Note di Matematica **20,** n. 1, 2000/2001, 95–110.

Minimal saddle towers with genus two

Cláudia Cueva Cândido
Universidade de São Paulo, Instituto de Matemática e Estatística, C. U. Armando de S. Oliveira, Caixa Postal 66281, 05315-970, São Paulo (SP)—Brasil, cueva@ime.usp.br

Maria Elisa Esteves Lopes Galvão
Universidade de São Paulo, Universidade de Mogi das Cruzes, Av. Dr. Cândido Xavier de Souza, 200, 08780-911, Mogi das Cruzes (SP)—Brasil

Received: 15 December 1999; accepted: 25 June 2000.

Abstract. In this work, we prove the existence of a complete, embedded, singly periodic minimal surface, whose quotient by a vertical translation has genus two and four Scherk type ends. With an adequate choice of parameters it can be shown that the surface is a covering of Scherk's saddle tower.

MSC 2000 classification: primary 53A10, secondary 53C42

1. Introduction

In the classical theory of minimal surfaces the well known Scherk saddle towers are minimal singly periodic surfaces which have genus zero and four Scherk type ends in the quotient space.

The aim of this work is to discuss the construction of saddle towers with genus two in the quotient space. In some cases we can assure that the surface is a covering of the original one or of the deformed Scherk tower [1, 4, 5].

The construction of singly periodic examples with genus two and four Scherk type ends, in the quotient space, will be made by considering a 2-torus $\overline{M}$ as a hyperelliptic curve. The functions of the Weierstrass representation will be defined on this 2-torus. In some cases we can solve the period problem thus having a well defined singly periodic minimal surface with planar Scherk ends.

2. The Riemann surface $\overline{M}$ and its automorphisms

We will consider a compact genus two Riemann surface $\overline{M}$ given by the set

$$\overline{M} = \left\{ (z,w) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : w^2 = \frac{(z+\lambda_1)(z-1)(z+\lambda)}{(z-\lambda_1)(z+1)(z-\lambda)},\ 0 < \lambda < 1 < \lambda_1 \right\} \tag{1}$$

By removing of $\overline{M}$ the four points corresponding to the solutions of the equation $z^2 + c^2 = 0$, $c > 0$, we will have an appropriated surface to define an immersion with vertical period. The equation above was chosen to have the normal map of the surface exactly described by the function w.

In the figure 1 we have the values of the function z associated to some points of the 2-torus $\overline{M}$. The symbol o identifies the points corresponding to the ends.

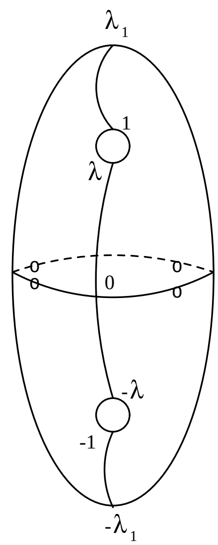


Figure 1. The 2-torus $\overline{M}$

The symmetries of $\overline{M}$ will be very useful to eliminate some of the periods.

First of all we observe that $w^2(\overline{z}) = \overline{w^2(z)}$; if z is real then $w^2(z)$ is real. Moreover, analysing the sign of w^2, we have:

- w is real if z belongs to $]-\infty, -\lambda_1[\ ,\ ]-1, -\lambda[\ ,\ ]\lambda, 1[\ $ or $]\lambda_1, +\infty[$
- w is purely imaginary if z belongs to $]-\lambda_1, -1[\ ,\ ]-\lambda, \lambda[\ $ or $]1, \lambda_1[$.

If z is imaginary, $z = -\overline{z}$ and $w^2(-\overline{z}) = \dfrac{1}{\overline{w^2(z)}}$ then $w(z) = \dfrac{1}{\overline{w}(z)}$, that is, the w has modulus equal to 1 on the imaginary axis. From these calculations we have the following lemma.

Lemma 1. *Let* $\overline{M} = \left\{(z, w) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : w^2 = \dfrac{(z+\lambda_1)(z-1)(z+\lambda)}{(z-\lambda_1)(z+1)(z-\lambda)}\right\}$ *be a 2-torus.*

Let: $\sigma, \delta, \tau \colon \overline{M} \to \overline{M}$ *be the maps given by*

$$\begin{aligned} \sigma(z,w) &= (\overline{z}, -\overline{w}) \\ \delta(z,w) &= (\overline{z}, \overline{w}) \\ \tau(z,w) &= \left(-\overline{z}, \frac{1}{\overline{w}}\right) \end{aligned}$$

These maps are automorphisms of $\overline{M}$.

The curves fixed by σ *are corresponding to*

$$\{z \in \mathbb{R} \mid -\lambda_1 \leq z \leq -1, \ -\lambda \leq z \leq \lambda \qquad \textit{and} \qquad 1 \leq z \leq \lambda_1\}.$$

The curves fixed by δ *are corresponding to*

$$\{z \in \mathbb{R} \mid -1 \leq z \leq -\lambda \qquad \textit{and} \qquad \lambda \leq z \leq 1\}.$$

The automorphism τ *fixes the curve of the* 2*-torus corresponding to the imaginary axis in the plane* z.

