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Four dimensional symplectic geometry over the field with three elements and a moduli space of Abelian surfaces

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Abstract. We study certain combinatorial structures related to the simple group of order 25920. Our viewpoint is to regard this group as $G = P\mathbf{Sp}(4, \mathbf{F}_3)$, and so we describe these configurations in terms of the symplectic geometry of the four dimensional space over the field with three elements. Because of the isogeny between $\mathbf{SO}(5)$ and $\mathbf{Sp}(4)$ we can also describe these in terms of an inner product space of dimension five over that same field. The study of these configurations goes back to the 19^{th}-century, and we relate our work to that of previous authors. We also discuss a more modern connection: these configurations arise in the theory of the Igusa compactification of the moduli space of principally polarized Abelian surfaces with a level three structure.

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1. Introduction

Let V be a four-dimensional vector space over $\mathbf{F}_3$, the field with three elements, equipped with the standard symplectic form

$$\langle v,\ w\rangle \ = \ v\, J\, {}^t w \quad \text{where } J \ = \ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{GL}(4, \mathbf{F}_3)$ act on V on the left by $g(v) = v.g^{-1}$. We obtain as isometry group of this form the symplectic group

$$\mathbf{Sp}(V) \ = \ \mathbf{Sp}(4, \mathbf{F}_3) \ = \ \{g \in \mathbf{GL}(4, \mathbf{F}_3) \mid g\, J\, {}^t g = J\}$$

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We let $G = P\mathbf{Sp}(V) = P\mathbf{Sp}(4, \mathbf{F}_3) = \mathbf{Sp}(4, \mathbf{F}_3)/\pm 1$. G is the unique simple group of order 25920. This group was already studied in the 19$^{\text{th}}$ century. Camille Jordan devoted a whole chapter of his *Traité des Substitutions* [22] to it. He knew that this group appeared in the symplectic form as we have described it above, and he also knew that it is a subgroup of index 2 inside the symmetry group of the configuration of 27 lines on a cubic surface. Since that time, a number of mathematicians, notably, Baker, Burkhardt, Coble, Coxeter, Dickson, Edge, Todd, have studied this group from a number of points of view. We became interested in this group in its role as the automorphism group of $\mathcal{A}_2(3)^*$, the Igusa compactification of the Siegel modular variety of degree two and level three, parametrizing principally polarized abelian surfaces with a level three structure [17]. A key point in our investigations was the identification of $\mathcal{A}_2(3)^*$ with $\tilde{B}$, the desingularization of Burkhardt's quartic B. See [20] for a careful and thorough study of this, relating the classical and modern viewpoints.

Various important subvarieties of this moduli space are naturally indexed by certain configurations in the 4-dimensional symplectic space V over $\mathbf{F}_3$. The configurations of special interest here are those whose stabilizer subgroups are maximal subgroups of G. These subgroups have index respectively 27, 36, 40, 40, 45. A glance at the entry of the Atlas of finite simple groups [7] corresponding to G shows a simple description, in the language of the symplectic vector space V, of these maximal subgroups for the indices 40, 40, 45. Part of the purpose of this paper is to give a description, in terms of the symplectic geometry of V, of the maximal subgroups of index 27 and 36. The key notion for this is the concept of a spread of nonsingular pairs, or nsp-spread, and the related concept of a double-six. We introduce these in section 2. of this paper. It turns out that there are 27 nsp-spreads and 36 double-sixes, and that these are the configurations whose stabilizer subgroups are the subgroups of index 27 and 36 respectively.

Both these concepts were discovered by the second author of this paper over a decade ago. However, the notion of an nsp-spread actually appeared in a not so well known paper by Coble [6]. In fact, lemma 1, and theorems 1 and 2 were known to A. B. Coble in 1908.

One of the goals of this paper is to explain some of the links between different descriptions given by various authors of these combinatorial structures, and to relate them to the geometry of the moduli space. In section 3. of this paper we introduce a combinatorial structure, the Tits building with scaffolding [19], derived from the symplectic description of G, and show how to interpret the geometry of our variety in terms of this structure, from three geometric descriptions—those given by Burkhardt, Baker and ourselves.

With this in hand, we return to the study of these configurations in section 4..

Our viewpoint enables us to clarify some of Coble's work, as well as that of L. E. Dickson [9].

In sections 5., 6. and 7. we carry out a more structural analysis of the maximal subgroups of G. The main result is that they are all of the form $H(\mathbf{F}_3)/\pm 1$ for algebraic subgroups $H \subset \mathbf{Sp}(4)$ defined over $\mathbf{F}_3$. In section 6. we discuss a Galois-twisted group that turns out to be the stabilizer of a double-six. In section 7. we reinterpret these results in the language of an inner product space of dimension 5 over $\mathbf{F}_3$, relating our work to that of Edge [10], [11]. We conclude with some remarks on the Weyl group of E_6 (which contains G as a subgroup of index 2.) Finally, we present a number of tables which have aided us in studying these structures. These tables may be of use to others wishing to study this group and its related combinatorial structures.

In this paper we do not give an encyclopedic account of this group G. There are many aspects hardly touched upon – for instance, characteristic 2 descriptions of G, or relations to the theory of polytopes (see [8, 12]). Even from the perspective taken here, that of symplectic geometry over $\mathbf{F}_3^4$, we have not analyzed every configuration studied previously, only those that seem most relevant to the geometry of the moduli space. In view of the importance of these configurations in the geometry of moduli spaces of abelian surfaces, and of their intrinsic beauty, we hope that this work elucidates some of these structures.

One caveat about notation. It is a standard practice to denote by PG the group of projective transformations induced by a group G of linear transformations. However, if G is now an algebraic group, and if $\mathbf{P}G$ denotes the algebraic subgroup of $\mathbf{PGL}(N)$ generated by G, it is not true in general that $PG(R) = \mathbf{P}G(R)$. This fails in fact for $\mathbf{Sp}(2g)$ (see the discussion in section 7.). We will use $\mathbf{PSp}(2g)$ denote the algebraic group which is the quotient $\mathbf{Sp}(2g)/\mu_2$. The symbol PG, with an unbold P, will denote the functor $G(R)/(\text{the scalars in } G(R))$.

It is a pleasure to thank James Hirschfeld for his insightful comments on an earlier version of this work, and in particular for bringing the work of Coble and Edge to our attention. For the most complete general reference concerning geometries over finite fields, consult his trilogy: [14, 15, 16]. To connect our paper with his books, note that a symplectic form is called a null polarity there.

2. Nsp-Spreads and related structures

Let $V = (\mathbf{F}_3)^4$. We let l denote any line in V through the origin, and note that l is specified by $v \in V - \{0\}$, well-defined up to sign. In this case we simply write $l = v$, where the ambiguity in v is understood. We thus observe that $\{l\} = (V - \{0\}) / \pm 1$ has cardinality 40. This set may be canonically identified with a finite projective space:

$$\{l\} \;=\; \mathbf{P}\,(V) \;=\; \mathbf{P}^3\,(\mathbf{F}_3)\,.$$

We let h denote any plane (through the origin) in V that is isotropic under $\langle\ ,\ \rangle$. Thus $\langle v,\ w\rangle = 0$ for any two $v, w \in h$. Note that h may be identified with a certain type of line in $\mathbf{P}^3\,(\mathbf{F}_3)$. Each h contains four l's, and may be identified with this set. If l_1, l_2 generate the plane h, we write $h = l_1 \wedge l_2$. We let δ denote a plane in V that is nonsingular under the symplectic form. So there exists $v, w \in \delta$ such that $\langle v,\ w\rangle \neq 0$. As before we write $\delta = l_1 \wedge l_2$ for a generating set. δ determines its orthogonal complement $\delta^{\perp}$ with respect to the symplectic form, and we have $V = \delta \oplus \delta^{\perp}$. We let Δ be the unordered pair $\{\delta, \delta^{\perp}\}$. We can identify Δ with the set of eight l's that it contains. Δ is called a nonsingular pair.

It is easy to check that every line l is contained in 4 isotropic planes and 9 nonsingular pairs, so $\{h\}$ has cardinality $40 \cdot 4/4 = 40$ and $\{\Delta\}$ has cardinality $40 \cdot 9/8 = 45$. We number the latter as $\Delta^1,\ \ldots,\ \Delta^{45}$ and give these in table I. (In the table, we abbreviate $v = (v_1,\ v_2,\ v_3,\ v_4)$ by $v_1\,v_2\,v_3\,v_4$; since $v_i \in \{0,\ 1,\ 2\}$ this is unambiguous.) It is straightforward to check that G acts transitively on $\{l\}$, $\{h\}$, and $\{\Delta\}$.

Definition 1. A spread of nonsingular pairs, or nsp-spread, is a set $\sigma = \{\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4,\ \Delta_5\}$ of nonsingular pairs with the property that for every l there exists an i such that $l \in \Delta_i$.

Remark 1. Since there are 8 l's in each Δ, and there are 40 l's total, $\{\Delta_1,\ \ldots,\ \Delta_5\}$ is an nsp-spread if and only if $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$.

It is a nontrivial fact that nsp-spreads exist; that they do is at the heart of our analysis.

Lemma 1. *Let Δ_1 and Δ_2 be any two nonsingular pairs with $\Delta_1 \cap \Delta_2 = \emptyset$. Then there exists a unique nsp-spread containing them: $\{\Delta_1,\ \ldots,\ \Delta_5\}$.*

Proof. It clearly suffices to prove this for a particular choice of Δ_1, so let $\Delta_1 = \Delta^1$. Then we must have

$$\Delta_2 = \Delta^i \ \text{ for } \ i \in \{26, 27, 28, 29, 30, 37, 38, 39, 42, 43, 44, 45\}.$$

Each of these choices of i yields a unique nsp-spread, the nsp-spreads being

$$\begin{array}{l} \{\Delta^1,\ \Delta^{26},\ \Delta^{37},\ \Delta^{44},\ \Delta^{45}\} \\ \{\Delta^1,\ \Delta^{27},\ \Delta^{30},\ \Delta^{38},\ \Delta^{42}\} \\ \{\Delta^1,\ \Delta^{28},\ \Delta^{29},\ \Delta^{39},\ \Delta^{43}\} \end{array}$$

$\boxed{QED}$

Theorem 1. *There are 27 nsp-spreads.*

Proof. Each nsp-spread contains 5 nonsingular pairs. As the proof of lemma 1 shows, each nonsingular pair is contained in 3 nsp-spreads. Since there are 45 nonsingular pairs, there are $45 \cdot 3/5 = 27$ nsp-spreads. $\boxed{QED}$

We number the nsp-spreads $\sigma^j, j = 1, \ldots, 27$, and give them in Table II. It is straightforward to check that G acts transitively on $\{\sigma\}$.

Definition 2. A doublet $\{\sigma_1, \sigma_2\}$ is a disjoint pair of nsp-spreads.

Definition 3. A double-six is a set consisting of two ordered collections of six nsp-spreads

$$\{(\sigma_{1,1},\ \sigma_{1,2},\ \sigma_{1,3},\ \sigma_{1,4},\ \sigma_{1,5},\ \sigma_{1,6}),\ (\sigma_{2,1},\ \sigma_{2,2},\ \sigma_{2,3},\ \sigma_{2,4},\ \sigma_{2,5},\ \sigma_{2,6})\}$$

such that

$$\sigma_{i_1,j_1} \cap \sigma_{i_2,j_2} \ = \begin{cases} \text{a single nonsingular pair} & \text{if } i_1 \neq i_2 \text{ and } \ j_1 \neq j_2 \\ \emptyset & \text{otherwise} \end{cases}$$

In identifying double-sixes $\{(\sigma_{1,i}), (\sigma_{2,i})\}$, we are free to interchange $(\sigma_{1,i})$ with $(\sigma_{2,i})$ and to perform the same permutation of six objects to $(\sigma_{1,i})$ and $(\sigma_{2,i})$. There are six doublets in a double-six, namely the $\{\sigma_{1,i}, \sigma_{2,i}\}$ for $i = 1, \ldots, 6$.

It is a nontrivial fact that double-sixes exist.

Theorem 2. *There are 216 doublets. If $\{\sigma_1, \sigma_2\}$ is a doublet, then there is a unique double-six $\theta = \{(\sigma_{1,i}), (\sigma_{2,i})\}$ with $\sigma_1 = \sigma_{1,1}$ and $\sigma_2 = \sigma_{2,1}$. There are 36 double-sixes.*

Proof. Consider an nsp-spread σ^i. This nsp-spread contains 5 nonsingular pairs, each of which is contained in 3 nsp-spreads, that is, in 2 nsp-spreads other than σ^i. Thus, σ^i intersects $5 \cdot 2 = 10$ other nsp-spreads (as σ^i is the only nsp-spread containing 2 of these pairs, by lemma 1), and as there are 27 nsp-spreads in all, σ^i is disjoint from 16 others. Therefore there are $27 \cdot 16/2 = 216$ doublets.

To show that double-sixes exist, we exhibit one. The doublet $\{\sigma^{22}, \sigma^{26}\}$ is contained in the unique double-six

$$\{(\sigma^{22},\ \sigma^{8},\ \sigma^{1},\ \sigma^{6},\ \sigma^{27},\ \sigma^{14}),\ \ (\sigma^{26},\ \sigma^{3},\ \sigma^{5},\ \sigma^{7},\ \sigma^{13},\ \sigma^{24})\}.$$

It is straightforward to check that G acts transitively on doublets, so each doublet is contained in a unique double-six. Since each double-six contains 6 doublets, there are $216/6 = 36$ double-sixes. $\boxed{QED}$

We list the doublets in Table III, grouped into double-sixes. Note that G acts transitively on the double-sixes.

The doublets and double-sixes can be understood in terms of the structure of the group G.

Proposition 1. *Let $F \subset G$ be a Sylow 5-subgroup, and consider the action of $F \cong \mathbf{Z}/5$ on $\Sigma = \{\sigma^1, \ldots, \sigma^{27}\}$. There are exactly 2 fixed points for this action: σ_1, σ_2, and these form a doublet. Of the other 5 orbits of F on Σ, each of which consists of 5 nsp-spreads, there are 3 orbits whose nsp-spreads are pairwise disjoint. Denote these by $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$, $i = 1, \ldots, 5$. Then, after proper renumbering, we have for all $i = 1, \ldots, 5$,*

$$\begin{array}{rclrcl} \#(\sigma_1 \cap a_i) & = & 0 & \qquad \#(\sigma_2 \cap a_i) & = & 1 \\ \#(\sigma_1 \cap b_i) & = & 1 & \qquad \#(\sigma_2 \cap b_i) & = & 0 \\ \#(\sigma_1 \cap c_i) & = & 1 & \qquad \#(\sigma_2 \cap c_i) & = & 1 \end{array}$$

so that

$$\{(\sigma_1, a_1, \ldots, a_5),\ (\sigma_2, b_1, \ldots, b_5)\}$$

forms a double-six.

