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Domains of $\mathbb{R}$-analytic existence in a real separable quojection

Murielle Mauer
Institut de Mathématique, Université de Liège
Grande Traverse, 12, Sart-Tilman, Bàtiment B37, B-4000 Liège 1 Belgium

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Abstract. We prove that if E is a real separable quojection, a non void domain Ω of E is a domain of $\mathbb{R}$-analytic existence if and only if Ω is open for a continuous semi-norm. We also prove that in a real separable Fréchet space, every non void domain is a domain of $\mathbb{R}$-analyticity if and only if E has a continuous norm.

Keywords: real-analytic function, domain of analyticity, domain of analytic existence, quojection.

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1 Introduction and statement of the results

Let E be a Hausdorff locally convex space.

Definition 1. Let Ω be an open subset of E. A function $f : \Omega \to \mathbb{R}$ is *analytic on* Ω if for every $x_0 \in \Omega$, there is a sequence $(P_k)_{k\in\mathbb{N}_0}$ of continuous k-homogeneous polynomials from E into $\mathbb{R}$ such that the following expansion

$$f(x) = \sum_{k\in\mathbb{N}_0} P_k(x - x_0)$$

holds on a neighbourhood of x_0.

We denote by $A(\Omega)$ the set of the analytic functions on Ω.

Definition 2. A *domain of analyticity in* E is a non void domain Ω of E such that, for every domain Ω_1 of E verifying $\Omega_1 \not\subset \Omega \not\subset E \setminus \Omega_1$ and for every connected component Ω_0 of $\Omega \cap \Omega_1$, there is $f \in A(\Omega)$ such that $f|_{\Omega_0}$ has no analytic extension onto Ω_1.

Definition 3. A *domain of analytic existence in* E is a non void domain Ω of E for which there is $f \in A(\Omega)$ such that, for every domain Ω_1 of E verifying $\Omega_1 \not\subset \Omega \not\subset E \setminus \Omega_1$ and every connected component Ω_0 of $\Omega \cap \Omega_1$, $f|_{\Omega_0}$ has no analytic extension onto Ω_1.

It is clear that every domain of analytic existence is a domain of analyticity.

In [7], J. Schmets and M. Valdivia solved the characterization of the domains of $\mathbb{R}$-analytic existence in a real separable normed space as follows.

Theorem 1. *For every non void domain Ω of a separable real normed space E, there is a C_∞-function f on E which is $\mathbb{R}$-analytic on Ω and has Ω as domain of $\mathbb{R}$-analytic existence. In particular, every non void domain of a separable real normed space is a domain of $\mathbb{R}$-analytic existence.*

In addition, they gave an example showing that for this result, the hypothesis of separability on the real normed space is really needed.

Example 1. If A is an uncountable set, then the open unit ball of $c_{0,\mathbb{R}}(A)$ is a domain of $\mathbb{R}$-analyticity but not a domain of $\mathbb{R}$-analytic existence.

We are interested in the characterisation of domains of $\mathbb{R}$-analyticity and of $\mathbb{R}$-analytic existence in more general real spaces than in normed spaces. In [6], we have considered the case of a real separable Fréchet space and proved the following proposition:

Proposition 1. *If E is a real separable Fréchet space, every non void domain Ω of E which is open for a continuous semi-norm p is a domain of $\mathbb{R}$-analytic existence. In fact, there is a function f which is C_∞ on E for the semi-norm p, $\mathbb{R}$-analytic on Ω for the semi-norm p and which has Ω as domain of $\mathbb{R}$-analytic existence.*

In this paper, we characterize domains of $\mathbb{R}$-analytic existence first in an arbitrary product of real separable Banach spaces, next in a real separable quojection. We prove that in such cases, the converse of the proposition 1 is valid. We also get a natural necessary and sufficient condition for a real separable Fréchet space to have every non void domain as domain of $\mathbb{R}$-analyticity. Here are the results.

Proposition 2. *In $E = \prod_{j \in J} E_j$ where E_j is a real separable Banach space for every $j \in J$, a non void domain Ω is a domain of $\mathbb{R}$-analytic existence if and only if Ω is open for a continuous semi-norm that is to say if and only if there are a finite subset A of J and a non void domain U of $\prod_{j \in A} E_j$ such that $\Omega = \{(e_j)_{j \in J} \in \prod_{j \in J} E_j : (e_j)_{j \in A} \in U\}$.*

Let us notice that this proposition generalises the theorem 5 in [5]. Actually, in [5], A. Hirschowitz had already got the result of the proposition 2 in the case of the space $\omega = \mathbb{R}^{\mathbb{N}}$. A different proof for the case of the space ω is also given in [6].