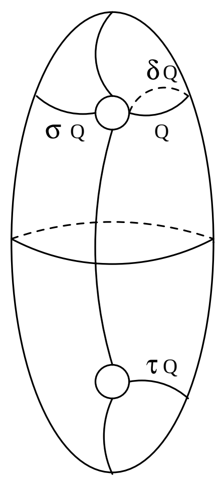


Figure 2. Automorphisms of the 2-torus $\overline{M}$

3. The Weierstrass Data

For each $c \in \mathbb{R}$, $c > 0$ let us define

$$M = \overline{M} \backslash \{(z,w) : z = ic \quad \text{or} \quad z = -ic\} \tag{2}$$

In order to construct the immersion of the 2-torus with four Scherk type ends we recall that according to [6] such ends are asymptotic to a vertical flat annulus and the winding number of M vanishes. From these remarks it follows that the total curvature is $c(M) = 2\pi(\chi(M) - W(M)) = 2\pi\chi(M) = -12\pi$, since that $\overline{M}$ has genus two and four points have been removed. Consequently, the degree of the Weierstrass function g will be 3. We will take the normal vector in the vertical direction and pointing up at the points of M such that $z = \lambda_1$, $z = \lambda$ or $z = -1$ and pointing down at the points such that $z = 1$, $z = -1$ or $z = -\lambda_1$.

This choice leads naturally to $g = w$, since the two functions have the same zeros and poles.

The holomorphic 1-form η must have double zeros in the points corresponding to $z = \lambda_1$, $z = \lambda$ e $z = -1$. In order to guarantee Scherk type ends, η must have, according to [6], simple poles at the points $z = \pm ic$ (corresponding to the ends of the surface).

By observing zeros and poles in the table that follows, where we have set $w_c = \sqrt{w^2(ic)}$ e $w_{-c} = \sqrt{w^2(-ic)}$, we define:

$$\eta = \frac{1}{(z^2+c^2)g}\mathrm{d}z \tag{3}$$

and we have

z	$-\lambda_1$	-1	$-\lambda$	λ	1	λ_1	$-ic$	$-ic$	ic	ic	∞	∞
$g = w$	0	∞	0	∞	0	∞	w_{-c}	$-w_{-c}$	w_c	$-w_c$	1	-1
$\frac{1}{z^2+c^2}$							∞	∞	∞	∞	0^2	0^2
$\frac{1}{g}$	∞	0	∞	0	∞	0						
$\mathrm{d}z$	0	0	0	0	0	0					∞^2	∞^2
η		0^2		0^2		0^2	∞	∞	∞	∞		

Let ϕ_1, ϕ_2 and ϕ_3 be the holomorphic 1-forms defined from the Weierstrass data $g = w$ e $\eta = \frac{1}{(z^2+c^2)g}\mathrm{d}z$:

$$\begin{aligned} \phi_1 &= \frac{1}{2}(1-g^2)\eta \\ \phi_2 &= \frac{i}{2}(1+g^2)\eta \\ \phi_3 &= g\eta \end{aligned}$$

In the theorem that follows we will verify that the surface obtained from the 1-form $\Phi = (\phi_1, \phi_2, \phi_3)$ defined on $\overline{M}$ is complete singly periodic and minimal.

More precisely, we will first demonstrate that on curves around the ends there are real periods in the vertical direction. Secondly we will verify that there exists a pair (λ, λ_1) such that $\mathrm{Re} \int_\gamma \Phi = 0$ for all closed curves γ in the homology group of M. To do this we will use the symmetries of M. We have:

Lemma 2. *The automorphisms σ, δ and τ of Lemma 1 satisfy*

$$\sigma^*\Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{\Phi}, \qquad \delta^*\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{\Phi}$$

and $$\tau^*\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overline{\Phi}.$$

From the previous lemma we can conclude that σ, δ and τ induce, in $\mathbb{R}^3$, reflexion symmetries with respect to orthogonal planes parallel to x_2Ox_3, x_1Ox_3 and x_1Ox_2, respectively. The images of the curves fixed by σ, δ and τ are planar geodesics contained in these orthogonal planes [3]. On the curves fixed by σ the function g assume imaginary values and on the curves fixed by δ we have g real; finally on the curves fixed by τ the function g assumes values with modulus one.

Theorem 1. *Let M be the compact Riemann surface with four points removed given by 2. For c in a neighbourhood of $c = 1$ there exists a pair (λ, λ_1) such that $X : M \to \mathbb{R}^3$ defined by $X(p) = \mathrm{Re} \int_{p_0}^{p} \Phi$, is a minimal singly periodic surface in $\mathbb{R}^3$.*

Proof. In the z plane we will consider the positively oriented circle $\widetilde{\gamma}$ with center in $z = ic$ and radius $r = \dfrac{c}{2}$. Let γ be the lifting of $\widetilde{\gamma}$ to M; we observe that $\tau\gamma = -\gamma$. We have:

$$\int_\gamma \Phi = -\int_{\tau\gamma} \Phi = -\int_\gamma \tau^*\Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{\int_\gamma \Phi}$$

therefore $\mathrm{Re} \int_\gamma \phi_1 = 0$ and $\mathrm{Re} \int_\gamma \phi_2 = 0$.