Proof. Since all 5-Sylow subgroups are conjugate, and cyclic, it will suffice to examine the action of a single element g of order 5. Choose

$$g = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

Then g fixes σ^{22} and σ^{26}, and the sets $\{a_i\}$ and $\{b_i\}$ are given by

$$\{\sigma^8,\ \sigma^1,\ \sigma^6,\ \sigma^{27},\ \sigma^{14}\} \quad \text{and} \quad \{\sigma^3,\ \sigma^5,\ \sigma^7,\ \sigma^{13},\ \sigma^{24}\}$$

respectively. $\boxed{QED}$

Remark 2. Since each double-six contains 12 nsp-spreads, each nsp-spread is contained in $36 \cdot 12/27 = 16$ double-sixes.

Remark 3. The nsp-spreads in each "half" of a double-six contain the same collection of nonsingular pairs, so of the 45 nonsingular pairs, 30 are contained with multiplicity 2 in the nsp-spreads forming a double - six, while the other 15 are not contained at all in these nsp-spreads.

Our analysis enables us to easily identify the structure of some of the stabilizers.

Lemma 2. *The subgroup of G stabilizing a double-six is isomorphic to S_6, the symmetric group on 6 elements. The isomorphism is given by the action of the stabilizer on the doublets in a double-six.*

Proof. Let H be the stabilizer of a double-six. Since G acts transitively on the 36 double-sixes, H has order $25920/36 = 720$. H obviously acts on the 6 doublets contained in a double-six that it fixes, so we obtain a homomorphism $\varphi : H \to S_6$. Since G acts transitively on the doublets, we see that H acts transitively on the 6 doublets in a double-six fixed by H. This is because each doublet is contained in a unique double-six. Hence $\mathrm{Im}(\varphi)$ is a transitive subgroup. We will show that $\mathrm{Im}(\varphi) = S_6$ by showing that it contains a 5-cycle and a transposition, [27], and this shows that φ is an isomorphism.

Consider the element g of proposition 1. The proof of proposition 1 shows that $\varphi(g)$ is a 5-cycle. Here we are thinking of H as the stabilizer of the double-six number 11 of table III.

Now consider the element of G:

$$t \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculation shows that $\varphi(t)$ is a transposition, this time regarding H as the stabilizer of the double-six number 33. $\boxed{QED}$

Lemma 3. *The subgroup of G stabilizing an nsp-spread is isomorphic to an extension of the alternating group A_5 by $(\mathbf{Z}/2)^4$.*

Proof. Let $K = K(\sigma)$ be the stabilizer of the nsp-spread

$$\sigma = \{\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4,\ \Delta_5\}.$$

We claim that there is an exact sequence

$$1 \longrightarrow (\mathbf{Z}/2)^4 \longrightarrow K \longrightarrow A_5 \longrightarrow 1.$$

We prove this by constructing homomorphisms

$$\pi : K \to S_5 \quad \text{and} \quad \rho : \mathrm{Ker}(\pi) \to (\mathbf{Z}/2)^5\,.$$

with $\mathrm{Image}(\pi) = A_5$ and $\mathrm{Image}(\rho) \cong (\mathbf{Z}/2)^4$. Since K has order $25920/27 = 960 = 16 \cdot 60$, this proves the claim.

Each element of K induces a permutation of the set $\{\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4,\ \Delta_5\}$ and this gives us π. Each Δ_i is a pair $\{\delta_i', \delta_i''\}$. For $k \in K$ leaving Δ_i invariant, we set $\varepsilon_i(k) = +1$ (resp. -1) if k fixes both δ_i' and δ_i'' (resp. k transposes δ_i' and δ_i''). For each $k \in \mathrm{Ker}(\pi)$, set $\rho(k) \;=\; (\varepsilon_1(k),\ \ldots,\ \varepsilon_5(k))$. Clearly ρ is a homomorphism.

We claim that $\mathrm{Image}(\pi) = A_5 \subset S_5$ and $\mathrm{Image}(\rho) = E \subset (\mathbf{Z}/2)^5$ where

$$E \;=\; \{(\varepsilon_1,\; \ldots,\; \varepsilon_5) \mid \prod \varepsilon_i = 1\}.$$

Consider the element t of lemma 2. Calculation shows that t is in the stabilizer of σ_1, the first nsp-spread of table II, and indeed, that t leaves each nonsingular pair in this nsp-spread invariant, so $t \in \mathrm{Ker}(\pi)$, $\pi : K(\sigma_1) \to S_5$. Further calculation shows that $\rho(t) = (1, -1, -1, -1, -1)$. Also, calculation shows that the element g of proposition 1, which leaves nsp-spread σ_{22} of table II invariant, acts as a 5-cycle on the nonsingular pairs contained therein. Since all stabilizers are conjugate, there is an element $g' \in K(\sigma_1)$ such that $\pi(g')$ is a 5-cycle. Then conjugating t by powers of g' gives elements $t_i \in \mathrm{Ker}(\pi)$ with $\rho(t_i) = (\varepsilon_1, \ldots, \varepsilon_5)$ where $\varepsilon_j = 1$ if $j = i$ and $\varepsilon_j = -1$ if $j \neq i$. These elements generate the subgroup E so $\mathrm{Im}(\rho) \supset E$. Therefore, 16 divides the order of $\mathrm{Im}(\rho)$ and hence the order of $\mathrm{Ker}(\pi)$.

Calculation shows that t leaves the nsp-spread σ_2 invariant, but it does not fix each of the nonsingular pairs that it contains. In fact $\pi(t)$ is a product of two disjoint transpositions, where this time $\pi : K(\sigma_2) \to S_5$.

Finally, calculation shows that the element of G of order 3:

$$s \;=\; \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

leaves σ_1 invariant, and $\pi(s)$ is a 3-cycle, $\pi : K(\sigma_1) \to S_5$.

We have seen that 16 divides the order of $\mathrm{Ker}(\pi)$ and that $\mathrm{Im}(\pi)$ has elements of order 2, 3, 5. Thus 30 divides the order of $\mathrm{Im}(\pi)$. Since K has order 960, either $\mathrm{Ker}(\pi)$ has order 16 and $\mathrm{Im}(\pi)$ has order 60 or $\mathrm{Ker}(\pi)$ has order 32 and $\mathrm{Im}(\pi)$ has order 30. But the latter case is impossible since S_5 has no subgroups of order 30. Thus we are in the former case, with $\mathrm{Im}(\rho)$ isomorphic with $(\mathbf{Z}/2)^4$, and $\mathrm{Im}(\pi)$ being A_5, the unique subgroup of S_5 of order 60. $\boxed{QED}$

Theorem 3. *There are G-equivariant identifications*

$$\begin{aligned} \{\textit{nsp-spreads}\} &\leftrightarrow \{\textit{lines on the cubic surface}\} \\ \{\textit{double-sixes}\} &\leftrightarrow \{\textit{double-sixes of lines on the cubic surface}\} \\ \{\textit{nonsingular pairs}\} &\leftrightarrow \{\textit{tritangent planes to the cubic surface}\} \end{aligned}$$

Proof. The cardinalities of these sets match up, being 27, 36, and 45 respectively, and in all cases G acts transitively on the set. Thus the stabilizer of each of these is a subgroup of index 27, 36, and 45 (= order 960, 720, and 576) respectively. However, Dickson [9] has shown that there is a unique conjugacy class of subgroups of G of each of these orders.

For a constructive proof, note that if the third identification holds, the first two follow (as an nsp-spread is determined by the pairs it contains, and a line on the cubic by the 5 tritangent planes on which it lies).

But now note that the stabilizer of the non singular pair

$$\{(1000) \wedge (0010),\ (0100) \wedge (0001)\}$$

is the image in G of the subgroup

$$\left\{ \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} 0 & a_1 & 0 & b_1 \\ a_2 & 0 & b_2 & 0 \\ 0 & c_1 & 0 & d_1 \\ c_2 & 0 & d_2 & 0 \end{pmatrix} \right\}$$

of $\mathbf{Sp}(4,\ \mathbf{F}_3)$, where

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathbf{SL}(2,\ \mathbf{F}_3) \quad \text{for} \quad i = 1,\ 2$$

and this is exactly the subgroup stabilizing a tritangent plane [7, p. 26]. (Note that [7] works projectively, so that what we describe as a line (resp. isotropic plane) is described there as a point (resp. isotropic line).) $\boxed{QED}$

Remark 4. Given theorem 3, we observe that lemmas 2 and 3 verify the structures of the stabilizers given in [7].

In addition to considering the symplectic group $\mathbf{Sp}(V)$, we may also consider the group of symplectic similitudes

$$\mathbf{GSp}(V) = \mathbf{GSp}(4, \mathbf{F}_3) = \{g \in \mathbf{GL}(4, \mathbf{F}_3) \mid gJ\,{}^t g = \lambda J, \quad \lambda \in \mathbf{F}_3^*\}$$

Let $\tilde{G} = \mathbf{GSp}(4, \mathbf{F}_3)/\pm 1$. Then $\tilde{G}$ is a group of order 51480 having G as a subgroup of index two. A representative of the nontrivial coset is the element of order two

$$\tilde{g}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that $\tilde{G}$ acts on all the objects that we have considered here (isotropic lines and planes, nonsingular pairs, nsp-spreads and double-sixes).

Proposition 2.

1. *The element $\tilde{g}_0$ leaves seven nonsingular pairs invariant and permutes the other 38 in pairs.*

2. *The element $\tilde{g}_0$ leaves three nsp-spreads invariant and permutes the other 24 in pairs.*

3. *The subgroup of $\tilde{G}$ stabilizing a double-six is isomorphic to $S_6 \times S_2$.*

4. *The subgroup of $\tilde{G}$ stabilizing an nsp-spread is isomorphic to an extension of S_5 by $(\mathbf{Z}/2)^4$.*

PROOF. a) Direct calculation. On non-singular pairs, $\tilde{g}_0$ acts as the permutation

$$(4\ 5)(6\ 8)(7\ 9)(10\ 13)(11\ 14)(12\ 15)(17\ 18)(20\ 21)(22\ 23)(24\ 25)$$
$$(27\ 28)(29\ 30)(31\ 40)(32\ 34)(33\ 35)(36\ 41)(38\ 39)(42\ 43)(44\ 45).$$

b) Direct calculation. On nsp-spreads, $\tilde{g}_0$ acts as the permutation

$$(2\ 3)(5\ 6)(8\ 9)(10\ 13)(11\ 14)(12\ 15)(16\ 22)(17\ 23)(18\ 24)(19\ 25)(20\ 26)(21\ 27).$$

c) Let $\tilde{H}$ be the stabilizer in $\tilde{G}$ of a double-six, and H the stabilizer in G of a double-six. Let $\tilde{\varphi} = (\varphi, \varphi') : \tilde{H} \to S_6 \times S_2$, where φ is the map of lemma 2 and $\varphi'(h)$ is the action of h on the two "halves" of a double-six (either leaving them invariant or interchanging them). It is easy to check, from the proof of lemma 2, that the image of H under $\tilde{\varphi}$ is the subgroup of $S_6 \times S_2$ consisting of permutations (α, α') with $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\alpha')$.

The element $\tilde{g}_0$ stabilizes the double-six θ_{20} and direct calculation shows that $\varphi(\tilde{g}_0)$ is a product of two transpositions, while $\varphi'(\tilde{g}_0)$ is a transposition, so $\tilde{\varphi}(\tilde{g}_0) \notin \mathrm{Im}(H)$ and hence $\tilde{\varphi} : \tilde{H} \to S_6 \times S_2$ is an isomorphism.

d) Let $\tilde{K}$ be the stabilizer in $\tilde{G}$ of an nsp-spread and K the stabilizer in G of an nsp-spread. Let π be the map of lemma 3. The element $\tilde{g}_0$ stabilizes the nsp-spread σ_1 and direct calculation shows that $\pi(\tilde{g}_0)$ is a transposition, hence not an element of $A_5 = \mathrm{Im}(K)$, so $\pi(\tilde{K}) = S_5$, and the result follows as in the proof of lemma 3. $\boxed{QED}$

We conclude this section by considering the notion of an nsp-spread in general.

Definition 4. Let V be a $2n$-dimensional vector space over a field $\mathbf{F}$, n even, equipped with a nonsingular symplectic form $\langle\ ,\ \rangle$. A nonsingular pair $\Delta = \{\delta_1,\ \delta_2\}$ is the image in $\mathbf{P}(V)$ of a pair of n-dimensional subspaces $\tilde{\delta}_1, \tilde{\delta}_2$, each of which is nonsingular under the restriction of $\langle\ ,\ \rangle$, and such that $V = \tilde{\delta}_1 \oplus \tilde{\delta}_2$. An nsp-spread is a set

$$\sigma \ = \ \{\Delta_i\}_{i\in I}$$

of pairwise disjoint nonsingular pairs such that

$$\bigcup_{i\in I} \Delta_i \ = \ \mathbf{P}(V)$$

where Δ_i is identified with the set of projective lines that it contains.

We can show that nsp-spreads exist in considerable generality. In particular, they exist for any even n and $\mathbf{F}$ any finite field of odd characteristic, or any number field. See [18].

3. The Tits building with scaffolding

We briefly recall the definition of the Tits building with scaffolding $\overline{\mathcal{T}}(V)$ of the symplectic vector space V [19, 17]. (Our description here is simplified a bit because the units of $\mathbf{Z}/3$ are just ± 1.)

$\overline{\mathcal{T}}(V)$ is a graph with vertices

$$\begin{array}{r l} \{l\} & \text{of cardinality } 40 \\ \{h\} & \text{of cardinality } 40 \\ \{\Delta\} & \text{of cardinality } 45 \end{array}$$

and edges

$$\begin{array}{r l} \{l,\ h\} & \text{if } l \subset h, \text{ of cardinality } 160 \\ \{l,\ \Delta\} & \text{if } l \subset \Delta, \text{ of cardinality } 360 \end{array}$$

(The full subcomplex $\{l\},\ \{h\},\ \{l,\ h\}$ is the usual Tits building $\mathcal{T}(V)$.)

The action of G on $V/\pm 1$ clearly induces an action on $\overline{\mathcal{T}}(V)$.

We have the following description of $\mathcal{A}_2(3)^*$, the Siegel modular variety of degree two and level three [17]: The variety $\mathcal{A}_2(3)^*$ is the Igusa compactification of $\mathcal{A}_2(3) = \Gamma(3)\backslash\mathfrak{S}_2$, where $\mathfrak{S}_2$ is the degree two Siegel space, and $\Gamma(3)$ is the principal congruence subgroup of level 3 in $\Gamma(1) = P\mathbf{Sp}(4,\ \mathbf{Z})$. The quotient $\Gamma(1)/\Gamma(3) = P\mathbf{Sp}(4,\ \mathbf{F}_3) = G$ acts on $\mathcal{A}_2(3)$ and this action extends to the compactification $\mathcal{A}_2(3)^*$. The "boundary" of $\mathcal{A}_2(3)^*$ is

$$\mathcal{A}_2(3)^* - \mathcal{A}_2(3) = \bigcup_l D(l)$$

where each $D(l)$ is Shioda's elliptic modular surface of level 3. We call such a $D(l)$ a corank 1 boundary component.