Proposition 3. *If E is a real separable quojection, a non void domain Ω of E is a domain of $\mathbb{R}$-analytic existence if and only if Ω is open for a continuous semi-norm.*

Proposition 4. *If E is a real separable Fréchet space, every non void domain Ω of E is a domain of $\mathbb{R}$-analyticity if and only if E has a continuous norm.*

2 About analytic functions on a Baire space

Let E be a Hausdorff locally convex space. Let us consider the following two definitions.

Definition 4. Let p be a continuous semi-norm on E and Ω an open subset of E for p. A function $f : \Omega \to \mathbb{R}$ is *analytic on* Ω *for* p if for every $x_0 \in \Omega$, there are $r > 0$ and a sequence $(P_k)_{k\in\mathbb{N}_0}$ of k-homogeneous polynomials from E to $\mathbb{R}$ which are continuous for p and such that the equality

$$f(x) = \sum_{k=0}^{+\infty} P_k(x - x_0)$$

holds for every $x \in b_p(x_0, \leq r)$.

We denote by $A_p(\Omega)$ the set of the analytic functions on Ω for p.

Definition 5. Let Ω be an open subset of E. A function $f : \Omega \to \mathbb{R}$ is *locally analytic on* Ω *for a semi-norm* if for every $x_0 \in \Omega$, there are $p \in \mathrm{cs}(E)$ and $r > 0$ such that $f \in A_p(b_p(x_0, < r))$.

Of course, every function which is locally analytic on Ω for a semi-norm is analytic on Ω.

We will need later on the following result which is proved in [6].

Proposition 5. *If* E *is a complex Baire space or a real Hausdorff locally convex space such that its complexification* Z_E *is a Baire space and if* Ω *is an open subset of* E *then a function* $f : \Omega \to \mathbb{R}$ *is analytic on* Ω *if and only if it is locally analytic on* Ω *for a semi-norm.*

3 Domains of ℝ-analytic existence in a product of real separable Banach spaces

We are going to prove the proposition 2.

First, it is quite easy to prove that if there are a finite subset A of J and a non void domain U of $\prod_{j\in A} E_j$ such that

$$\Omega = \{(e_j)_{j\in J} \in \prod_{j\in J} E_j : (e_j)_{j\in A} \in U\}$$

then Ω is a domain of $\mathbb{R}$-analytic existence. In fact, U is a non void domain in a real separable Banach space and by the theorem 1, there is $f \in A(U)$ which has U as domain of $\mathbb{R}$-analytic existence. It is then a direct matter to verify that the function

$$g : \Omega \to \mathbb{R} \quad (e_j)_{j\in J} \mapsto f((e_j)_{j\in A})$$

is $\mathbb{R}$-analytic on Ω and has Ω as domain of $\mathbb{R}$-analytic existence.

To prove the converse, we need the following lemma.

Lemma 1. *If $E = \prod_{j\in J} E_j$ is a product of real or complex Banach spaces and Ω a non void domain of E then for each analytic function f on Ω, there exists a finite subset A of J such that for every $x_0 \in \Omega$, there is a sequence $(P_k)_{k\in\mathbb{N}_0}$ of continuous k-homogeneous polynomials from E into $\mathbb{R}$ such that the following expansion*

$$f(x) = \sum_{k\in\mathbb{N}_0} P_k\left(\sum_{j\in A} \varepsilon_j(x_j - x_{0,j})\right)$$

with $(\varepsilon_j)_k = \delta_{jk}$ for every $j,k \in J$ holds on a neighbourhood of x_0.

Proof. Let us prove the result for the complex case. The real case will follow directly from the fact that if E is a real locally convex space such that its complexification Z_E is a Baire space then for any $\mathbb{R}$-analytic function $f : \Omega \to \mathbb{R}$, one may find an open subset $\widetilde{\Omega}$ of Z_E and an analytic function $\tilde{f} : \widetilde{\Omega} \to \mathbb{C}$ such that $\Omega \subset \widetilde{\Omega}$ and $\tilde{f}|_\Omega = f$ (cf. Theorem 7.1, p. 103 in [1]).

Let f be an analytic function on Ω. Since $E = \prod_{j\in J} E_j$ is a complex Baire space, by using the proposition 5, one gets that f is locally analytic on Ω for a semi-norm. Let us take a point $a \in \Omega$. There are then $p \in \mathrm{cs}(E)$ and $r > 0$ such that $f \in A_p(b_p(a, < r))$ and in particular, f is continuous for p on $b_p(a, < r)$. Let us denote by $||.||_j$ the norm of E_j for every $j \in J$ and let us say that p is the semi-norm

$$p(e) = \sum_{j\in A} ||e_j||_j, \quad e \in E$$

where A is a finite subset of J. Therefore, the function f depends on the components $(e_j)_{j\in A}$ on $b_p(a, < r)$.