From the Cauchy integral formula it follows that $\mathrm{Re} \displaystyle\int_\gamma \phi_3 = \frac{\pi}{c}$. Hence, the period vector at the ends $(z, w) = (ic, \pm w_c)$ is $(0, 0, \dfrac{\pi}{c})$ and it is easy to see that the period at the ends $(z, w) = (-ic, \pm w_{-c})$ is $(0, 0, -\dfrac{\pi}{c})$.

To study the periods on the homology curves we consider, in the z-plane, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ circles centered in $A = \dfrac{1+\lambda}{2}$ and $B = \dfrac{1+\lambda_1}{2}$, respectively, both with radius $r = \dfrac{\lambda_1 - \lambda}{2}$. We will denote by γ_1^* and γ_2^* the liftings of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ to $\overline{M}$.

The curves γ_1^*, γ_2^*, $\tau\gamma_1^*$ and $\tau\gamma_2^*$ constitute a basis of the homology group of $\overline{M}$.

The curve γ_1^* is homotopic to the closed curve γ_1 on $\overline{M}$ corresponding to the interval $I_1 = \{z \in \mathbb{R} \mid \lambda \le z \le 1\}$. One branch of the function $w = w(z)$

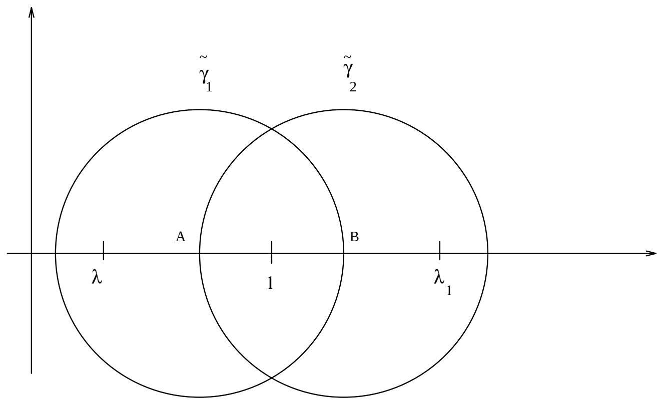


Figure 3. The z plane

such that $w(1) = 0$ and $w(\lambda) = \infty$, is real and positive on this interval; the other branch is real and negative. Let C_1 be the curve in $\overline{M}$ given by the points (z, w) such that $z \in I_1$ and $w(z) \geq 0$. If (z, w) belongs to C_1 then $\sigma(z, w) = (\overline{z}, -\overline{w}) = (z, -w)$ and σC_1 describes the portion of γ_1 corresponding to the negative branch of the function w. According the orientation we can write $\gamma_1 = C_1 \cup (-\sigma C_1)$.

The curve γ_2^* is homotopic to the closed curve γ_2 in $\overline{M}$ corresponding to the interval $I_2 = \{z \in \mathbb{R} | 1 \leq z \leq \lambda_1\}$. On this curve the function $w = w(z)$ assumes imaginary values; denoting by C_2 the portion of γ_2 corresponding to one of the branches of w we can write $\gamma_2 = C_2 \cup (-\delta C_2)$.

From the above considerations, it follows that $\sigma\gamma_1 = -\gamma_1$ and $\delta\gamma_2 = -\gamma_2$ and hence:

1. $\displaystyle\int_{\gamma_1} \Phi = -\int_{\sigma\gamma_1} \Phi = -\int_{\gamma_1} \sigma^*\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overline{\int_{\gamma_1} \Phi}$; we have $\mathrm{Re}\int_{\gamma_1} \phi_2 = 0$ and $\mathrm{Re}\int_{\gamma_1} \phi_3 = 0$;

2. $\displaystyle\int_{\gamma_2} \Phi = -\int_{\delta\gamma_2} \Phi = -\int_{\gamma_2} \delta^*\overline{\Phi} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overline{\int_{\gamma_2} \Phi}$; we have $\mathrm{Re}\int_{\gamma_2} \phi_1 = 0$ and $\mathrm{Re}\int_{\gamma_2} \phi_3 = 0$.

These two integrals (i) and (ii) depend on λ and λ_1. From now our aim is to prove that there exists a pair $(\lambda, \lambda_1) \in \Omega = \{(\lambda, \lambda_1) : 0 < \lambda < 1 < \lambda_1\}$ such that $\mathrm{Re}\int_{\gamma_1} \phi_1 = \mathrm{Re}\int_{\gamma_2} \phi_2 = 0$.