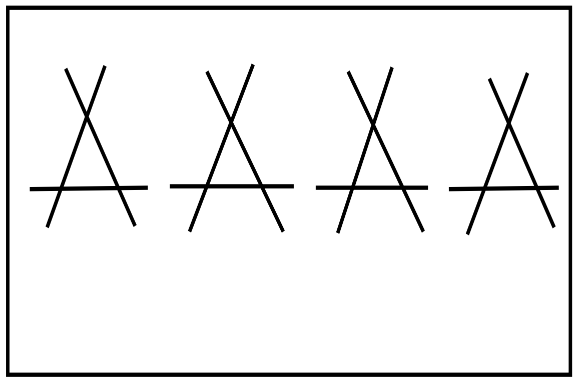

Figure 1. Corank 1 boundary component $D(l)$

Each of these is a family of genus 1 curves, parameterized by a copy $M(l) \cong \mathbf{P}^1$ of the modular curve of level 3. Over four points of $M(l)$, the "cusps", the general fiber degenerates into a triangle of $\mathbf{P}^1$'s. Hence, each $D(l)$ has four triangles of rational lines on it. There are also nine cross-sections of the fibration $D(l) \to M(l)$, not pictured in figure 1.

The $D(l)$ are not pairwise disjoint; if two of them intersect, the intersection is one of the $\mathbf{P}^1$'s in the 4 distinguished triangles. The union of all these intersections form 40 connected components $C(h)$, each of which is a "tetrahedron" of $\mathbf{P}^1$'s. Moreover, $D(l_1)$ intersects $D(l_2)$ if and only if $h = l_1 \wedge l_2$. In this case

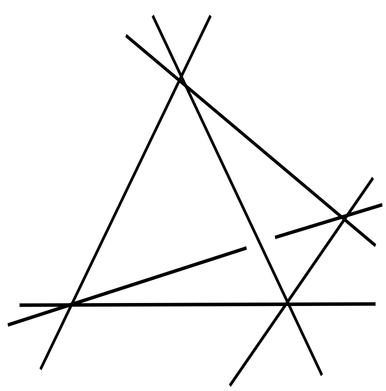

Figure 2. Corank 2 boundary component $C(h)$

we say that $\{l_1, l_2\}$ is an incident pair.

$\mathcal{A}_2(3)^*$ contains 45 Humbert surfaces $H(\Delta)$ (of discriminant 1), each of which is isomorphic with $M(l) \times M(l) \cong \mathbf{P}^1 \times \mathbf{P}^1$. Each of these has 8 distinguished rational lines, namely the cusp $\times M(l)$ and the $M(l) \times$ cusp.

These are mutually disjoint, and $H(\Delta) \cap D(l)$ if nonempty, is equal one of the 9 sections of the elliptic modular surface $D(l)$, and it is also equal to one of the 8 distinguished rational lines on $H(\Delta)$. The intersection is nonempty if and only if l is in Δ.

$\mathcal{A}_2(3)^*$ is identified with $\tilde{B}$, and blowing down each of the 45 Humbert surfaces yields a quartic projective threefold with 45 ordinary double points (Burkhardt's quartic). We refer the reader to [20] for a further discussion of B.

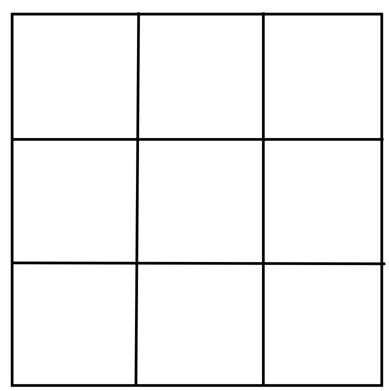

Figure 3. Humbert surface $H(\Delta)$

We now present our dictionary, showing how to translate between Burkhardt's language [3, 4, 5], Baker's language [1], and ours [17]: (We have labeled each object with its cardinality.)

Dictionary

(40) Vertex l of $\overline{\mathcal{T}}(V)$	: Burkhardt–Hauptebene : Baker–Jacobi plane : $\mathcal{A}_2(3)^*$-corank 1 boundary component $D(l)$
(40) Vertex h of $\overline{\mathcal{T}}(V)$	: Burkhardt–Hauptraum erster Art : Baker–Steiner tetrahedron : $\mathcal{A}_2(3)^*$-corank 2 boundary component $C(h)$
(45) Vertex Δ of $\overline{\mathcal{T}}(V)$	: Burkhardt–Hauptraum zweiter Art : Baker-node : $\mathcal{A}_2(3)^*$-Humbert surface $H(\Delta)$
(160) Edge $(l,\ h)$ of $\overline{\mathcal{T}}(V)$	: $\mathcal{A}_2(3)^*$-exceptional fiber in $D(l)$ (or "face" of a tetrahedron in $C(h)$)
(360) Edge $(l,\ \Delta)$ of $\overline{\mathcal{T}}(V)$	: $\mathcal{A}_2(3)^*$-section of $D(l)$
(240) Incident pair $\{l_1, l_2\}$ of $\overline{\mathcal{T}}(V)$	: Burkhardt–Hauptgerade : Baker-κ line : $\mathcal{A}_2(3)^*$-exceptional $\mathbf{P}^1$ $= D(l_1) \cap D(l_2) \subset C(l_1 \wedge l_2)$
(27) Nsp-spread $\{\Delta_1, \ldots, \Delta_5\}$ of $\overline{\mathcal{T}}(V)$	: Burkhardt-Pentatope zweiter Art : Baker–Jordan pentahedron : $\mathcal{A}_2(3)^*$-set of 5 Humbert surfaces $H(\Delta_1), \ldots, H(\Delta_5)$ intersecting every $D(l)$

4. Coble and Dickson

We now turn to the work of Coble [6] and Dickson [9]. V is therefore a four dimensional symplectic vector space. For the rest of this section we will work projectively, so that l, h, and δ become an (isotropic) point, an isotropic line and a nonsingular line, and $\delta^\perp$ the complementary line to δ. We still call Δ, σ, and θ a nonsingular pair, nsp-spread, and double-six.

Although Coble did not refer to Dickson in his work, we shall first investigate Dickson's work and then use our results to explain Coble's. We begin with some general notation and language.

Definition 5. Let a group A operate on a set X. For $x \in X$, let $P(x)$ denote the stabilizer subgroup of x,

$$P(x) = \{a \in A \mid a(x) = x\}.$$

If A acts transitively on X then the index of $P(x)$ in A is the cardinality of X.

Lemma 4. *Let a group A operate transitively on a set X.*

1. *For any $x_1,\ x_2 \in X$, $P(x_1)$ and $P(x_2)$ are mutually conjugate in A.*

2. *The following are equivalent:*

   i. For any $x_1 \neq\ x_2 \in X$, $P(x_1) \neq P(x_2)$.

   ii. For any $x \in X$, $B = P(x)$ satisfies $B = N(B)$, i.e., B is its own normalizer in A.

3. *Define an equivalence relation $\sim$ on X by $x_1 \sim\ x_2$ if $P(x_1) = P(x_2)$. Then every equivalence class has $[N(B) : B]$ elements, where $B = P(x)$.*

Definition 6. If B is a subgroup of A such that $N(B) = B$, then we say that B is self-conjugate (in A). (Here we follow Dickson's language.)

Definition 7. Let X be a set.

1. A subset of m elements of X is an m-ad. For $m = 2$ or 3, an m-ad is a duad or a triad, respectively.

2. If X has n elements, and n is a multiple of m, an unordered partition of X into n/m m-ads is an m-adic syntheme. (This language was introduced by Sylvester in 1844.)

Theorem 4. *(Dickson 1904 [9]) $G = P\mathbf{Sp}(4, \mathbf{F}_3)$ has exactly 114 conjugacy classes of proper subgroups, of which 18 are self-conjugate.*

Dickson's approach was purely algebraic. Our aim here is to understand these 18 self-conjugate subgroups by using the geometry of the vector space V, or more precisely, by using configurations arising from objects we have already considered. We follow Dickson's notation for these subgroups. In particular, the subscript attached to a subgroup denotes its order.

Theorem 5. *The 18 self-conjugate subgroups of G arise as follows:*

1. G_{20}, *of index 1296, is the normalizer of a 5 - Sylow subgroup.*
2. G^*_{24}, *of index 1080, is the stabilizer of a pair* (h, σ).
3. K^*_{36}, *of index 720, is the stabilizer of a pair* (h, Δ) *with* $h \cap \Delta \neq \emptyset$.
4. G_{48}, *of index 540, is the stabilizer of a duadic syntheme of doublets in a double-six* θ.
5. G_{72}, *of index 360, is the stabilizer of a pair* (l, Δ) *with* $l \subset \Delta$.
6. G^{**}_{72}, *of index 360, is the stabilizer of a triadic syntheme of doublets in a double-six* θ.
7. H_{96}, *of index 270, is the stabilizer of a duad of nonsingular pairs in an nsp-spread* σ.
8. G_{120}, *of index 216, is the stabilizer of a doublet.*
9. G'_{120}, *of index 216, is the image of* G_{120} *under the outer automorphism of* G_{720}.
10. G_{160}, *of index 162, arises as follows:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\mathbf{Z}/2)^4 & \longrightarrow & G_{960} & \xrightarrow{\pi} & A_5 & \longrightarrow & 1 \\ & & || & & \bigcup & & \bigcup & & \\ 1 & \longrightarrow & (\mathbf{Z}/2)^4 & \longrightarrow & G_{160} & \longrightarrow & N_{10} & \longrightarrow & 1 \end{array}$$

where N_{10} *is the normalizer (in* A_5*) of a 5-Sylow subgroup.*

11. G_{162}, *of index 160, is the stabilizer of a pair* (l, h) *with* $l \subset h$.
12. G_{192}, *of index 135, is the stabilizer of a pair* (Δ, σ) *with* $\Delta \in \sigma$.
13. H_{216}, *of index 120, is the stabilizer of a duadic syntheme of* l*'s in an* h.
14. G_{576}, *of index 45, is the stabilizer of a nonsingular pair* Δ.
15. G_{648}, *of index 40, is the stabilizer of a point* l.

16. *H_{648}, of index 40, is the stabilizer of an isotropic line h.*

17. *G_{720}, of index 36, is the stabilizer of a double-six θ.*

18. *G_{960}, of index 27, is the stabilizer of an nsp-spread σ.*

Remark 5. a) Since, by lemma 1, any two disjoint nonsingular pairs are contained in a unique nsp-spread, H_{96} can also be described as the stabilizer of a duad of disjoint nonsingular pairs.

b) The symmetric group of order 6 has a unique (up to inner automorphisms) outer automorphism, as observed by Todd [26]. By lemma 2, the subgroup G_{720} is isomorphic to S_6, and G_{120} is a subgroup of G_{720}, so applying this automorphism takes G_{120} to G'_{120}.

c) The top line in the construction of G_{160} is the exact sequence in the proof of lemma 3. Also, $N_{10} = \pi(G_{10})$ where G_{10} is the unique (up to conjugacy) subgroup of G of order 10. (Note that $G_{10} = G_{20} \cap \pi^{-1}(A_5)$).

Proof. The difficult part of the theorem is discovering the configurations that are stabilized. Once this is done, verifying that the stabilizers are as claimed is a (more or less) routine calculation, and so we omit it. $\boxed{QED}$

Remark 6. In comparing the entries in the "Dictionary" of section 3. with the configurations of the above theorem, one is missing, that of an incident pair $\{l_1, l_2\}$. But, l_1 and l_2 are incident if and only if they are both contained in some isotropic line h. Now h contains 4 l's, say $\{l_1, l_2, l_3, l_4\}$. Thus, $P(\{l_1, l_2\}) \subset P(h)$, and then if $\{l_1, l_2\}$ is stabilized, so is $\{l_3, l_4\}$. Thus $P(\{l_1, l_2\}) = P(\{l_3, l_4\})$, and this subgroup is not self-conjugate. However, it is of index two in $P(\{\{l_1, l_2\}, \{l_3, l_4\}\}) = H_{216}$, of index 120, which does appear in the theorem.

Some self-conjugate subgroups can be realized in more than one way as the stabilizer of a configuration. Perhaps the most interesting of these is G_{48} of index 540.

Proposition 3. *a) Given a pair (h_1, Δ) with $h_1 \cap \Delta = \emptyset$, then there is a unique $h_2 \neq h_1$ with $h_2 \cap \Delta = \emptyset$ and $P(h_1, \Delta) = P(h_2, \Delta)$. Furthermore, h_1 and h_2 are disjoint.*

b) Given a duad (h_1, h_2) of disjoint isotropic lines, there is a unique nonsingular pair Δ with $P(h_1, \Delta) = P(h_2, \Delta)$. Furthermore, $h_1 \cap \Delta = \emptyset$ and $h_2 \cap \Delta = \emptyset$.

c) In this situation, $P(h_i, \Delta)$ is a normal subgroup of index 2 in

$$P(\{h_1, h_2\}, \Delta) = P(h_1, h_2)$$

Also, $P(\{h_1, h_2\}, \Delta) = G_{48}$.

Proof. First let us see that the cardinality of each of these sets is 540. In a) there are 45 Δ's, and each is disjoint from 24 h's, so when the h's are paired we get $540 = 45 \cdot 24/2$ sets $\{\{h_1, h_2\}, \Delta\}$. In b), there are 40 h's, and each is disjoint from 27 other h's, and this pair determines a unique Δ, so we get $540 = (40 \cdot 27)/2) \cdot 1$ sets $\{\{h_1, h_2\}, \Delta\}$. In c), there are 36 double-sixes, and a set of 6 objects has

$$\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15 \quad \text{duadic synthemes,}$$

so we get $540 = 36 \cdot 15$ duadic synthemes of doublets in a double-six.

Thus to construct a bijective correspondence between these objects we need only construct an injective mapping between them.

Also, given a duad $\{h_1, h_2\}$ of disjoint h's, there is an element of G which interchanges h_1 and h_2. Assuming b), the uniqueness of Δ implies that any element of G which stabilizes $\{h_1, h_2\}$ must also stabilize Δ, verifying all but the last claim in c).

First we show how to pass from a duadic syntheme of doublets in a double-six to a set $\{\{h_1, h_2\}, \Delta\}$. Consider a duadic syntheme $(ij)(kl)(mn)$ of doublets in a fixed double-six θ. Then the doublets i and j, regarded as sets of nonsingular pairs, intersect in two nonsingular pairs Δ^1_{ij} and Δ^2_{ij}, and similarly for the doublets k and l and the doublets m and n. The nonsingular pairs Δ^1_{ij} and Δ^2_{ij} are disjoint and so determine a unique nsp-spread σ_{ij}, and similarly for k and l and m and n. Then, as sets of nonsingular pairs,

$$\sigma_{ij} \cap \sigma_{kl} \cap \sigma_{mn} = \{\Delta\}$$

for a unique Δ. (Otherwise σ_{ij}, σ_{kl}, σ_{mn} are pairwise disjoint.) As sets of l's,

$$\Delta^1_{ij} \cup \Delta^2_{ij} \cup \ldots \cup \Delta^2_{mn} = \{24\ l\text{'s}\}.$$

Δ contains 8 l's, disjoint from the above 24. Together these give 32 of the 40 l's. The remaining 8 can be grouped into a unique way into h_1 and h_2.

