dot{y}' = -\varepsilon^{N+1} \frac{\partial f^{(N)}}{\partial x}(y', x') \ .$$

Then, denoting by F_N an upper bound on the norm of $\frac{\partial f^{(N)}}{\partial x}$, one obtains that the variation of the actions is given by

$$\begin{aligned} |y(t) - y(0)| &\le |y(t) - y'(t)| + |y'(t) - y'(0)| + |y(0) - y'(0)| \\ &\le 2\varepsilon\tilde{\rho} + \varepsilon^{N+1} F_N \ |t| \ , \end{aligned}$$

where $\tilde{\rho}$ is an upper bound on the norm of $\Delta^{(N)}$. Let $\rho \equiv 3\varepsilon\tilde{\rho}$; if

$$|t| \le \frac{\tilde{\rho}}{\varepsilon^N F_N} \ ,$$

then we obtain the bound

$$|y(t) - y(0)| \le \rho \ .$$

To get estimates for exponential times, let $\tau > 0$ and fix the order N of the normal form as $N = [\frac{K\tau}{|\log \varepsilon|}]$ (where $[\cdot]$ denotes the integer part), so that

$$\varepsilon^{-N} = e^{K\tau} \ ;$$

then the stability time becomes

$$|t| \le \frac{\tilde{\rho}}{F_N} e^{K\tau} \ .$$

In conclusion, in the non–resonant regime we obtain the confinement of the actions for exponential times, by properly choosing the order N of normalization, namely

$$|y(t) - y(0)| \le \rho \quad \text{for} \quad |t| \le \frac{\tilde{\rho}}{F_N} \ e^{K\tau}$$

provided

$$N = [\frac{K\tau}{|\log\varepsilon|}] \ .$$

Notice that in the non–resonant case we did not require the convexity or quasi–convexity assumption, which is an essential hypothesis when dealing with a resonant frequency (see [24], [25]).

The stability estimates provided by Nekhoroshev's theorem are particularly relevant in Celestial Mechanics. In fact, they can be used to provide bounds on the elliptic elements for an exponentially long time, possibly comparable with the age of the solar system, namely 5 billion years. In Celestial Mechanics, effective estimates have been developed for several models, like the three–body problem, the triangular Lagrangian points, the resonant D'Alembert problem and the perturbed Euler rigid body (see, e.g., [3], [4], [12], [13], [18], [19]).

6.2 Exponential estimates for dissipative systems

Let us now consider a conformally symplectic (dissipative) n–dimensional, time–dependent vector field of the form

$$\begin{aligned} \dot{x} &= \omega(y) + \varepsilon\frac{\partial f(y,x,t)}{\partial y} + \mu r(y,x,t) \\ \dot{y} &= -\varepsilon\frac{\partial f(y,x,t)}{\partial x} + \mu\left(g(y,x,t) - \eta(y,x,t)\right) \ , \end{aligned} \tag{20}$$

where $y \in \mathbb{R}^n$, $(x,t) \in \mathbb{T}^{n+1}$, $\varepsilon \in \mathbb{R}_+$, $\mu \in \mathbb{R}_+$, ω and η are real–analytic functions, f, g, r are periodic, real–analytic functions. Having fixed the initial condition $y(0) = y_0 \in \mathbb{R}^n$, we denote by $V \subset \mathbb{R}^n$ an open neighborhood of y_0. Assume that for some $K \in \mathbb{Z}_+$ the vector function $\omega = \omega(y)$ satisfies the *non–resonance* condition

$$|\omega(y)\cdot k + m| > 0 \quad \text{for all } y \in V \ , \quad (k,m) \in \mathbb{Z}^{n+1}\backslash\{0\} \ , |k| + |m| \leq K \ . \tag{21}$$

Then, we can state the following stability result (see [14], [15] for complete details).