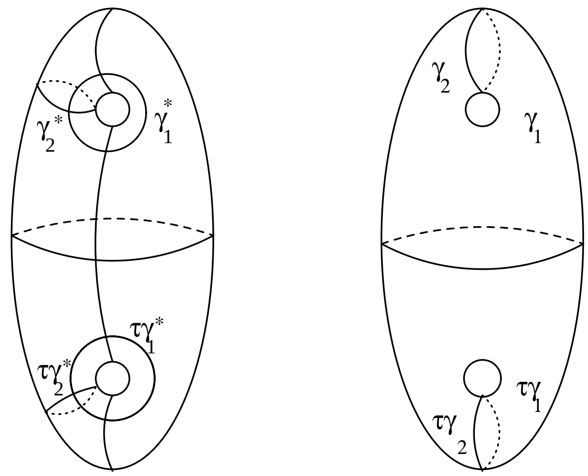


Figure 4. Homology curves of $\overline{M}$

We denote

$$\pi_1(\lambda, \lambda_1) = \mathrm{Re} \int_{\gamma_1} \phi_1 = \int_{\lambda}^{1} \frac{(\lambda_1 + \lambda - 1)z^2 - \lambda\lambda_1}{\sqrt{(\lambda_1^2 - z^2)(1 - z^2)(z^2 - \lambda^2)}(z^2 + c^2)} \mathrm{d}z. \qquad (4)$$

$$\pi_2(\lambda, \lambda_1) = \mathrm{Re} \int_{\gamma_1} \phi_2 = \int_{1}^{\lambda_1} \frac{-z^3 + (\lambda_1 + \lambda - \lambda\lambda_1)z}{\sqrt{(\lambda_1^2 - z^2)(z^2 - 1)(z^2 - \lambda^2)}(z^2 + c^2)} \mathrm{d}z. \qquad (5)$$

To verify the existence of a common zero to these two integrals we will develop the same arguments founded in the work of Wölgemuth [8]: by constructing a closed curve in the region Ω we will show that its image under $\pi = (\pi_1, \pi_2)$ is a plane curve such that the winding number around the point $(0, 0)$ is distinct from zero. The conclusion will follow from the next result, demonstrated in [2, p. 86]:

Theorem. Let $f : D \to P$ be a continuous mapping of a disc into the plane, let C be the boundary circle of D, and let y be a point of the plane not on $f(C)$. If the winding number of $f|_C$ about y is not zero, then $y \in f(D)$; i. e. there is a point $x \in D$ such that $f(x) = y$.

Deriving π_1 with respect to λ_1 and π_2 with respect to λ we obtain:

$$\frac{\partial \pi_1}{\partial \lambda_1}(\lambda, \lambda_1) = \int_{\lambda}^{1} \frac{z^2(\lambda_1 + \lambda - \lambda\lambda_1 - z^2)}{\sqrt{(\lambda_1^2 - z^2)^3(1 - z^2)(z^2 - \lambda^2)}(z^2 + c^2)} \mathrm{d}z.$$

$$\frac{\partial \pi_2}{\partial \lambda}(\lambda, \lambda_1) = \int_{1}^{\lambda_1} \frac{z[\lambda\lambda_1 - (\lambda_1 + \lambda - 1)z^2]}{\sqrt{(\lambda_1^2 - z^2)(z^2 - 1)(z^2 - \lambda^2)^3}(z^2 + c^2)} \mathrm{d}z.$$

Recalling that $0 < \lambda < 1 < \lambda_1$ we have $\lambda_1 + \lambda - \lambda\lambda_1 > 1$, that is, $\lambda\lambda_1 < \lambda_1 + \lambda - 1$ and it follows:

• the first integrand is positive, for $\lambda < z < 1$ and

• the second integrand is negative, for $1 < z < \lambda_1$.

Hence $\dfrac{\partial \pi_1}{\partial \lambda_1} > 0$ and $\dfrac{\partial \pi_2}{\partial \lambda} < 0$, that is, π_1 is strictly increasing as a function of λ_1 and π_2 is strictly decreasing as a function of λ.

The values of π_1 and π_2 in some points (λ, λ_1) can be estimated directly by computer and in the special case $c = 1$ we found:

$$\pi_1(0.1,\ 2) = 0.131057 > 0 \quad \pi_2(0.1,\ 2) = -0.0183368 < 0$$
$$\pi_1(0.9,\ 44) = -0.00333298 < 0 \quad \pi_2(0.9,\ 44) = 0.0345741 > 0$$

From these values and from the properties of the functions π_1 and π_2 it follows that:

$$\pi_1(\lambda, \lambda_1) > 0, \quad if \quad (\lambda, \lambda_1) \in l_1 = \{(\lambda, \lambda_1) : \lambda = 0.1 \quad and \quad 2 \leq \lambda_1 \leq 44\}$$
$$\pi_2(\lambda, \lambda_1) < 0, \quad if \quad (\lambda, \lambda_1) \in l_2 = \{(\lambda, \lambda_1) : 0.1 \leq \lambda \leq 0.9 \quad and \quad \lambda_1 = 2\}$$
$$\pi_1(\lambda, \lambda_1) < 0, \quad if \quad (\lambda, \lambda_1) \in l_3 = \{(\lambda, \lambda_1) : \lambda = 0.9 \quad and \quad 2 \leq \lambda_1 \leq 44\}$$
$$\pi_2(\lambda, \lambda_1) > 0, \quad if \quad (\lambda, \lambda_1) \in l_4 = \{(\lambda, \lambda_1) : 0.1 \leq \lambda \leq 0.9 \quad and \quad \lambda_1 = 44\}$$

Let α be the closed curve in Ω (figure 5) obtained as the union of the segments l_j, $j = 1, \ldots, 4$. It follows that $\pi(\alpha)$ turns once around $(0,0)$. Moreover, α is homotopic to the unitary circle and $\pi : \Omega \to \mathbb{R}^2$ is continuous; from the theorem [2] mentioned before we have that there exists a pair (λ, λ_1) in the region limited by α such that $\pi(\lambda, \lambda_1) = (0,0)$.