All these statements can be verified by direct calculation. Because of the transitivity of the action of G on duadic synthemes of doublets in a double-six, it suffices to verify them for one such. Consider the double-six θ^{36} in Table 3. Number the doublets therein 1–6 from left to right. Consider the duadic syntheme (12)(46)(35). Then

$$\begin{aligned}
\{\Delta^1_{12}, \Delta^2_{12}\} &= \{\Delta^{32}, \Delta^{34}\} \subset \sigma^7 \\
\{\Delta^1_{46}, \Delta^2_{46}\} &= \{\Delta^{33}, \Delta^{35}\} \subset \sigma^4 \\
\{\Delta^1_{35}, \Delta^2_{35}\} &= \{\Delta^{44}, \Delta^{45}\} \subset \sigma^1
\end{aligned}$$

and $\sigma^7 \cap \sigma^4 \cap \sigma^1 = \{\Delta^{26}\}$ so $\Delta = \Delta^{26}$. The remaining 8 l's are

$$\{(0001), (0010), (0011), (0012), (0100), (1000), (1100), (1200)\}$$

which can be grouped into

$$h_1 = \{(0001), (0010), (0011), (0012)\} \text{ and } h_2 = \{(0100), (1000), (1100), (1200)\}.$$

Next we show how to pass from (h_1, Δ) to (h_2, Δ) and thus to (h_1, h_2). Let $\Delta = \{\delta, \delta^\perp\}$ and consider

$$\{\alpha l + \beta l' \mid l \in \delta, l' \in \delta^\perp, \alpha/\beta = \pm 1\}$$

This set consists of 32 l's, all those not in Δ. Then h_2 is defined by the condition $\alpha l + \beta l' \subset h_1 \Leftrightarrow \alpha l - \beta l' \subset h_2$. Again this can be checked by direct computation in the above example, and by transitivity it suffices to consider a single example.

Thus we have correspondences between the various objects in the proposition, and it is routine to verify that they are injective, completing the proof. $\boxed{QED}$

Definition 8. If $\{h_1, h_2\}$ and Δ are as in the above proposition, i.e., $P(h_1, \Delta) = P(h_2, \Delta)$, then $\{h_1, h_2\}$ and Δ are called related.

While this result deals with the case $h \cap \Delta = \emptyset$, in connection with parts 2 and 3 of theorem 5, and for our work below, we wish to expand on the case $h \cap \Delta \neq \emptyset$.

Proposition 4. 1. *Suppose that $h \cap \Delta \neq \emptyset$. Then, after proper numbering, $h \cap \Delta = \{l_1, l_2\}$ where $h = \{l_1, l_2, l_3, l_4\}$, $\Delta = \{\delta, \delta^\perp\}$ and $l_1 \in \delta$, $l_2 \in \delta^\perp$.*

2. *For any fixed duad $\{l_1, l_2\} \subset h$, there are exactly three Δ's with $h \cap \Delta = \{l_1, l_2\}$.*

3. *Fix h and σ arbitrarily. Then there are exactly two (necessarily disjoint) nonsingular pairs $\Delta_1, \Delta_2 \subset \sigma$ with $\Delta_i \cap h \neq \emptyset$. In that case, after proper renumbering $\Delta_1 \cap h = \{l_1, l_2\}$, and $\Delta_2 \cap h = \{l_3, l_4\}$ with $h = \{l_1, l_2, l_3, l_4\}$.*

In the situation of c) we call the duads $\{l_1, l_2\}$ and $\{l_3, l_4\}$ complementary.

Proof. Routine. $\boxed{QED}$

Now we come to the work of Coble [6]. Let us begin by comparing our language and notation with his. We denote points by l; Coble denoted them P. We have isotropic lines h; Coble called these complex lines C. We have nonsingular lines δ, which determine their orthogonal complements $\delta^\perp$ and nonsingular pairs $\Delta = \{\delta, \delta^\perp\}$; Coble had non-complex lines N which determine conjugate non-complex lines N' and skew pairs NN'. An isotropic line h with $\Delta \cap h \neq \emptyset$ or in Coble's language a complex line C with $C \cap NN' \neq \emptyset$ he called a transversal of NN'. We have spreads of nonsingular pairs σ; Coble had sets of five skew pairs of conjugate lines F. Finally, we have double-sixes θ, and Coble also called these double-sixes.

First we observe that Coble knew proposition 4 and used it to count nsp-spreads (in our language) as follows: Since an nsp-spread contains every l once, it intersects every h twice. Fix an $h = \{l_1, \ldots, l_4\}$. Then there are three ways to choose a duadic syntheme of l's in h. Fix one, $\{\{l_1, l_2\}, \{l_3, l_4\}\}$. Then there are three ways to choose Δ_1 with $\Delta_1 \cap h = \{l_1, l_2\}$ and three ways to choose Δ_2 with $\Delta_2 \cap h = \{l_3, l_4\}$. The nonsingular pairs Δ_1 and Δ_2 are necessarily disjoint and so extend to a unique nsp-spread σ. Thus there are $3 \cdot 3 \cdot 3 = 27$ nsp-spreads.

To proceed further, let us recall the tetrahedron $C(h)$ constructed in the beginning of section 3.. The four faces of the tetrahedron correspond to the four l's in an h. The six edges correspond to the six duads $\{l_i, l_j\}$ of l's in an h (being the intersection of the faces corresponding to l_i and l_j). In a tetrahedron each edge has a unique opposite edge, i.e., the edge to which it is not adjacent. The edge corresponding to the duad $\{l_1, l_2\}$ has as its opposite edge the edge corresponding to the complementary duad $\{l_3, l_4\}$.

Now fix $h = \{l_1, \ldots, l_4\}$. For any duad $\{l_i, l_j\}$ of l's in h, there are three nonsingular pairs Δ_1, Δ_2, Δ_3 with $\Delta_k \cap h = \{l_i, l_j\}$, $k = 1, 2, 3$. Coble calls $\{\Delta_1, \Delta_2, \Delta_3\}$ a triad. If $\{l_{i'}, l_{j'}\}$ is the complementary duad to $\{l_i, l_j\}$, then there are three other nonsingular pairs Δ_4, Δ_5, Δ_6 with $\Delta_k \cap h = \{l_{i'}, l_{j'}\}$, $k = 4, 5, 6$. Coble calls $\{\Delta_4, \Delta_5, \Delta_6\}$ the conjugate triad to $\{\Delta_1, \Delta_2, \Delta_3\}$. Thus each of the three duadic synthemes of l's in an h determines a pair of conjugate triads. Coble calls the union of these three pairs of conjugate triads a triad complex. We see that the triad complexes are in bijective correspondence with the isotropic lines, hence there are 40 of them. (Coble constructed them in a slightly different way.)

We have observed earlier (as did Coble) that any two disjoint nonsingular pairs Δ_1, Δ_2 are contained in a unique nsp-spread $\sigma = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\}$, and so determine $\{\Delta_3, \Delta_4, \Delta_5\}$. We shall call $\{\Delta_3, \Delta_4, \Delta_5\}$ the complementary triad to the duad $\{\Delta_1, \Delta_2\}$.

We now consider certain polygons, by which we mean configurations of points and lines. We must distinguish between two kinds, polygons in $\mathbf{P}^3 = \mathbf{P}(V)$, where point and lines have their usual meaning, and "abstract" polygons, where we have notions of point, line, and incidence (but these are not subsets of $\mathbf{P}^3$). We shall be considering two general types, *quadrilaterals*, which may be in $\mathbf{P}^3$ or abstract, and *complex hexagons*, which are abstract.These polygons were defined by Coble. Our point here is to clarify their meaning. Again, we shall (mostly) omit proofs of these results; once the facts have been discovered their verification is routine. We follow Coble's notation for these polygons.

Definition 9. A quadrilateral q in $\mathbf{P}^3$ is a quadrilateral with one pair of opposite edges δ and $\delta^\perp$ and another pair of opposite edges h_1 and h_2 with intersection behavior as in figure 4.

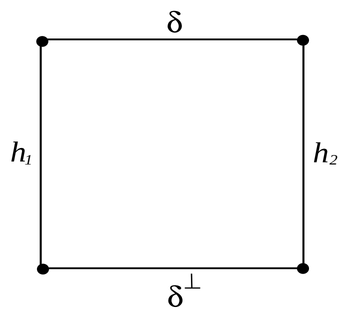

Figure 4. Quadrilateral q

Lemma 5. *Suppose δ, $\delta^\perp$, and h_1, and h_2 form a quadrilateral q. Let $\Delta = \{\delta, \delta^\perp\}$. Then $\{h_1, h_2\}$ and Δ are related. Conversely, all quadrilaterals q arise in this way. Thus, there are 540 quadrilaterals q.*

Definition 10. A quadrilateral Q in $\mathbf{P}^3$ is a quadrilateral all of whose edges are isotropic lines, i. e., a set of isotropic lines $\{h_i, h_j, h_{i'}, h_{j'}\}$ with intersection behavior as in figure 5.

Lemma 6. *Each quadrilateral Q determines a nonsingular pair $\Delta = \Delta(Q)$ as follows: Label the intersection points l_{ij}, $l_{ji'}$, $l_{i'j'}$, $l_{j'i}$ clockwise from the upper right-hand corner. Then l_{ij} and $l_{i'j'}$ generate a nonsingular line δ, and $l_{ji'}$ and $l_{j'i}$ generate its orthogonal complement, yielding a nonsingular pair $\Delta = \{\delta, \delta^\perp\}$. Conversely, given any nonsingular pair $\Delta = \{\delta, \delta^\perp\}$, any duad of points $\{l_{ij}, l_{i'j'}\}$ in δ and any duad of points $\{l_{ji'}, l_{j'i}\}$ in $\delta^\perp$ there is a unique quadrilateral Q with l_{ij}, $l_{ji'}$, $l_{i'j'}$, $l_{j'i}$ as vertices. Thus there are* $1620 = 45 \cdot 6^2$ *quadrilaterals Q.*

Given a quadrilateral Q determined by $\Delta = \{\delta, \delta^\perp\}$, a duad of points in δ and a duad of points in $\delta^\perp$, there is another quadrilateral Q' determined by the same Δ, but by the complementary duad of points in δ and the complementary duad of points in $\delta^\perp$. We call Q' complementary to Q.

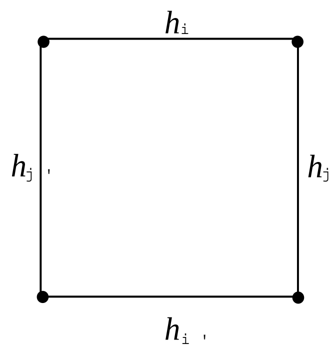

Figure 5. Quadrilateral Q

We now come to the abstract polygons. In all cases, the vertices will be quadrilaterals Q. The edges will be of two types: edges $E \leftrightarrow \{h_1, h_2\}$ where h_1 and h_2 are disjoint isotropic lines, and edges $D \leftrightarrow \Delta$, a nonsingular pair. The edge E will be incident to the the vertex Q if h_1 and h_2 are a pair of opposite edges of the quadrilateral Q, and the edge D will be incident to the the vertex Q if $\Delta = \Delta(Q)$.

Definition 11. A hyperboloid H is an abstract quadrilateral as in figure 6

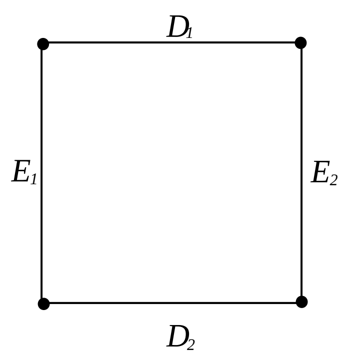

Figure 6. Hyperboloid H

Lemma 7. *If $D_1 \leftrightarrow \Delta_1$ and $D_2 \leftrightarrow \Delta_2$ are opposite edges in a hyperboloid H, then Δ_1 and Δ_2 are disjoint. Conversely, any duad of disjoint nonsingular pairs $\{\Delta_1, \Delta_2\}$ determines a unique hyperboloid H. Thus there are 270 hyperboloids H.*

Definition 12. A hyperboloid h is an abstract quadrilateral as in figure 7.

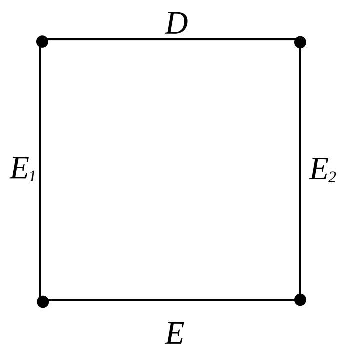

Figure 7. Hyperboloid h

This is the one place where Coble's notation conflicts with ours. The meaning of h should be clear in context.

Lemma 8. *If $D \leftrightarrow \Delta$ and $E \leftrightarrow \{h_1, h_2\}$ are opposite edges in a hyperboloid h, then $\{h_1, h_2\}$ and Δ are related. Conversely, if $\{h_1, h_2\}$ and Δ are related, they determine a unique hyperboloid h. Thus there are 540 hyperboloids h.*

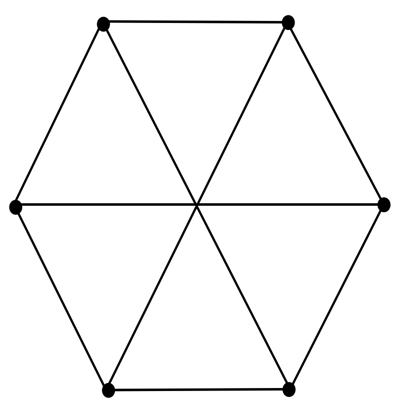

Figure 8. Complex hexagon

We now come to the most elaborate of Coble's configurations, the complex hexagon.

Definition 13. A complex hexagon K is an abstract configuration as in figure 8, where the six edges are lines of type E and the three diagonals are lines of type D. We let Q_i, $i = 0, \ldots, 5$ be the vertices, $E_{i,i+1}$ the edge joining Q_i and Q_{i+1}, and $D_{i,i+3}$ the diagonal joining Q_i and Q_{i+3} (all subscripts modulo

6).

Coble constructs complex hexagons as follows: Pick an arbitrary quadrilateral Q and let $Q_0 = Q$. Label its edges $\{h_0, h_1, h'_0, h'_1\}$; this involves a choice, but we will see that it is irrelevant. There is exactly one other quadrilateral Q having h_1 and h'_1 as a pair of opposite edges. Let Q_1 be this quadrilateral, and label its edges $\{h_1, h_2, h'_1, h'_2\}$. Continue in this way. It turns out that Q_5 has edges $\{h_5, h_0, h'_5, h'_0\}$ so the hexagon "closes up". If in labeling the edges of Q_0 we had switched the subscripts, we would have obtained the same complex hexagon but would have traversed its vertices in the opposite order. It then turns out that Q_i and Q_{i+3} are complementary, so we set $D_{i,i+3} = \Delta(D_i) = \Delta(D_{i+3})$.

Lemma 9. *Let K be a complex hexagon, with diagonals $D_{14} \leftrightarrow \Delta_1$, $D_{25} \leftrightarrow \Delta_2$, $D_{30} \leftrightarrow \Delta_3$. Then $\{\Delta_1, \Delta_2, , \Delta_3\}$ is a disjoint triad of nonsingular pairs. Conversely, given any disjoint triad $\{\Delta_1, \Delta_2, , \Delta_3\}$, there is a unique complex hexagon having diagonals $D_1 \leftrightarrow \Delta_1$, $D_2 \leftrightarrow \Delta_2$, $D_3 \leftrightarrow \Delta_3$. There are 270 complex hexagons.*

Proof. We showed in lemma 6 that there are 1620 quadrilaterals Q. Since each quadrilateral Q determines a unique complex hexagon K, and each complex hexagon has 6 quadrilaterals Q as vertices, there are $270 = 1620/6$ complex hexagons.