Next, for every $x_0 \in \Omega$ and for every $k \in \mathbb{N}$, we denote by $(c_k)_{x_0}$ the continuous symmetric k-linear mapping from E^k into $\mathbb{C}$ which generates the continuous k-homogeneous polynomial $\hat{D}^k_{x_0} f$. For every $k \in \mathbb{N}$ and for every $x^{(1)}, \ldots, x^{(k)} \in E$, one has

$$(c_k)_{x_0}(x^{(1)}, \ldots, x^{(k)}) = D_{\xi_1} \cdots D_{\xi_k} f(x_0 + \xi_1 x^{(1)} + \cdots + \xi_k x^{(k)})|_{(\xi_1,\ldots,\xi_k)=(0,\ldots,0)}.$$

By using this expression, it is easy to see that if we fix $x^{(1)}, \ldots, x^{(k)} \in E$ such that one of the $x^{(j)}$'s is such that $(x^{(j)})_k = 0$ for every $k \in A$, one has

$$(c_k)_{x_0}(x^{(1)}, \ldots, x^{(k)}) = 0 \quad \forall\, x_0 \in b_p(a, < r).$$

In addition, the function

$$x_0 \in \Omega \mapsto (c_k)_{x_0}(x^{(1)}, \ldots, x^{(k)})$$

is analytic on Ω. And since it is vanishing on a non void open subset $b_p(a, < r)$ of the connected open subset Ω, it is vanishing everywhere on Ω. Consequently, for every $x_0 \in \Omega$, one has

$$f(x) = \sum_{k \in \mathbb{N}_0} \frac{\hat{D}^k_{x_0} f}{k!} \left(\sum_{j \in A} \varepsilon_j (x_j - x_{0,j}) \right)$$

on a neighbourhood of x_0. QED

Finally, let us prove that if a non void domain Ω of a product of real Banach spaces is not open for a continuous semi-norm then Ω is not a domain of $\mathbb{R}$-analytic existence.

Let f be a $\mathbb{R}$-analytic function on Ω. For this function f, let us consider the finite subset A of J given by the lemma 1 and the continuous semi-norm $p(e) = \sum_{j \in A} ||e_j||_j$. The domain Ω is not open for p. Therefore, there is a point $x_0 \in \Omega$ such that for every $r > 0$, the semi-ball $b_p(x_0, < r)$ is not included in Ω. We consider next the expansion of f at x_0. There are $r > 0$ and $q \in cs(E)$ such that

$$f(x) = \sum_{k \in \mathbb{N}_0} P_k \left(\sum_{j \in A} \varepsilon_j (x_j - x_{0,j}) \right)$$

on $b_q(x_0, < r)$. One may assume $p < q$ or $p = q$.

Let us take $\Omega_1 = b_p(x_0, < r)$ and Ω_0 the connected component of x_0 in $\Omega \cap \Omega_1$. The function g defined by

$$g(x) = \sum_{k \in \mathbb{N}_0} P_k \left(\sum_{j \in A} \varepsilon_j (x_j - x_{0,j}) \right)$$

is $\mathbb{R}$-analytic on $\Omega_1 = b_p(x_0, < r)$ and is equal to f on Ω_0. In fact, f and g are two $\mathbb{R}$-analytic functions on a connected open subset which are equal on $b_q(x_0, < r)$ a non void open subset of Ω_0. Hence the conclusion.

4 Domains of ℝ-analytic existence in a real separable quojection

Let E be a real separable quojection. We are going to prove the proposition 3. By the proposition 1, every non void domain Ω of E which is open for a continuous semi-norm is a domain of $\mathbb{R}$-analytic existence. To prove the converse, we will need the lemma 1 and the following result proved by J. Bonet, M. Maestre, G. Metafune, V. B. Moscatelli and D. Vogt in [3].

Theorem 2. *If E is a quojection which is not a Banach space, there exists an index set I such that E is isomorphic to a quotient of $\ell_1(I)^{\mathbb{N}}$.*

For the proof of the proposition 3, we may assume that E is not a Banach space. Otherwise, it is the theorem 1 of Schmets and Valdivia. There is then I such that E is isomorphic to a quotient of $F = \ell_1(I)^{\mathbb{N}}$.

Let Ω be a non void domain Ω of E which is not open for a continuous semi-norm and f a $\mathbb{R}$-analytic function on Ω. Let us denote by s the canonical quotient mapping from F to E. The open subset $U = s^{-1}(\Omega)$ of F is connected (cf. Theorem 6.1.28, p. 440 in [4]) and the function $g = f \circ s$ is $\mathbb{R}$-analytic on U. Therefore, by using the lemma 1, there exists a finite subset A of J such that for every $y_0 \in U$, the continuous k-homogeneous polynomials $(Q_k)_{k\in\mathbb{N}}$ from F to $\mathbb{R}$ of the Taylor series expansion of g at y_0 depend only on the components related to indexes in A.