The zero level curves of π_1 and π_2 can be drawn by computer and we can see that this pair is unique for c equal to 1. The continuity of π as function of c, guarantees the resolution of the period problem for c in a neighbourhood of $c = 1$ and assures that X is a singly periodic minimal immersion. $\boxed{QED}$

4. The geometry of the quotient surface

The surface obtained from Theorem 1 has its symmetries related to the automorphisms σ, δ and τ. As a consequence of lemma 2 the images of the curves fixed by the symmetries σ, δ and τ are planar symmetry curves contained in planes parallel to x_2Ox_3, x_1Ox_3 e x_1Ox_2, respectively. Let γ_1 and γ_2 be the closed curves of the 2-torus corresponding to the intervals $[\lambda, 1]$ and $[1, \lambda_1]$ as in theorem 1 and let γ_3 be the closed curve corresponding to the interval $[-\lambda, \lambda]$.

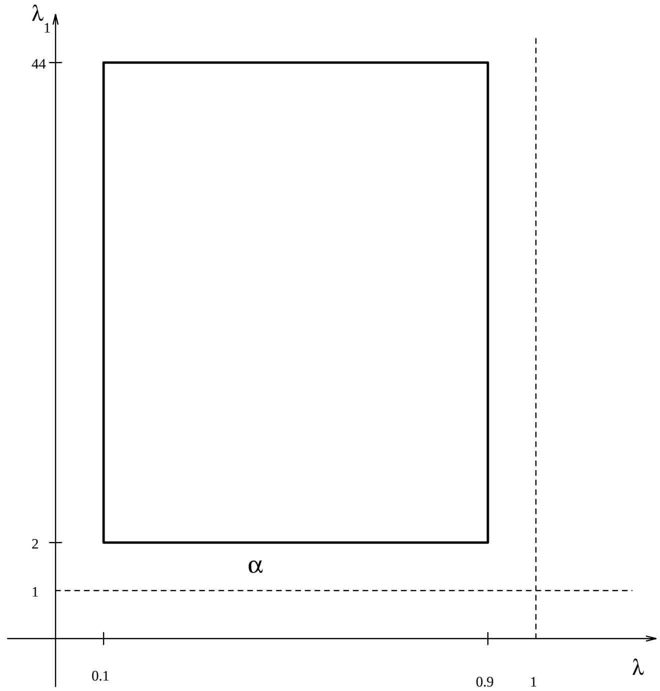


Figure 5. The curve α in Ω

The curves γ_2 and γ_3 are fixed by σ and γ_1 is fixed by δ. Thus the plane of $X(\gamma_1)$ is orthogonal to the planes of $X(\gamma_2)$ and $X(\gamma_3)$. Moreover, by theorem 1, we have a vertical period only on the closed curves around the ends. Hence, the images $X(\gamma_1)$, $X(\gamma_2)$ and $X(\gamma_3)$ are closed curves in $\mathbb{R}^3$. It follows that the intersection points $X(\lambda) = X(\gamma_1) \cap X(\gamma_3)$ and $X(1) = X(\gamma_1) \cap X(\gamma_2)$ are in the intersection line determined by the planes of $X(\gamma_1)$ and $X(\gamma_2)$.

We can conclude that the intersection points $X(-\lambda_1)$, $X(-\lambda)$, $X(-1)$, $X(\lambda)$, $X(1)$, $X(\lambda_1)$ belong to the vertical axis Ox_3.

In the points where $z = ic$ or $z = -ic$ we have simple poles of η and the Scherk ends are asymptotic to a pair of vertical planes. There exists θ_0, $0 < \theta_0 < \dfrac{\pi}{2}$ such that $g(ic, w_c) = \cos\theta_0 + i\sin\theta_0$, $g(ic, -w_c) = -\cos\theta_0 - i\sin\theta_0$, $g(-ic, w_{-c}) = \cos\theta_0 - i\sin\theta_0$ e $g(-ic, -w_{-c}) = -\cos\theta_0 + i\sin\theta_0$. Moreover, the curves fixed by τ go to the ends.