We shall exhibit a single complex hexagon explicitly, and we will see that the diagonals of it form a disjoint triad of nonsingular pairs. Clearly, distinct triads are the diagonals of distinct complex hexagons. Since there are 270 such triads, these are then in bijective correspondence with the complex hexagons. We can most economically exhibit a complex hexagon by giving the pairs $\{h_i, h_{i'}\}$, $i = 0, \ldots, 5$. We write

$$\{h, h'\} = \{l_1 \wedge l_2, l'_1 \wedge l'_2\} \text{ with } l_j \subset h,\ l'_j \subset h',\ j = 1, 2.$$

With this notation:

$$\begin{aligned}
\{h_0, h'_0\} &= \{(1000) \wedge (0001), (0100) \wedge (0010)\} \\
\{h_1, h'_1\} &= \{(1000) \wedge (0100), (0010) \wedge (0001)\} \\
\{h_2, h'_2\} &= \{(1100) \wedge (0012), (1200) \wedge (0011)\} \\
\{h_3, h'_3\} &= \{(1010) \wedge (0102), (1020) \wedge (0101)\} \\
\{h_4, h'_4\} &= \{(1010) \wedge (0101), (1020) \wedge (0102)\} \\
\{h_5, h'_5\} &= \{(1001) \wedge (0110), (1002) \wedge (0120)\}
\end{aligned}$$

This complex hexagon has diagonals

$$D_{14} \leftrightarrow \Delta^{37},\ D_{25} \leftrightarrow \Delta^{26},\ \text{and } D_{30} \leftrightarrow \Delta^{1}.$$

Note that these three nonsingular pairs are indeed disjoint, and are part of the nsp-spread σ^1. $\boxed{QED}$

Remark 7. Note that any triad of disjoint nonsingular pairs $\{\Delta_1, \Delta_2, \Delta_3\}$ extends to a unique nsp-spread $\sigma = \{\Delta_1, \ldots, \Delta_5\}$ and so has a unique complementary duad of disjoint nonsingular pairs $\{\Delta_4, \Delta_5\}$. (For example, $\{\Delta^{44}, \Delta^{45}\}$ is the complementary duad to $\{\Delta^1, \Delta^{26}, \Delta^{37}\}$.) Since a duad of disjoint nonsingular pairs determines a unique hyperboloid H, we have a bijective correspondence between complex hexagons K and hyperboloids H. Coble argues in a reverse way to us. He simply states that there are 270 complex hexagons, as described in lemma 9, and uses this to conclude that there are $1620 = 270 \cdot 6$ quadrilaterals Q.

Remark 8. Let $C(27, 45)$ be the finite geometry whose points are given by lines on, and whose lines are given by the tritangents to, a nonsingular cubic surface in projective 3-space (or equivalently, whose points are the nonsingular pairs and whose lines are the nsp-spreads in projective 3-space over $\mathbf{F}_3$), with incidence relation that a point is on a line if the corresponding line is contained in the corresponding tritangent plane (or, equivalently, if the nonsingular pair is an element of the nsp-spread). Consider

$$\mathrm{Aut}\,(C(27, 45))\,,$$

the automorphism group of the geometry. Then, as was observed by Jordan [22], this group has order 51840, and so it must be our group $\tilde{G}$. There is an imbedding of $\tilde{G}$ into S_{27} given by its action on points, and the image of this imbedding is contained in the alternating group A_{27}. There is also an imbedding of $\tilde{G}$ into S_{45} given by its action on lines, and the image of this is *not* contained in A_{45}. (Compare with parts a) and b) of proposition 2.) The inverse image of A_{45} under this imbedding is our group G, the unique simple group of order 25,920. (Thus G should rightly be called “the group of even automorphisms of the 45 tritangent planes to a nonsingular cubic surface” but the name “group of even automorphisms of the 27 lines on a cubic surface” has stuck.) The group $\tilde{G}$ is known to be isomorphic to $W(E_6)$, the Weyl group of E_6, and $\tilde{G}$ is analyzed from that viewpoint in [12], [8]. See also section 8..

5. Maximal subgroups

In this section we will let G denote the algebraic group $\mathbf{Sp}(4)$ defined over the field $\mathbf{F}_3$, so that the group formerly denoted by G is now the group of rational points $G(\mathbf{F}_3)/\pm 1$.

$G(\mathbf{F}_3)/\pm 1$ has 5 maximal subgroups of index 27, 36, 40, 40, 45, respectively. Each of these forms one conjugacy class, and we have identified each of them with the stabilizer subgroup $P(z, \mathbf{F}_3) \subset G(\mathbf{F}_3)/\pm 1$ for a certain configuration z in $\mathbf{P}(V)$, where V is the 4-dimensional symplectic vector space over $\mathbf{F}_3$. The configurations are $z = \sigma,\ \theta,\ l,\ h,\ \Delta$ (nsp-spread, double-six, point, isotropic line, nonsingular pair), respectively. In this section and the next two we study these subgroups more closely. In particular, each of them will be shown to be of the form $H(\mathbf{F}_3)/\pm 1$ for an algebraic subgroup $H \subset \mathbf{Sp}(4)$ over $\mathbf{F}_3$. It is well-known that G is essentially isomorphic with $\mathbf{SO}(5)$, a fact visible at the level of Dynkin diagrams ($B_2 = C_2$). The symplectic descriptions of the maximal subgroups has already been given, and so we will present a corresponding orthogonal picture. Much of this material seems to be known; the Atlas [7] provides both symplectic and orthogonal interpretations for some of the maximal subgroups. Unfortunately, it provides no references for any of this, and in any case it does not describe all the maximal subgroups this way.

We recall that G has 4 conjugacy classes of parabolic subgroups: G itself, and the stabilizers of a point, isotropic line, and maximal isotropic flag, or, in the notations introduced, $P(\emptyset),\ P(l),\ P(h),\ P(l \subset h)$. The last one is a Borel subgroup. The dimensions of these algebraic groups are respectively, 10, 7, 7, 6. Like all parabolic groups, each of these has a Levi decomposition $P = U \ltimes M$ as a semidirect product of a unipotent and a reductive part. The reductive factors are respectively

$$\mathbf{Sp}(4), \quad \mathbf{GL}(1) \times \mathbf{SL}(2), \quad \mathbf{GL}(2), \quad \mathbf{GL}(1) \times \mathbf{GL}(1).$$

These parabolics are well-known, so we will not explicate them in any more detail. (For a description, see for instance [25].) Returning to the maximal subgroups, we have:

$P(l, \mathbf{F}_3)/\pm 1$. This is a maximal subgroup of index 40.

$P(h, \mathbf{F}_3)/\pm 1$. This is a maximal subgroup of index 40.

$P(\Delta, \mathbf{F}_3)/\pm 1$. This is a maximal subgroup of index 45, the stabilizer of a nonsingular pair. For reasons that will become apparent later, we refer to the Δ as a *split* nonsingular pair. The structure of $P(\Delta)$ as an algebraic group is easily found by considering the special Δ of the form

$$\{1000 \wedge 0010, 0100 \wedge 0001\}.$$

namely,

$$P(\Delta) \cong (\mathbf{SL}(2) \times \mathbf{SL}(2)) \ltimes \mathbf{Z}/2$$

where the $\mathbf{Z}/2$-factor acts by the interchange of factors $\mathbf{SL}(2)$ (see the proof of theorem 3). As an algebraic group, this has dimension 6.

$P(\sigma,\, \mathbf{F}_3)/\pm 1$. This is a maximal subgroup of index 27, the stabilizer of an nsp-spread. As an abstract group this has been described in lemma 3. It turns out that the corresponding algebraic group is the constant finite group with this value, so has dimension 0. This will become clear in the orthogonal description.

$P(\theta,\, \mathbf{F}_3)/\pm 1$. This is a maximal subgroup of index 36, the stabilizer of a double-six. In lemma 2 this was shown to be the symmetric group on six letters in a natural way. The structure of this as an algebraic group is not apparent in this view. This can also be identified with the the stabilizer of a *twisted* or *non-split* nonsingular pair, and in this interpretation, one sees that $P(\theta) \cong \Sigma\mathbf{L}_{\mathbf{F}_9/\mathbf{F}_3}(2)$, and so has dimension 6. This will be explained in the next two sections.

6. $\Sigma\mathbf{p}$

Let F be a field and E a Galois extension of F of degree n with Galois group $\Gamma = \mathrm{Gal}(E/F)$. Let G be an algebraic group defined over F. Then Γ acts on the set of E-rational points $G(E)$, so we may form the semi-direct product

$$H(E) \;=\; G(E) \ltimes \Gamma$$

Notice that if G is a linear algebraic group, then this defines a linear algebraic group H defined over F. Indeed, this is the semi-direct product of Weil's restriction-of-scalars

$$\mathrm{Res}_F^E(G)$$

with the constant group Γ, where the Galois action is by F-group automorphisms.

Definition 14. In the above situation, if $G = \mathbf{Sp}(2g)$, we set $H(E) = \Sigma\mathbf{p}(2g,\; E)$. In case $g = 1$, we write $\Sigma\mathbf{L}(2,\; E)$ for $\Sigma\mathbf{p}(2,\; E)$.

We remark that the notation is slightly ambiguous in that the symbol $\Sigma\mathbf{p}(2g,\; E)$ depends not only on E but on a Galois extension (see remark 9); in all cases considered here, the Galois extension will be clear.

Proposition 5. *Let E/F be a Galois extension of degree n. Then there is a natural imbedding*

$$\Sigma\mathbf{p}(2g,\; E) \;\hookrightarrow\; \mathbf{Sp}(2gn,\; F)$$

well-defined up to conjugacy.

Proof. If Z is a finite-dimensional K-vector space with a non-degenerate K-bilinear form θ, we let $\mathrm{Aut}_K(Z,\; \theta)$ denote the group of isometries, ie., the K-linear automorphisms of Z fixing θ.

In our case, let V be a $2g$-dimensional E-vector space with a non-degenerate alternating form ψ. Since all these are isomorphic, we may choose $V = E^{2g}$, and

$\psi(x,\, y) = {}^t xJy$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that for this choice, $\gamma(\psi(x,\, y)) = \psi(\gamma(x),\, \gamma(y))$ for every $\gamma \in \Gamma = \mathrm{Gal}(E/F)$.

We may regard V as a $2gn$-dimensional F-vector space, and equip it with the alternating form

$$\varphi(x,\, y) \;=\; \mathrm{Tr}_{E/F}(\psi(x,\, y)).$$

As ψ is non-degenerate, for fixed y, $x \mapsto \psi(x,\, y)$ is a surjective map $V \to E$, and as E/F is separable, $\mathrm{Tr}_{E/F} : E \to F$ is surjective, and so the composition $x \mapsto \varphi(x,\, y)$ is surjective onto F, hence φ is non-degenerate.

Clearly, every E-linear map preserving ψ defines an F-linear map preserving φ, so we have an injection

$$\mathbf{Sp}(2g,\ E) \;\cong\; \mathrm{Aut}_E(V,\ \psi) \longrightarrow \mathrm{Aut}_F(V,\ \varphi) \;\cong\; \mathbf{Sp}(2gn,\ F).$$

Since the extreme right and left-hand identifications depend only on the choice of symplectic bases, the isomorphism is well-defined up to conjugation. We wish to extend this imbedding to the larger group $\Sigma\mathbf{p}(2g,\ E)$, and for this it will suffice to show that the elements of the Galois group Γ act so as to preserve φ. But

$$\begin{aligned} \varphi(\gamma(x),\, \gamma(y)) &= \mathrm{Tr}_{E/F}(\psi(\gamma(x),\, \gamma(y))) \\ &= \mathrm{Tr}_{E/F}(\gamma\psi(x,\, y)) \\ &= \mathrm{Tr}_{E/F}(\psi((x,\, y)) \;=\; \varphi(x,\, y) \end{aligned}$$

as required. $\boxed{QED}$

Corollary 1. *The imbedding in the above theorem induces an imbedding*

$$P\Sigma\mathbf{P}(2g,\ E) \;=\; (\mathbf{Sp}(2g,\ E)/\pm 1) \ltimes \Gamma \hookrightarrow P\mathbf{Sp}(2gn,\ F)$$

Remark 9. The above construction generalizes to an F-conjugacy class of imbeddings of algebraic groups over F:

$$\Sigma\mathbf{p}_{E/F}(2g) \hookrightarrow\; \mathbf{Sp}(2gn)$$

where by definition

$$\Sigma\mathbf{p}_{E/F}(2g,\ R) \;=\; \mathbf{Sp}(2g,\ R \otimes_F E) \ltimes \Gamma$$

on the category of F-algebras R. In our previous notation we have

$$\Sigma\mathbf{p}_{E/F}(2g,\ F) = \Sigma\mathbf{p}(2g,\ E).$$

Corollary 2. *Under the above construction, the image of* $P\Sigma\mathbf{L}(2,\ \mathbf{F}_9)$ *in* $P\mathbf{Sp}(4,\ \mathbf{F}_3)$ *is the stabilizer of a double-six.*

Proof. This image is a subgroup of order 720, and there is a unique conjugacy class of these, the stabilizers of the double-sixes. $\boxed{QED}$

Remark 10. From the corollary we recover the isomorphisms

$$P\Sigma\mathbf{L}(2,\ \mathbf{F}_9) \cong S_6 \quad \text{and} \quad P\mathbf{SL}(2,\ \mathbf{F}_9) \cong A_6.$$

See [14, Section 6.4(vi)].