Theorem 4. *Consider the vector field* (20) *defined on* $V \times \mathbb{T}^{n+1}$ *and let* $y \in V$ *be such that* $\omega = \omega(y)$ *satisfies* (21)*. Let* $\rho_0 > 0$, $\tau_0 > 0$*; there exist* $\varepsilon_0 > 0$, $\mu_0 > 0$*, such that for any* $\varepsilon \leq \varepsilon_0$, $\mu \leq \mu_0$ *and for a suitable drift* η *, one has:*

$$|y(t) - y(0)| \leq \rho_0 \qquad \textit{for} \quad |t| \leq C\ e^{K\tau_0} \ ,$$

for some positive constant C*.*

Like in the conservative case, the proof is based on the construction, up to an optimal order N, of a double coordinate change of variables, say

$$(X,Y) = \Xi_d^{(N)} \Xi_c^{(N)}(x,y) \ ,$$

where we refer to $\Xi_c^{(N)}$ as the conservative transformation and to $\Xi_d^{(N)}$ as the dissipative transformation. The first transformation implements a classical normal form to remove the conservative terms to the N–th order; the change of variables generated by $\Xi_d^{(N)}$ removes the dissipative terms up to the order N. In conclusion, one obtains a normal form described by the equations

$$\begin{aligned} \dot X &= \Omega^{(N)}(Y) + O_1^{>K}(\varepsilon,\mu) + O_{N+1}(\varepsilon,\mu) \\ \dot Y &= O_1^{>K}(\varepsilon,\mu) + O_{N+1}(\varepsilon,\mu) \ , \end{aligned} \tag{22}$$

where $\Omega^{(N)}$ is the new frequency of motion, $O_1^{>K}(\varepsilon,\mu)$ denotes terms of first order in the parameters, but with Fourier index greater than K, $O_{N+1}(\varepsilon,\mu)$ denotes terms of order $N+1$ in the parameters. We remark that $\Xi_c^{(N)}$ and $\Xi_d^{(N)}$ can be constructed by solving suitable normal form equations; in the conservative case such equations looks similar to (10) (extended to the non–autonomous case) and their solution is found provided the frequency satisfies the non–resonant assumption. In the dissipative case the normal form equation is again similar to (10), but it involves also the drift η, which must be suitably chosen in order to fulfill some compatibility conditions.

The stability estimates are found like in the conservative case; more precisely, let $\lambda = \max(\varepsilon,\mu)$ and choose N such that $\lambda^N = e^{-K\tau_0}$ for some $\tau_0 > 0$. A bound on the diffusion of the normal form variables can be obtained from (22) as

$$|Y(t) - Y(0)| \leq |\dot Y| \ t \leq \big(|O_1^{>K}(\varepsilon,\mu)| + |O_{N+1}(\varepsilon,\mu)|\big) \ t \leq C_1 e^{-K\tau_0} \ t \ ,$$

for some constant $C_1 > 0$, due to the decay of the Fourier coefficients and to the choice of the normal form order. A bound on the original variables is obtained by estimating the deformation and the diffusion as follows, where $C_2\lambda$ denotes a bound on the transformation with $C_2 > 0$:

$$\begin{aligned} |y(t) - y(0)| &\leq |y(t) - Y(t)| + |Y(t) - Y(0)| + |y(0) - Y(0)| \\ &\leq 2C_2\lambda + C_1 e^{-K\tau_0} \ t \\ &\leq C_3\lambda \qquad \text{for } t \leq C_4 e^{K\tau_0} \ , \end{aligned}$$

for some constants $C_3 > 0$, $C_4 > 0$. Choosing properly τ_0 such that $K\tau_0 \leq (\frac{1}{\lambda})^c$ for $c > 0$, one gets a bound on the actions for exponential times, say

$$t \leq C_4 e^{(\frac{1}{\lambda})^c} \ .$$

We conclude by mentioning that the proof can also be extended to the resonant case; we refer for details to [14], [15].

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