Proposition 1. *For all c in a neighbourhood of $c = 1$ the surface obtained in Theorem 1 is embedded.*

Proof. By lemma 2, $X(M)$ consists of eight congruent parts. Each one of them is the image of a quadrant in the z plane punctured in $z = ic$ or

$z = -ic$. Let R be the lifting to M of the first quadrant, such that $w(\infty) = 1$ and $w(0) = i$. The image of R by X, $F = X(R)$, is one of the eight congruent parts and we will show that F is a graphic over the plane x_1Ox_3. The boundary of F is constituted by six symmetry plane curves, fixed by reflexions, namely:

α_1: image of $\{z = iy,\ 0 \leq y < c\}$
α_2: image of $\{z = x,\ 0 \leq x \leq \lambda\}$
α_3: image of $\{z = x,\ \lambda \leq x < 1\}$
α_4: image of $\{z = x,\ 1 \leq x < \lambda_1\}$
α_5: image of $\{z = x,\ x \geq \lambda_1\}$
α_6: image of $\{z = iy,\ y \geq c\}$

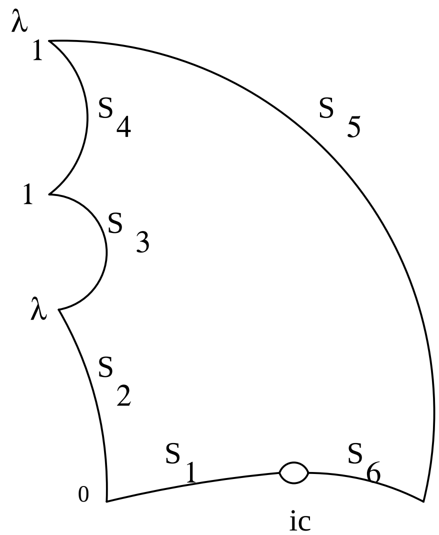


Figure 6. The region R in M

In the figure 6, we denote by s_j the curve in the 2-torus corresponding to the curve α_j, in $\mathbb{R}^3$, $j = 1, \ldots, 6$.

The curves α_1 and α_6 are contained in distinct planes parallel to x_1Ox_2; moreover they diverge towards the end. In fact, restrict to these curves the function g has its image in the unitary circle. We observe that on the curve α_1 we can write $g(iy) = \cos\theta(y) + i\sin\theta(y)$, $0 < \theta(y) < \theta_0$. The function θ is continuous, $\theta(0) = 0$ and $\theta(c) = \theta_0$. The third coordinate function on α_1 is constant whereas the first and the second are given by:

$$x_1(iy) = \int_0^y \frac{\sin\theta(t)}{c^2 - t^2}dt$$
$$x_2(iy) = \int_0^y \frac{(-\cos\theta(t))}{c^2 - t^2}dt$$

The two above integrals diverge when y is close to c therefore α_1 diverge to the end. Analogously it is easy to see that α_6 also diverges to the end.

The restriction of the projection to the plane x_1Ox_3 to the boundary of F is injective. The projection of α_1 is the coordinate $x_1(iy)$ given above and its derivative $x_1' = \dfrac{\sin\theta(y)}{c^2-y^2}$ is strictly positive if $z \neq 0$. The function α_6 has the same properties and the projections of α_1 e α_6 to x_1Ox_3 are injective.

In order to see that the projection of the reunion of the curves α_2 and α_5 has no self-intersections, it is sufficient to observe that this union is the image of the positive real axis. With some calculations we obtain the third coordinate for z real, $z \geq 0$ as:

$$x_3(z) = \int_0^z \frac{1}{(c^2+z^2)}dz = \frac{1}{c}\arctan\frac{z}{c}$$

Therefore the function $x_3(z)$ is strictly increasing, $0 \leq x_3(z) \leq \dfrac{\pi}{2c}$, and it follows that the projections of α_2 to α_5 over x_1Ox_3 are injective. We conclude that the projection of the boundary of F over the plane x_1Ox_3 has no self-intersections.

In the interior of F there is no point with normal pointing in a direction parallel to x_1Ox_3. This follows from the fact that $g = w$ and w is real only at the curves fixed by σ, whose images are the curves α_2 and α_4 contained in the boundary of F. The imaginary part of w is strictly positive in all interior points of F; it follows that the projection of F over the plane x_1Ox_3 is injective. Consequently F is a graphic over x_1Ox_3 and $X(M)$ is an embedded surface in $\mathbb{R}^3$. $\boxed{QED}$

5. Remarks

We observe that the Scherk towers have genus zero and four Scherk type ends [4]. In the classical example, the ends correspond to the solutions of the equation $z^4 = -1$. The Weierstrass data are:

$$g(z) = z \qquad \eta = \frac{1}{(z^4+1)}\mathrm{d}z.$$

The symmetrically deformed examples are obtained from the classical one by moving the punctures; the ends correspond to $z = \pm e^i\alpha$, $0 < \alpha < \frac{\pi}{4}$, and the Weierstrass data are:

$$g(z) = z \qquad \eta = \frac{1}{(z^2+z^{-2}-2\cos 2\alpha)z^2}\mathrm{d}z.$$

The question that arises immediately is: are the surfaces obtained in Theorem 1 coverings of the classical or deformed Scherk saddle towers?

In these examples, if the ends are branch points of the map $g = w$, the parameters λ and λ_1 are uniquely determined. Moreover, on the compact surface $\overline{M}$ given by 1 we will have two more automorphisms. Using the induced symmetries by these automorphisms the period problem becomes trivial. Furthermore $X(M)$ is a triple covering of the genus zero Scherk saddle tower.