Remark 11. There is a useful variant of the construction in proposition 5. Let q be an odd prime power, and suppose that $F = \mathbf{F}_q$, $E = \mathbf{F}_{q^n}$, $K = \mathbf{F}_{q^{2n}}$. $\mathrm{Gal}(K/F)$ is generated by the Frobenius $x \to x^q$. Set $\overline{x} = x^{q^n}$. The fixed field of $x \to \overline{x}$ is E. Let $c \in E$ be any element such that $\overline{c} = -c$. (These exist; take any $d \in K,\ d \notin E$, and let $c = d - \overline{d}$.) Define

$$\psi(x,\ y)\ =\ \mathrm{Tr}_{K/E}(c\,x\,\overline{y}),$$

which is easily shown to be an E-symplectic nonsingular bilinear form on $V = K$. Then

$$\varphi(x,\ y)\ =\ \mathrm{Tr}_{E/F}(\psi(x,\ y))\ =\ \mathrm{Tr}_{K/F}(c\,x\,\overline{y})$$

is an F-alternating nonsingular bilinear form on V. We are thus in the situation of proposition 5, except that ψ is not Galois invariant. In any case, the first part of the proposition goes through and gives an imbedding

$$\mathbf{Sp}(2g,\ \mathbf{F}_{q^n})\ \hookrightarrow\ \mathbf{Sp}(2gn,\ \mathbf{F}_q).$$

Applied to the case $q = 3, n = 2, g = 1$ this gives rise to an injection

$$P\mathbf{SL}(2,\ \mathbf{F}_9)\ \hookrightarrow\ P\mathbf{Sp}(4,\ \mathbf{F}_3).$$

This imbedding is useful for constructing elements of $P\mathbf{Sp}(4,\ \mathbf{F}_3)$ with certain properties. An example is the following. Note that $\mathbf{F}_{81} = \mathbf{F}_3(\zeta)$ where ζ is a primitive 10^{th} root of unity. The transformation $x \to \zeta x$ of $V = K$ is $F = \mathbf{F}_3$-symplectic:

$$\begin{aligned}\varphi(\zeta x,\ \zeta y) &= \mathrm{Tr}(c\,\zeta\,x\,(\overline{\zeta\,y})) \\ &= \mathrm{Tr}(c\,\zeta\,x\,\zeta^9\,\overline{y})\ =\ \mathrm{Tr}(c\,x\,\overline{y})\ =\ \varphi(x,\ y).\end{aligned}$$

Expressing the action of ζ in a symplectic basis gives an element of order 10 in $\mathbf{Sp}(4,\ \mathbf{F}_3)$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Now recall that $\mathbf{SL}(2)$, as an algebraic group over $\mathbf{F}_q$, has two conjugacy classes of maximal tori, the split torus $T \cong \mathbf{G}_m$ consisting of the diagonal matrices, and the anisotropic torus T' consisting of the elements of norm 1 inside $\mathrm{Res}_{\mathbf{F}_q}^{\mathbf{F}_{q^2}}(\mathbf{G}_m)$. (Here, norm = $\mathrm{Norm}_{\mathbf{F}_{q^2}/\mathbf{F}_q}$.) Applied to $q = 9$, we see that

$$\begin{aligned} T'(\mathbf{F}_9) &= \{\text{elements of norm 1 in } \mathbf{F}_{81}^\times\} \\ &= \langle \zeta \rangle = \text{the group generated by } \zeta. \end{aligned}$$

Under the imbedding $\mathrm{Res}_{\mathbf{F}_3}^{\mathbf{F}_9}(\mathbf{SL}(2)) \to \mathbf{Sp}(4)$, $\mathrm{Res}_{\mathbf{F}_3}^{\mathbf{F}_9}(T')$ maps to the maximal torus T_1 of $\mathbf{Sp}(4)$ over $\mathbf{F}_3$ consisting of the elements of $\mathrm{Res}_{\mathbf{F}_3}^{\mathbf{F}_{81}}(\mathbf{G}_m)$ such that $\overline{x} = x^{-1}$, and $\langle \zeta \rangle$ is the group of $\mathbf{F}_3$-rational points of this maximal torus.

The same idea applied to the image of T under this imbedding yields elements of order 8, since $T(\mathbf{F}_9) = \mathbf{F}_9^\times$ is cyclic of order 8. For example we obtain in this way the element

$$\begin{pmatrix} -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

The image of $\mathrm{Res}_{\mathbf{F}_3}^{\mathbf{F}_9}(T)$ is also a maximal torus T_2 of $\mathbf{Sp}(4)$ over $\mathbf{F}_3$, isomorphic with $\mathrm{Res}_{\mathbf{F}_3}^{\mathbf{F}_9}(\mathbf{G}_m)$. There are three other conjugacy classes of maximal tori in $\mathbf{Sp}(4)$ over a finite field. There is the group of diagonal symplectic matrices $T_3 \cong \mathbf{G}_m \times \mathbf{G}_m$, and there is $T_4 = T' \times T'$ obtained from the canonical imbedding $\mathbf{SL}(2) \times \mathbf{SL}(2) \to \mathbf{Sp}(4)$, where this time T' is regarded as a group-scheme over $\mathbf{F}_3$. Finally there is $T_5 = \mathbf{G}_m \times T'$.

Note that the two explicit matrices we have constructed are in fact in $\mathbf{Sp}(4,\ \mathbf{Z})$ and have the same order as their mod 3 reductions.

7. SO(5) picture

We now reinterpret our results in the language of an inner product space of dimension 5 over $\mathbf{F}_3$. Much of what follows has a large overlap with the work of Edge (see [10, 11, 12]). We will not systematically establish all the points of contact between his work and ours, but we will mention some of them. In [11, sect. 16, pp. 141–142] Edge gives a detailed comparison between certain configurations he discusses in connection with the geometry of quadrics over $\mathbf{F}_3$ and the configurations described by Baker and Todd based on the Burkhardt quartic. In view of the dictionary in section 3., these can then be related to our configurations. Edge uses the language of classical projective geometry in his work. In what follows we will describe those configurations of interest to us, but using the language of modern algebraic geometry. A general introduction to quadrics over finite fields can be found in chapter 5 of [14].

First some generalities on quadrics: We consider a smooth quadric $Q \subset \mathbf{P}^N$ over a field F of characteristic not equal to 2. This is the locus of zeros of a quadratic form in $N+1$ variables, $Q(x)=0$. We let $Q(x,\ y) = {}^t xQy$ denote the associated symmetric bilinear form. Now Q may be regarded as a square matrix, and then the nonsingularity of the quadric is expressed by $\det Q \neq 0$. This multiple use of the symbol Q should cause no confusion. For example, $x \in Q \Leftrightarrow Q(x)=0$. Also, we often identify a K-rational point in projective space with a vector $x \in K^{N+1}$ that represents it. The points of the quadric $Q(x)=0$ are called *isotropic* vectors (or Q-isotropic vectors). If $x \notin Q$, we say that x is *anisotropic*; in this case the nonzero value $Q(x)$ is not well defined, but its square-class in $K^\times / (K^\times)^2$ is. If $Q(x)$ belongs to the square-class α, we call x an α-point. In this way we obtain a partition of the anisotropic K-rational vectors into sets indexed by the square-classes of K. This will only be used here in the case where the ground-field F is a finite field of odd characteristic. Then there are only two square-classes, the class of 1 (the squares) and the nonsquares. If $Q(x)$ is a nonzero square, we call x a *plus-point*; if it is a nonsquare then we call it a *minus-point*. It is clear that the orthogonal group of the quadratic form $\mathbf{O}(Q)$ operates so as to preserve the sets of isotropic vectors and of the α-points for any fixed α.

Given a point $v \in \mathbf{P}^N$, the hyperplane

$$H(Q,\ v)\ =\ \{w \in \mathbf{P}^N\ :\ Q(w,\ v)=0\}$$

consists of vectors orthogonal to v, and the intersection

$$\Pi(Q,\ v)\ =\ Q \cap H(Q,\ v)$$

is called the *polar* with respect to v. This is a quadric of dimension one less. If $v \in Q$, the polar is the intersection of Q with the tangent plane to Q at v, and is a cone with vertex v. If $v \notin Q$ then $H(Q,\ v)$ is transverse to Q and the polar is nonsingular. The tangent planes $T_P Q$ as P varies over $\Pi(Q,\ v)$ all pass through v.

Recall also that if $v \notin Q$ the map

$$r(Q,\ v)\ :\ x \mapsto x - 2\frac{Q(x,\ v)}{Q(v,\ v)}v$$

is the orthogonal reflection in the plane $H(Q,\ v)$. It sends $v \to -v$ as a vector, hence fixes it in projective space. It fixes the plane $H(Q,\ v)$ as well.

Our aim is to explicate the isogeny between $\mathbf{SO}(5)$ and $\mathbf{Sp}(4)$ in geometric form. This discussion is taken from [13, p. 278]. Let $V = \mathbf{A}^4$, and let $\mathbf{Gr}(2,\ 4)$

be the Grassmannian variety of 2-dimensional vector subspaces of V. This is also the Grassmannian of lines in $\mathbf{P}^3 = \mathbf{P}(V)$, sometimes denoted $\mathbf{G}(1,3)$. It is isomorphic to a quadric in $\mathbf{P}^5 = \mathbf{P}\left(\bigwedge^2 V\right)$, via the Plücker imbedding. If L is a line generated by vectors $x,\ y$, the map is

$$\mathbf{Gr}(2,\ 4) \ni L \mapsto p(L) = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$$

where

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$$

These satisfy Plücker's relation

$$Q(p) \ = \ p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} \ = \ 0$$

and this gives rise to the isomorphism $\mathbf{Gr}(2,\ 4) \cong Q$. All of this works in the category of schemes over $\mathbf{Z}$.

If V has a non-degenerate alternating bilinear form φ, we can choose a basis in which φ is the standard symplectic form, and the condition that L be φ-isotropic, $\varphi(x,\ y) = 0$, is that $p_{02}+p_{13} = 0$. In other words, the variety $\mathrm{Iso}(V,\ \varphi)$ of isotropic 2-planes in V (= maximal isotropic subspaces) is isomorphic with the quadric q in $(p_{01}, p_{02}, p_{03}, p_{12}, p_{23})$-space with equation

$$q(p) \ = \ p_{02}^2 + p_{01}p_{23} + p_{03}p_{12} \ = \ 0.$$

In the language of polars, this can be stated as $q = \Pi(Q,\ j)$, where $j = (0,1,0,0,1,0)$. If P is the parabolic subgroup of $\mathbf{Sp}(4)$ stabilizing a maximal isotropic subspace (in previous notation, $P = P(h)$), then

$$\mathrm{Iso}(V,\ \varphi) \ = \ \mathbf{Sp}(4)/P$$

where the quotient is understood in the category of sheaves for the flat topology. This works over $\mathrm{Spec}(\mathbf{Z})$ for the standard form φ.

Let $T \in \mathbf{Sp}(V,\ \varphi) = \mathbf{Sp}(4,\ F)$. Then T acts on (the F-rational points of) $\mathbf{Gr}(2,4)$, $\mathrm{Iso}(V,\ \varphi)$, and also $\wedge^2 T$ acts on $\wedge^2 V$. The Plücker imbedding is equivariant for these actions, and the symplectic group by its definition preserves the hyperplane $p_{02} + p_{13} = 0$. This means that we have an action of $\mathbf{Sp}(4)$ on $(p_{01}, p_{02}, p_{03}, p_{12}, p_{23})$-space preserving the quadric q. This gives a homomorphism from the symplectic group to the group of orthogonal similitudes of the form q, but as the symplectic group has no rational character, the image lands in $\mathbf{O}(q)$, and in fact in $\mathbf{SO}(q)$ because the symplectic group is connected. We therefore have a homomorphism

$$\mathbf{Sp}(4) \ \longrightarrow \ \mathbf{SO}(q).$$

This is is an isogeny with kernel $\mathbf{Z}/2$ (really μ_2), the spin covering of the special orthogonal group. Thus we have an isomorphism of algebraic groups $\mathbf{PSp}(4) = \mathbf{Sp}(4)/\mu_2 \cong \mathbf{SO}(q)$. Now assume that our ground ring is a perfect field F. Taking F-rational points, we get the exact sequence in Galois cohomology:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{Sp}(4,\, F) \;\longrightarrow\; \mathbf{SO}(q,\, F) \;\xrightarrow{N}\; \mathrm{H}^1(\mathrm{Gal}(\overline{F}/F),\; \mathbf{Z}/2).$$

The map N is the spinorial norm.

This gives an isomorphism

$$P\mathbf{Sp}(4,\, F) = \mathbf{Sp}(4,\, F)/\pm 1 \;\cong\; \{g \in \mathbf{SO}(q,\, F) : N(g) = 1\}.$$

Note that we do not have $\mathbf{PSp}(4,\, F) = \mathbf{Sp}(4,\, F)/\pm 1$. For instance if F is a finite field the right-hand side in the exact sequence is $\mathbf{Z}/2$, and the image of $P\mathbf{Sp}(4,\, F)$ is of index 2 inside $\mathbf{PSp}(4,\, F) = \mathbf{SO}(5,\, F)$ as the group of elements of spinorial norm 1. When $F = \mathbf{F}_3$ this is a finite simple group, which in the notations of the Atlas is denoted $\Omega_5^+(3) = P\Omega_5^+(3) = O_5^+(3)$.

We will sketch the construction of the reverse correspondence. Let $F_{1,q}$ be the Fano variety of lines on the 3-dimensional quadric q. It is known that $F_{1,q} \cong \mathbf{P}^3 = \mathbf{P}(V)$ as follows. Each line $m \subset q$ is a collection $\{l_x : x \in m\}$ of φ-isotropic lines $l_x \subset \mathbf{P}\,(V)$. In fact, this is the pencil of isotropic lines through a unique point $v \in \mathbf{P}\,(V)$. If $v \in \mathbf{P}\,(V)$ is any point, the hyperplane

$$H(\varphi,\; v) \;=\; \{w \in \mathbf{P}\,(V) : \varphi(v,\; w) = 0\}$$

contains v and the isotropic lines through v are exactly the lines through v contained in the plane $H(\varphi,\; v)$, and any such hyperplane defines a line in q via the Plücker imbedding. The isomorphism $F_{1,q} \cong \mathbf{P}(V)$ sends m to the axis v of the pencil $\{l_x : x \in m\}$. Now any element of $\mathbf{O}(q)$ preserves q, and the lines on it, and therefore induces an automorphism of $\mathbf{P}(V)$ preserving all the hyperplanes $H(\varphi,\; v)$, therefore giving an element of the group of symplectic similitudes, and this gives the map in the other direction

$$\mathbf{SO}(q) \;=\; \mathbf{PSO}(q) \;\longrightarrow\; \mathbf{PSp}(4).$$

We have only sketched these arguments, because the equality

$$\mathbf{Spin}(q) \;\cong\; \mathbf{Sp}(4)$$

is classical. One can find a complete proof of the equality in the generality claimed here, namely in the category of group schemes over $\mathbf{Z}$, in [21, pp. 28-32].

We let $W = H(Q,\ j)$ denote the hyperplane of isotropy, with coordinates $(p_{01}, p_{02}, p_{03}, p_{12}, p_{23})$.

Next we turn to the nonsingular pairs $\Delta = \{\delta,\ \delta^{\perp}\}$. The Plücker coordinates $p(\delta),\ p(\delta^{\perp})$ lie on $Q = \mathbf{Gr}(2, 4)$ but not on the hyperplane of isotropy $p_{02}+p_{13} = 0$, that is, not on $H(Q,\ j)$. In fact they are reflected points of each other relative to this hyperplane, that is

$$r(Q,\ j)p(\delta)\ =\ p(\delta^{\perp})$$

This is easily seen for $\delta = \mathrm{span}(e_1,\ e_3)$, $\delta^{\perp} = \mathrm{span}(e_2,\ e_4)$ in a standard symplectic basis $\{e_1, e_2, e_3, e_4\}$, and the general case follows from this because the symplectic group acts transitively on the Δ's. Let $w = w(\Delta) \in H(Q,\ j)$ be the unique point of intersection of $H(Q,\ j)$ with the line connecting $p(\delta),\ p(\delta^{\perp})$. In fact, start with any F-rational point $w \in H(Q,\ j) - q$ and move parallel to j in the space $\wedge^2 V$ until you hit the quadric Q and you get the Plücker coordinates of a nonsingular pair. This intersection point is of the form $w + t\,j$, and satisfies $Q(w+t\,j,\ w+t\,j) = 0$. Remembering that $Q(w,\ j) = 0$, and $Q(j,\ j) = -1$, this becomes $t^2 = q(w,\ w)$. As to the field of definition of the vector $w(\Delta)$ there are two possibilities (F a finite field of odd characteristic):

1. w is a plus point. In that case $t^2 = q(w,\ w)$ has a solution in F. Since the Plücker coordinates $w = p(\delta)$ are F-rational, the line δ, and also $\delta^{\perp}$, is an F-rational line in V. This $\Delta = \{\delta,\ \delta^{\perp}\}$ is what has been previously called a nonsingular pair, but will now be called a split nonsingular pair.