Moreover, for $x_0 \in \Omega$, there is $y_0 \in U$ such that $x_0 = s(y_0)$ and if we denote by $(P_k)_{k\in\mathbb{N}_0}$ the polynomials from E into $\mathbb{R}$ of the Taylor series expansion of f at x_0 and by $(Q_k)_{k\in\mathbb{N}_0}$ the polynomials from F into $\mathbb{R}$ of the Taylor series expansion of g at y_0, one has $Q_k = P_k \circ s$ for every $k \in \mathbb{N}_0$.

Next, we consider the continuous semi-norm p on F given by

$$p(y) = \sum_{j\in A} ||y_j||_{\ell_1(I)}, \quad \forall\, y \in F$$

and the continuous semi-norm $\tilde{p}$ on E given by

$$\tilde{p}(s(y)) = \inf_{h\in\ker(s)} p(y+h), \quad \forall\, y \in F.$$

The open subset Ω of E is not open for $\tilde{p}$. We then fix $x_0 \in \Omega$ such that for every $r > 0$, $b_{\tilde{p}}(x_0, < r)$ is not contained in Ω and we consider the expansion of f at x_0. One has

$$f(x) = \sum_{k\in\mathbb{N}_0} P_k(x - x_0)$$

on a neighbourhood of x_0. If $y_0 \in U$ is such that $s(y_0) = x_0$ and if $(Q_k)_{k\in\mathbb{N}}$ are the polynomials of the expansion of g at y_0, since one has $Q_k = P_k \circ s$ for every $k \in \mathbb{N}$, it is easy to see that there is $r > 0$ such that the series $\sum_{k\in\mathbb{N}_0} P_k(x - x_0)$ converges on $s(b_p(y_0, < r))$. In addition, one gets directly $b_{\tilde{p}}(x_0, < r) \subset s(b_p(y_0, < r))$. Therefore, the function g defined by

$$g(x) = \sum_{k\in\mathbb{N}_0} P_k(x - x_0)$$

is $\mathbb{R}$-analytic on $b_{\tilde{p}}(x_0, < r)$. To conclude, we take $\Omega_1 = b_{\tilde{p}}(x_0, < r) \not\subset \Omega$ and Ω_0 the connected component of x_0 in $\Omega \cap \Omega_1$ and we just notice that g is a $\mathbb{R}$-analytic extension of $f|_{\Omega_0}$ onto Ω_1.

5 Characterisation of real separable Fréchet spaces where every domains are domains of ℝ-analyticity

We are going to prove the proposition 4. First, let us assume that E is a real separable Fréchet space which has a continuous norm $||.||$. Then the polar $b_{||.||}(1)^{\triangle}$ of the open unit ball is $\sigma(E', E)$-separable and there is a countable subset $\{\omega_n : n \in \mathbb{N}\}$ which is $\sigma(E', E)$-dense in $b_{||.||}(1)^{\triangle}$. We consider the function φ on E defined by

$$\varphi : E \to \mathbb{R} \quad e \mapsto \varphi(e) = \sum_{n=1}^{+\infty} \frac{\langle e\,,\,\omega_n\rangle^2}{n!}.$$

The function φ is ℝ-analytic on E (it is in fact a continuous homogeneous polynomial of degree 2) and is vanishing only at 0. Therefore, the function $1/\varphi$ is ℝ-analytic on $E \setminus \{0\}$ and has no ℝ-analytic extension onto E. It is enough to prove that every non void domain Ω of E is a domain of ℝ-analyticity.

Let us assume now that E is a real separable Fréchet space without any continuous norm. We are going to prove that $\Omega = E \setminus \{0\}$ is not a domain of ℝ-analyticity. It is the same than to prove that $\Omega = E \setminus \{0\}$ is not a domain of ℝ-analytic existence. We are going to prove that every ℝ-analytic function on $\Omega = E \setminus \{0\}$ has a ℝ-analytic extension onto the whole space E.

It is well known then that if E is a Fréchet space without any continuous norm, E contains the space $\omega = \mathbb{R}^{\mathbb{N}}$ (see theorem 2.6.13, pg 71 in [2]). In fact, there exists a continuous linear projector P from E into E such that $\mathrm{im}(P)$ is isomorphic to ω. Therefore, E is isomorphic to the non finite product $\ker(P) \times \omega$.

By using then the result 2 about products, if $E \setminus \{0\}$ is a domain of ℝ-analytic existence, there must exist some $n \in \mathbb{N}$ and a non void domain U of $\ker(P) \times \mathbb{R}^n$ such that

$$E \setminus \{0\} = U \times \omega.$$

Since it is not the case, $E \setminus \{0\}$ is not a domain of ℝ-analyticity. Hence the conclusion.

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