We will list further details concerning these remarks:

- *If the points given by $z = \pm ic$ are branch points of third order of $g = w$, then*

  i) $\dfrac{\sqrt{3}}{3} < c < \sqrt{3}$ *ii)* $\lambda = \dfrac{c(\sqrt{3}c - 1)}{\sqrt{3} + c}$ *iii)* $\lambda_1 = \dfrac{c(\sqrt{3}c + 1)}{\sqrt{3} - c}$.

  We recall that a point $z = z_0$ is a branch point of order three of $g = w$ if and only if $\dfrac{\partial g}{\partial z}(z_0) = 0$ and $\dfrac{\partial^2 g}{\partial z^2}(z_0) = 0$. We have

  $$g^2 = \frac{P(z)}{Q(z)} = \frac{z^3 + Az^2 - Bz - C}{z^3 - Az^2 - Bz + C}$$

  where $A = \lambda + \lambda_1 - 1$, $B = \lambda + \lambda_1 - \lambda\lambda_1$ and $C = \lambda\lambda_1$.

  It follows that z_0 is a third order branch point of g if and only if it is solution of the equations

  $$P'(z)Q(z) - P(z)Q'(z) = 0 \quad \text{and} \quad P''(z)Q(z) - P(z)Q''(z) = 0.$$

  These equations are respectively equivalent to:

  $$Az^4 - (3C - AB)z^2 + BC = 0 \tag{6}$$

  and

  $$z[2Az^2 - (3C - AB)] = 0. \tag{7}$$

  We have $BC \neq 0$ and hence $z_0 = 0$ is not solution of (6). Thus, the solutions of (7) are given by $z_0^2 = \dfrac{3C - AB}{2A}$ and by replacing in (6), we obtain $(3C - AB)^2 = 4ABC$. Finally, by assuming $z_0 = \pm ic$, we have the results established in *i)*, *ii)* and *iii)*.

  In the special case in which $\overline{M}$ is given by the parameters λ and λ_1 above, we have more symmetries in the 2-torus.

- *If the points given by $z = \pm ic$ are branch points of third order of $g = w$, then there are two more automorphisms in*

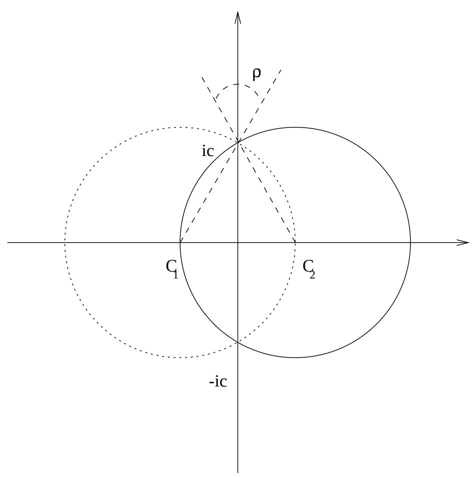
  $$\overline{M} = \left\{ (z, w) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : w^2 = \frac{(z + \lambda_1)(z - 1)(z + \lambda)}{(z - \lambda_1)(z + 1)(z - \lambda)} \right\},$$

$\alpha, \beta\colon \overline{M} \to \overline{M}$ given by

$$\alpha(z,w) = \left(\frac{c(\overline{z}+c\sqrt{3})}{\sqrt{3}\overline{z}-c}, \frac{1}{\overline{w}}\right)$$
$$\beta(z,w) = \left(\frac{c(\overline{z}-c\sqrt{3})}{\sqrt{3}\overline{z}+c}, \frac{1}{\overline{w}}\right)$$

Moreover, these automorphisms satisfy

$$\alpha^*\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overline{\Phi} \qquad \text{and} \qquad \beta^*\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overline{\Phi}.$$



In the figure 7, we have the correspondent curves fixed by the automorphisms α and β in the 2-torus $\overline{M}$ and the corresponding curves in z-plane. In this plane the circle with center $C_1 = (\frac{-c\sqrt{3}}{3}, 0)$ and radius $r = \frac{2c\sqrt{3}}{3}$ corresponds to the curve in $\overline{M}$ fixed by α and the circle with center $C_2 = (\frac{c\sqrt{3}}{3}, 0)$ and same radius r corresponds to the curve fixed by the automorphism β. The normal lines to the circles at the points $(0, \pm ic)$ make the angle of $\rho = 60^o$.

- *Let the points $z = \pm ic$ be branch points of third order of $g = w$. The surface*

$$X : M \to \mathbb{R}^3,$$

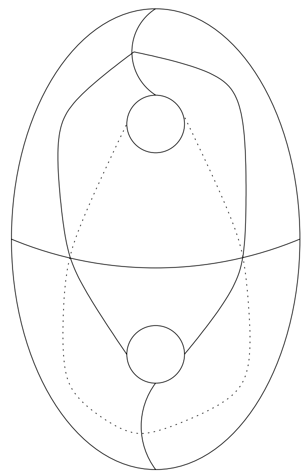


Figure 7. The curves fixed by α and β in the z-plane and in $\overline{M}$

described in theorem 1, with $\lambda = \lambda(c) = \dfrac{c(\sqrt{3}c-1)}{\sqrt{3}+c}$ *and* $\lambda_1 = \lambda_1(c) = \dfrac{c(\sqrt{3}c+1)}{\sqrt{3}-c}$ *is a triple covering of a deformed Scherk saddle tower.*

The reflexion symmetries induced by α and β fix two curves in horizontal planes (see figure 8); these curves divide $X(M)$ in three congruent parts that are invariant by vertical translation. Each one of these parts has four Scherk type ends and is invariant by three more symmetries that fix three orthogonal planes. Hence each one corresponds to a basic piece of the genus zero Scherk saddle tower, that is, $X(M)$, given by the solution $(\lambda = \lambda(c), \lambda_1 = \lambda_1(c))$, covers three times the classical Scherk saddle tower.