2. w is a minus point. In that case $\sqrt{q(w,\ w)}$ defines a nontrivial quadratic extension E of F. The $\delta,\ \delta^{\perp}$ are a Galois-conjugate pair of E-rational lines in V: $\overline{\delta} = \delta^{\perp}$. We say that Δ is a twisted, or nonsplit, nonsingular pair.

For example, when $F = \mathbf{F}_3$, there are 45 split nonsingular pairs, and 36 twisted nonsingular pairs. We will show that these 36 correspond naturally to the double-sixes.

We denote by $\Delta(w) = \{\delta(w),\ \delta^{\perp}(w)\}$ the nonsingular pair constructed by the above process from $w \in H(Q,\ j) - q$. For any such w consider the polar variety $q_w := \Pi(q,\ w)$ inside W. Since $w = \delta + t\,j$ and $w = \delta^{\perp} - t\,j$ for some $t \neq 0$, we see that

$$q_w\ =\ Q \cap H(Q,\ \delta) \cap H(Q,\ \delta^{\perp})$$

where we are using the abbreviation $H(Q,\ \delta) = H(Q,\ p(\delta))$. This smooth quadric surface is isomorphic over the algebraic closure $\overline{F}$ to a product of projective lines. This isomorphism does not necessarily hold over the field F. In fact,

1. w is a plus-point $\Rightarrow\ q_w \cong \mathbf{P}^1_F \times \mathbf{P}^1_F$.
2. w is a minus-point $\Rightarrow\ q_w \cong \mathrm{Res}^E_F\left(\mathbf{P}^1_E\right)$ where $E = F(\sqrt{q(w,w)})$.

Note in particular that $q_w(F)$ is respectively $\mathbf{P}^1(F) \times \mathbf{P}^1(F)$ and $\mathbf{P}^1(E)$. Since $q_w \subset q = \mathrm{Iso}(V,\ \varphi)$, the points of these quadrics correspond to certain collections of isotropic lines in V. These are easily identified:

1. If w is a plus-point, then $\Delta(w) = \{\delta,\ \delta^\perp\}$ for a pair of F-rational anisotropic lines. Take any F-rational points $a \in \delta$, $b \in \delta^\perp$. Then the line ab is isotropic, and the $\mathbf{P}^1(F) \times \mathbf{P}^1(F)$ indexed collection of isotropic lines so obtained is $q_w(F)$. Generalizing this construction to R-valued points for any F-algebra R gives the isomorphism of schemes $q_w \cong \delta(w) \times \delta^\perp(w)$.

2. If w is a minus-point, then $\Delta(w) = \{\delta,\ \delta^\perp\}$ for a pair of E-rational anisotropic lines with $\overline{\delta} = \delta^\perp$. Take any E-rational point $a \in \delta$. Its Galois conjugate $\overline{a}$ lies on $\delta^\perp$. The line $a\overline{a}$ is isotropic, and F - rational since fixed by Galois. The $\mathbf{P}^1(E)$ indexed collection of isotropic lines obtained in this way is $q_w(F)$. Generalizing this construction to R-valued points gives the isomorphism of schemes $q_w \cong \mathrm{Res}^E_F\left(\delta(w)\right) \cong \mathrm{Res}^E_F\left(\delta^\perp(w)\right)$.

Let us call the lines constructed above *cross-lines*. Given two distinct cross-lines on a nonsingular pair one of two alternatives holds: either they are skew or they intersect in a point necessarily on Δ. This can be seen as follows: If there were an intersection, not on either δ or $\delta^\perp$, the plane spanned by these cross-lines would contain both δ and $\delta^\perp$, since it would contain two points on each of these. But this contradicts $\delta \oplus \delta^\perp = V$, thinking of the projective lines now as planes through the origin in V. However, for a twisted nonsingular pair, the second alternative cannot occur. The reason is that the cross-lines are F-rational lines, and hence the intersection, if it existed, would be an F-rational point, and by the previous remark, on δ or $\delta^\perp$. But neither δ nor $\delta^\perp$ has any F-rational points because $\delta \cap \overline{\delta} = \delta \cap \delta^\perp = \emptyset$.

These observations are valid over any field of characteristic not 2, with the notion of plus and minus points replaced by α-points, but we will only need this for $\mathbf{F}_3$, where the statements above can be verified by direct calculation for any one representative, the transitivity of the symplectic group giving the generality of the claim. It is worth observing that over any finite field of odd characteristic, the quadratic forms Q and q are equivalent respectively to $Q_0 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2$ and $q_0 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$. The reason for this is that terms such as uv are equivalent to $x^2 - y^2$, so that Q and q are equivalent to sums and differences of squares, and the only invariant of a quadratic form over a finite field aside from its rank is its discriminant, which is -1 for Q and 1 for q.

There are 45 points in $\mathbf{P}^4(\mathbf{F}_3)$ where $q_0(x) = 1$ and 36 points where $q_0(x) = -1$. For the plus-point $w = (1, 0, 0, 0, 0)$ the quadric q_w is $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$, which has 16 points, reflecting $q_w(\mathbf{F}_3) = \mathbf{P}^1(\mathbf{F}_3) \times \mathbf{P}^1(\mathbf{F}_3)$. For the minus-point $w = (1, 1, 0, 0, 0)$ the quadric q_w is $2x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$, which has 10 points, reflecting $q_w(\mathbf{F}_3) = \mathbf{P}^1(\mathbf{F}_9)$.

When $F = \mathbf{F}_3$ we will identify the twisted nonsingular pairs with the double-sixes. Note that in this case each twisted nonsingular pair defines a set of 10 cross-lines, all disjoint, each containing four $\mathbf{F}_3$-rational points, hence constituting a partition of the 40 points of $\mathbf{P}^3(\mathbf{F}_3)$. We could call this an isotropic spread.

We now analyze the concept of an nsp-spread in the orthogonal setting. First consider the vector space $W = F^{N+1}$ with a quadratic form q. By a *base* we mean a collection $\{v_0,\ \ldots, v_N\}$ of plus-points of $\mathbf{P}(W)$ such that $q(v_i,\ v_j) = 0$ for $i \neq j$. This is almost the same thing as an orthonormal basis of W; we can choose liftings $\tilde{v}_i$ of v_i such that $q(\tilde{v}_i,\ \tilde{v}_i) = 1$, and the $\{\tilde{v}_i\}$ will indeed be an orthonormal basis.

Let $F = \mathbf{F}_3$, $W = H(Q,\ j)$, and let q be the quadratic form given by the restriction of the Plücker relation to the φ-isotropic 2-planes, as in the previous paragraphs. As we have mentioned, q is equivalent to a sum of 5 squares. In this case there are 27 bases. We will identify them with the 27 nsp-spreads. Recall the assignment

$$(\mathbf{P}(W) - q) \ni w \ \mapsto\ \Delta(w) \subset \mathbf{P}(V).$$

We have seen that this maps plus-points to split nonsingular pairs. The following lemma links the nsp-spreads to the bases:

Lemma 10. 1. *Over any field F of characteristic not equal to two, if w_1 and w_2 are q-anisotropic vectors, then*

$$q(w_1,\ w_2) = 0 \implies \Delta(w_1) \cap \Delta(w_2) = \emptyset.$$

2. *Over $F = \mathbf{F}_3$, the converse of the above implication is true.*

Proof. Given two anisotropic lines $\delta_1,\ \delta_2 \subset \mathbf{P}^3$, we have that $\delta_1 \cap \delta_2 = \emptyset \Rightarrow \delta_1^\perp \cap \delta_2^\perp = \emptyset$. Thus given two nonsingular pairs $\Delta_1 = \{\delta_1, \delta_1^\perp\}$ and $\Delta_2 = \{\delta_2, \delta_2^\perp\}$, either they are disjoint or they intersect in 2 points, i. e., after possibly exchanging a line with its dual, $\delta_1 \cap \delta_2 =$ point and $\delta_1^\perp \cap \delta_2^\perp =$ point. It is a general fact about Grassmannians that given two lines $l,\ m \subset \mathbf{P}^3$, then $l \cap m \neq \emptyset$ if and only if the line containing their Plücker coordinates $p(l),\ p(m) \in \mathbf{Gr}(2, 4) = Q$ is entirely contained in the Grassmannian. Recall also that the line determined by two points $x,\ y$ on a quadric Q is contained in Q if and only if they are orthogonal. Hence two lines $l,\ m$ are incident if and only if $Q(p(l),\ p(m)) = 0$, which we will abbreviate as $Q(l,\ m) = 0$.

Thus

$$\Delta_1 \cap \Delta_2 \neq \emptyset \Longleftrightarrow Q(\delta_1, \delta_2) = 0 \text{ or } Q(\delta_1, \delta_2^{\perp}) = 0.$$

The vectors $w_1 = w(\Delta_1)$ and $w_2 = w(\Delta_2)$ in $\mathbf{P}^4 = \mathbf{P}(H(Q,\ j))$ are determined by

$$w_i \ = \ \delta_i - \frac{Q(\delta_i,\ j)}{Q(j,\ j)}\, j$$

where we are identifying a line with its Plücker coordinate. Also

$$\delta_i^{\perp} \ = \ r(Q,\ j)\delta_i \ = \ \delta_i - 2\,\frac{Q(\delta_i,\ j)}{Q(j,\ j)}\, j.$$

One computes

$$q(w_1,\ w_2) \ = \ Q(w_1,\ w_2) \ = \ Q(\delta_1,\ \delta_2) - \frac{Q(\delta_1,\ j)Q(\delta_2,\ j)}{Q(j,\ j)}. \qquad (1)$$

Since δ_1 and δ_2 are anisotropic, $Q(\delta_1,\ j) \neq 0$ and $Q(\delta_2,\ j) \neq 0$. Notice that the expression on left-hand side depends only on the orthogonal projections $w_1,\ w_2$ of the vectors δ_1, δ_2 to the plane $H(Q,\ j)$. Since $\delta_i^{\perp}$ has the same projection to this plane as δ_i we can replace δ_i by $\delta_i^{\perp}$ for $i = 1$ or 2. We conclude from this that

$$\Delta_1 \cap \Delta_2 \neq \emptyset \Longrightarrow q(w_1, w_2) \neq 0$$

The contrapositive of this is part 1 of the lemma.

We cannot conclude the converse implication of this in general, but for the finite field with three elements we reason as follows. From $q(w_1, w_2) \neq 0$ we wish to conclude that $Q(\delta_1, \delta_2) = 0$ or $Q(\delta_1, \delta_2^{\perp}) = 0$, hence $\Delta_1 \cap \Delta_2 \neq \emptyset$. A simple calculation shows that

$$Q(\delta_1,\ \delta_2^{\perp}) \ = \ Q(\delta_1,\ \delta_2) + \frac{Q(\delta_1,\ j)Q(\delta_2,\ j)}{Q(j,\ j)} \qquad (2)$$

The second summand in both equations 1 and 2 is nonzero, as mentioned. If $q(w_1, w_2) \neq 0$ and $Q(\delta_1,\ \delta_2) \neq 0$ then the only possibilities for equation 1 are $2 - 1$ or $1 - 2$. But then equation 2 becomes $1 + 2 = 0$. $\boxed{QED}$

An nsp-spread is a mutually disjoint collection of 5 Δ's, and corresponds by the above to a collection of 5 mutually orthogonal plus - points in the 5-dimensional space W, which is a base by definition. It is clear that the stabilizer in $\mathbf{SO}(5, \mathbf{F}_3)$ of a base consists of the orthogonal matrices generated by the following matrices: the even permutation matrices of size 5, and the diagonal matrices with ± 1 down the diagonal, the product of these being 1. Conjugation by the permutation matrices acts as permutations of the diagonal entries in these diagonal generators. This gives a precision of the result in lemma 3, namely the stabilizer of an nsp-spread is the semi-direct product $(\mathbf{Z}/2)^4 \ltimes A_5$. Also as an algebraic group it is clear that the stabilizer of a base has the same matrices with coordinates in any extension of $\mathbf{F}_3$, so this stabilizer is the constant group, of dimension 0, with this value.

Finally we describe the double-sixes in the orthogonal language. In this paragraph, $F = \mathbf{F}_3$. The statements that follow were verified by computer calculations. As mentioned before, the quadratic form q is equivalent to a sum of 5 squares over any finite field of odd characteristic. We show that the minus-points correspond naturally to double-sixes. Let $B = \{e_1, \ldots, e_5\}$ be one of the 27 bases. This is essentially an orthonormal basis for q. Let $w = w(\Delta) = (w_1, \ldots, w_5) \in \mathbf{F}_3^5$ be a minus-point, expressed in this basis; given B, the coordinates w_i are ambiguous only up to $\pm w_i$ and permutations. Since $q(w) = w_1^2 + \ldots + w_5^2 = 2 \bmod 3$, it is clear that one of two possibilities occur: (1) exactly two vectors e_i in B have $q(w, e_i) \neq 0$ or (2) all five vectors in B have $q(w, e_i) \neq 0$. If the first case occurs then we say that B is a 2-base relative to w, and in the second case that B is a 5-base relative to w. Given w this gives a division of the 27 bases into the 2-bases and the 5-bases. It is clear that the stabilizer subgroup of w inside $\mathbf{O}(5, \mathbf{F}_3)$ preserves this partition into 2- and 5-bases relative to w. For any minus-point there are fifteen 2-bases and twelve 5-bases. We claim that the twelve 5-bases arrange themselves into a double-six. In fact, consider the orthogonal reflection in the hyperplane $H(q,\ w) \subset \mathbf{P}^4(\mathbf{F}_3)$. This orthogonal transformation fixes w and hence acts on the twelve 5 - bases. This involution acts fixed-point free on the twelve and the orbits

$$\{B_1,\ B_1'\}, \{B_2,\ B_2'\}, \{B_3,\ B_3'\}, \{B_4,\ B_4'\}, \{B_5,\ B_5'\}, \{B_6,\ B_6'\}$$

are the six doublets in a double-six. Let us call the projective involutions induced by these θ_w. They are elements of the group denoted $PGO_5(3)$ in the Atlas. We remark that these 36 involutions, one for each minus-point, are exactly the 36 F-inversions that appear in [11, p. 145].

Here is another observation, due to Edge (see also [14, Section 6.4(ii)] and [15, Theorem 15.3.17]):

Proposition 6. *There are isomorphisms:* $PGO_4^-(3) \cong S_6$ *and* $PSO_4^-(3) \cong A_6$.

This appears in [10, p. 277]. Recall that $PGO_4^-(3)$ is the group of projective transformations induced from the linear transformations that preserve a non-degenerate quadratic form in 4 variables over $\mathbf{F}_3$ of Witt defect 1. Here we are following the notations and terminology of the Atlas.