It remains to know whether this solution is unique or if there are other pairs (λ, λ_1) solving the period problem and rendering $X(M)$ distinct of the covering described above. Some facts point to the negative answer. For values of c such as $0.6 \leq c \leq 1.6$, by making computational estimates we draw, in the region Ω, the zero level curves of the functions π_1 and π_2, defined in Theorem 1. For each c, we can see that the zero level curves have only one intersection agreeing with the pair $(\lambda = \lambda(c), \lambda_1 = \lambda_1(c))$ obtained under the hypothesis that the ends are branch points of g. Thus, we can assure that for $c = 0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4, 1.5$ and 1.6, there is a unique solution and the surface obtained $X(M)$ is a triple covering either the classical or the deformed Scherk saddle tower.

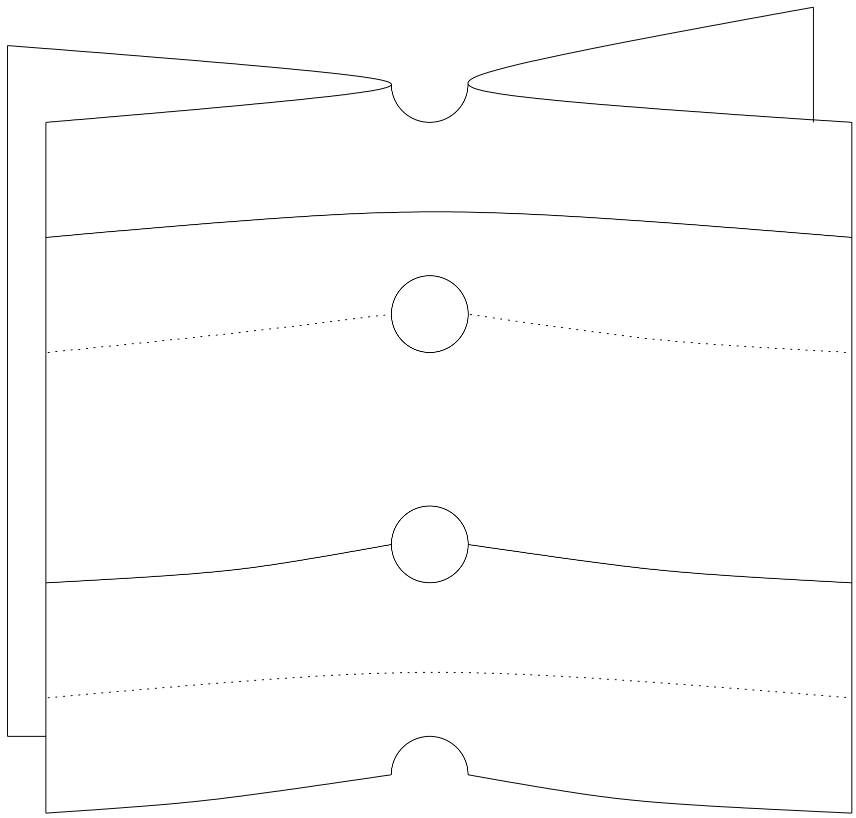


Figure 8. The saddle tower

References

[1] C. C. Cândido: Superfícies Mínimas Simplesmente Periódicas, Tese de Doutorado, IME–USP, São Paulo, 1998.

[2] W. G. Chinn, N. E. Steenrod: *First Concepts of Topology: the geometry of mappings of segments, curves, circles and disks*, The Mathematical Association of America, 1966.

[3] U. Dierkes, S. Hildebrandt, A. Kuster, O. Wohlrab: *Minimal Surfaces I*, A Series of Comprehensive Studies in Mathematics, vol. 295, Springer Verlag, 1992.

[4] H. Karcher: *Embedded minimal surfaces derived from Scherk's examples*, Manusc. Math., 83–114, 1988.

[5] H. Karcher: Construction of minimal Surfaces, Vorlesungsreihe, Bonn, 1989.

[6] W. H. Meeks, H. Rosenberg: *The geometry of periodic minimal surfaces*, Comm. Math. Helv. **68** (1993), 538–578.

[7] W. H. Meeks, H. Rosenberg: *The global theory of doubly periodic minimal surfaces*, Invent. Math. **97** (1989), 351–379.

[8] M. Wöhlgemuth: *Higher genus minimal surfaces of finite total curvature*, preprint, 1996.