Proof. We have already (see corollary 2 and remark 10) shown that the stabilizer inside $G = \Omega_5^+(3)$ of a double-six is isomorphic with the symmetric group S_6. We have identified the double-sixes with the minus points in the space $\mathbf{F}_3^5$ with the quadratic form $q = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$. Let w be any minus-point, eg. $(1,1,1,1,1)$. Then any element in the stabilizer G_w of w in G will fix the hyperplane $H(q,w)$ orthogonal to w, and induce an element in the orthogonal group of the quadratic form $q|w$ obtained by restricting q to $H(q,w)$. It is easily checked that $q|w$ has Witt defect 1, so we have a map $G_w \to PGO_4^-(3)$. This maps the subgroup $A_6 \cong G'_w$ consisting of elements of determinant one to $PSO_4^-(3)$. These two groups have the same cardinality, so it is an isomorphism by simplicity of A_6, and one also sees the isomorphism $G_w \cong PGO_4^-(3)$. $\boxed{QED}$

We can summarize the results of this section as the $\mathbf{Sp}(4) \sim \mathbf{SO}(5)$ - equivariant identifications in the table below. The stabilizers are algebraic subgroups in $\mathbf{Sp}(4)$, and their dimensions are given.

symplectic	orthogonal	stabilizer	dimension
isotropic line	point of q	parabolic $P(h)$	7
point	line in q	parabolic $P(l)$	7
split ns. pair	plus-point	$(\mathbf{SL}(2) \times \mathbf{SL}(2)) \ltimes \mathbf{Z}/2$	6
twisted ns. pair/dbl 6	minus-point	$\Sigma\mathbf{L}_{\mathbf{F}_9/\mathbf{F}_3}(2)$	6
nsp-spread	base	$2.\left((\mathbf{Z}/2)^4 \ltimes A_5\right)$	0

8. The Weyl group of E_6

In Coxeter [6], which is based on Edge [12], an isomorphism is established between the group $SO_6^-(2)$ and the Weyl group of the root system E_6. One also knows that this group is isomorphic with $PGO_5(3)$.This "exceptional" isomorphism will be explained below, but for now, we can identify the 36 involutions corresponding to the root hyperplanes in the root system E_6: they are the 36 involutions associated to the minus-points in our discussion of double-sixes in the previous section. We can also identify a set of six involutions corresponding to a basis for the root-system. Coxeter shows how to construct involutions he calls R_i, $i = 1, \ldots, 6$, associated to the nodes of the Dynkin diagram, and he identifies them with their action on the 27 lines on a cubic surface.

In a classical notation, these 27 lines are labeled a_i, b_i, c_{ij}, $1 \le i,j \le 6$, $i < j$. This notation presupposes a basic double-six has been chosen and the (a_i, b_i) $i = 1, \ldots, 6$ are the doublets in this double-six. For instance, take for a reference double-six entry 11 of Table 3. Thus, in the lexicographic ordering of the indices,

$$a_i = 1,\ 6,\ 8,\ 14,\ 22,\ 27$$
$$b_i = 5,\ 7,\ 3,\ 24,\ 26,\ 13$$
$$c_{ij} = 4,\ 2,\ 20,\ 18,\ 10,\ 9,\ 16,\ 11,\ 21,\ 12,\ 19,\ 17,\ 23,\ 15,\ 25$$

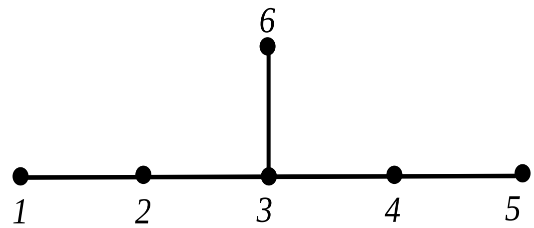

Figure 9. Dynkin diagram for E_6

In Table 3, the involution θ_w is identified with one of the double-sixes, and its effect on the nsp-spreads within it is to flip the top and bottom row of that entry. With this in mind we see that Coxeter's involutions correspond to the involutions in Table 3 as $R_1 \leftrightarrow \theta_1$, $R_2 \leftrightarrow \theta_{12}$, $R_3 \leftrightarrow \theta_{24}$, $R_4 \leftrightarrow \theta_{36}$, $R_5 \leftrightarrow \theta_{35}$, $R_6 \leftrightarrow \theta_{19}$.

There are 36 choices for the reference double-six. Once chosen there are 6! ways to identify the a_i with the first row in the double-six. But we could also flip the roles of the a_i and b_i. In all there are $36 \cdot 720 \cdot 2 = 51840$ ways to choose a generating set of 6 involutions like this. This is the right number: a choice of these 6 involutions is equivalent to the choice of a basis for the root system, or equivalently, of a Weyl chamber for E_6. It is known that the Weyl group acts simply transitively on the Weyl chambers, and as the Weyl group has 51840 elements, we get these many bases.

To close, we outline a conceptual proof of the isomorphism $SO_6^-(2) \cong PGO_5(3)$. This appears as an exercise in Bourbaki [2, Exercice 2, §4, Ch. VI, p. 228-229], but the method appears to have been discovered independently by Kneser [23], who used it to establish many other exceptional isomorphisms of finite groups of Lie type. We will use Bourbaki's notation, which does not agree with that of the Atlas. Let R be the root system of type E_6. We have the root lattice $Q(R) \cong \mathbf{Z}^6$ generated by R. This is contained in the lattice of weights $P(R)$. We denote the inner product on $V = Q(R) \otimes_{\mathbf{Z}} \mathbf{R}$ invariant under the Weyl group $W(R)$ by $(x \mid y)$. This is $\mathbf{Z}$-valued on $Q(R)$. We also let $A(R)$ be the group of all automorphisms of V that preserve R. It is known that $A(R) = W(R) \times \{1, -1\}$ in this case.

$P(R)/Q(R)$ is isomorphic with $\mathbf{Z}/3$. Thus, $E = Q(R)/3P(R)$ is a 5-dimensional vector space over $\mathbf{F}_3$. $(x \mid y)$ induces a quadratic form φ on E, since $(x \mid y) \in \mathbf{Z}$ for all $x \in Q(R)$, $y \in P(R)$. One can show that it is non-degenerate by direct computation. The group $\mathbf{SO}^+(\varphi)$ of orthogonal transformations of φ of determinant 1 and of spinorial norm 1 is known to be simple of order 25920. On the other hand, each element of $A(R)$ preserves $(x \mid y)$ and both lattices

$P(R)$, $Q(R)$. Thus we obtain a homomorphism $\lambda : A(R) \to \mathbf{O}(\varphi)$. One checks by direct computation that the reduction map $Q(R) \to E$ is carries R bijectively onto its image, from which it easily follows that λ is injective. A comparison of orders shows that it is an isomorphism.Thus $\lambda(W(R))$ is a subgroup of $\mathbf{O}(\varphi)$ of index 2. The group $\mathbf{O}(\varphi)/\mathbf{SO}^+(\varphi)$ is of type $(2,2)$. The subgroups of index 2 are

$$\{\pm 1\} \times \mathbf{SO}^+(\varphi), \qquad \mathbf{SO}(\varphi), \qquad \Omega(\varphi)$$

the last group being characterized by being of index two, not contained in $\mathbf{SO}(\varphi)$, and not containing -1. Clearly this is isomorphic with $P\mathbf{O}(\varphi)$. Note that $\Omega(\varphi)$ is isomorphic with the group we have denoted by $\tilde{G}$ in section 2.. This is the image of $W(R)$ under λ, because reflections have determinant -1 and it is known that $-1 \notin W(R)$ (see [2, p. 220]). Note also that λ maps the subgroup $W^+(R)$ of $W(R)$ consisting of elements of determinant 1 isomorphically onto $\mathbf{SO}^+(\varphi)$.

Now let $F = Q(R)/2Q(R)$, which is an $\mathbf{F}_2$-vector space of dimension 6. One verifies that $\frac{1}{2}(x \mid y)$ defines a non-degenerate quadratic form q_6 on F. As before we obtain a homomorphism $\mu : A(R) \to \mathbf{O}(q_6)$. Visibly, -1 is in the kernel. We claim that this generates the kernel, and thus we have an isomorphism of $W(R)$ with $\mathbf{O}(q_6)$. Perhaps the easiest way to see this is to note that the restriction to $W^+(R)$ is nontrivial, hence injective by simplicity of this group. The image is the subgroup of index 2, $\mathbf{O}^+(q_6)$. Then each reflection maps to a reflection in F. In fact, if $\overline{R}$ denoted the set of equivalence classes of R for the relation $r \sim -r$, μ maps $\overline{R}$ isomorphically onto its image in F. As $W(R)$ is generated by $W^+(R)$ and any reflection, this shows that $W(R)$ maps onto all of $\mathbf{O}(q_6)$, hence isomorphically onto it, by a comparison of orders.

Table 1. Non-singular pairs (a)

Δ	δ	$\delta^{\perp}$
1	1000 0010 1010 1020	0100 0001 0101 0102
2	1000 0011 1011 1022	1200 0001 1201 1202
3	1000 0012 1012 1021	1100 0001 1101 1102
4	1000 0110 1110 1220	0100 1001 1101 1201
5	1000 0120 1120 1210	0100 1002 1102 1202
6	1000 0111 1111 1222	1200 0101 1001 1102
7	1000 0112 1112 1221	1100 0102 1202 1001
8	1000 0122 1122 1211	1200 1002 1101 0102
9	1000 0121 1121 1212	1100 0101 1201 1002
10	1001 0010 1011 1021	0110 0001 0111 0112
11	1001 0011 1012 1020	1220 0001 1221 1222
12	1001 0012 1010 1022	1110 0001 1111 1112
13	1002 0010 1012 1022	0120 0001 0121 0122
14	1002 0011 1010 1021	1210 0001 1211 1212
15	1002 0012 1011 1020	1120 0001 1121 1122
16	1100 0010 1110 1120	0100 0012 0112 0121
17	1100 1010 1220 0120	0100 1012 1112 1212
18	1100 1020 1210 0110	0100 1021 1121 1221
19	1200 0010 1210 1220	0100 0011 0111 0122
20	1200 1010 1120 0110	0100 1011 1111 1211
21	1200 1020 1110 0120	0100 1022 1122 1222
22	1101 0010 1111 1121	0110 0012 0122 0101
23	1102 0010 1112 1122	0120 0012 0102 0111
24	1201 0010 1211 1221	0110 0011 0121 0102
25	1202 0010 1212 1222	0120 0011 0101 0112
26	1001 0120 1121 1211	0110 1002 1112 1222
27	1001 0121 1122 1210	1110 0111 1221 1002
28	1001 0122 1120 1212	1111 1220 1002 0112
29	1202 0110 1012 1122	0120 1101 1221 1011

Table 1. Non-singular pairs (b)

Δ	δ	$\delta^{\perp}$
30	1102 0110 1212 1022	0120 1201 1021 1111
31	1222 0121 1010 1101	1012 0101 1110 1211
32	1201 0112 1010 1122	1110 0102 1212 1011
33	1102 0122 1221 1010	1210 1012 1111 0102
34	1020 0121 1111 1202	1120 0101 1221 1022
35	1020 0111 1101 1212	1220 0101 1021 1122
36	1010 0111 1121 1202	1210 0101 1011 1112
37	1200 0012 1212 1221	1100 0011 1111 1122
38	1202 0012 1211 1220	1101 0011 1112 1120
39	1201 0012 1210 1222	1102 0011 1110 1121
40	1020 0112 1102 1211	1120 0102 1222 1021
41	1020 0122 1112 1201	1220 0102 1022 1121
42	1200 0112 1012 1121	1100 1011 1222 0122
43	1200 0121 1021 1112	1100 1022 1211 0111
44	1101 0112 1210 1022	1110 0122 1202 1021
45	1201 1012 1120 0111	1102 1011 1220 0121

Table 2. Nsp-spreads

σ	Δ
1	1 26 37 44 45
2	1 27 30 38 42
3	1 28 29 39 43
4	2 16 26 33 35
5	2 17 22 27 40
6	2 18 23 28 31
7	3 19 26 32 34
8	3 20 25 27 41
9	3 21 24 28 36
10	4 13 36 37 40
11	4 14 23 34 42
12	4 15 25 33 43
13	5 10 31 37 41
14	5 11 22 32 43
15	5 12 24 35 42
16	6 13 18 32 38
17	6 14 16 29 41
18	6 15 17 24 44
19	7 13 20 35 39
20	7 14 21 22 45
21	7 15 19 30 31
22	8 10 17 34 39
23	8 11 16 30 36
24	8 12 18 25 45
25	9 10 21 33 38
26	9 11 20 23 44
27	9 12 19 29 40

Table 3. double-sixes

θ	doublets						θ	doublets					
1	1	5	9	11	16	21	19	2	4	9	14	22	27
	6	7	2	18	20	10		7	3	5	25	15	23
2	1	5	11	19	23	25	20	2	6	10	14	17	22
	22	26	4	2	20	10		13	23	3	11	5	16
3	1	5	15	17	21	25	21	2	6	14	18	19	27
	27	13	20	2	4	18		26	15	21	3	5	16
4	1	6	8	15	23	25	22	2	9	12	13	19	27
	9	2	4	26	13	24		10	21	15	3	25	8
5	1	6	12	19	23	27	23	3	7	13	18	20	23
	21	10	26	24	3	4		21	17	9	14	22	1
6	1	8	11	14	21	25	24	2	9	14	17	19	24
	12	20	13	2	26	7		20	16	8	15	23	3
7	1	8	15	16	21	22	25	4	8	10	14	21	22
	19	18	7	13	24	2		13	23	7	19	5	12
8	1	9	12	16	22	27	26	4	8	16	20	22	27
	25	24	7	3	10	18		24	25	5	7	17	19
9	1	6	12	15	17	22	27	4	9	11	14	21	24
	11	18	13	3	24	4		20	12	23	6	15	7
10	2	4	10	14	18	24	28	4	8	11	14	18	27
	12	16	15	1	23	5		26	15	5	17	7	12
11	1	6	8	14	22	27	29	4	9	11	13	16	27
	5	7	3	24	26	13		10	17	25	7	15	6
12	2	6	7	12	17	19	30	5	9	12	13	16	26
	4	8	3	16	21	11		25	14	6	18	8	10
13	2	6	12	20	22	27	31	5	9	11	13	19	24
	24	25	5	3	21	11		15	22	8	20	6	10
14	2	7	10	17	24	26	32	6	7	18	19	23	25
	27	23	8	1	11	16		22	26	4	9	16	21
15	1	9	11	14	19	27	33	5	9	17	21	24	26
	15	26	3	10	7	20		23	27	6	8	18	20
16	2	9	12	17	22	26	34	10	14	17	21	24	25
	18	11	23	25	8	3		20	27	12	16	13	23
17	3	4	8	10	18	20	35	11	13	18	19	23	27
	5	9	1	17	19	12		21	26	10	17	15	22
18	1	6	14	17	19	25	36	11	14	18	19	24	25
	16	20	4	26	13	3		16	22	20	12	26	15

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