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The linear natural operators transforming affinors to tensor fields of type $(0, p)$ on Weil bundles

Włodzimierz M. Mikulski
Institute of Mathematics, Jagiellonian University, Kraków, Reymonta 4, Poland,
wlodzimierz.mikulski@im.uj.edu.pl

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Abstract. All linear natural operators transforming affinors to tensor fields of type $(0, p)$ on Weil bundles are classified.

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Introduction

Let $F : \mathcal{M} \to \mathcal{FM}$ be a product preserving bundle functor and let $A = F(\mathbf{R})$ be its Weil algebra, [3].

If $\lambda : A \to \mathbf{R}$ is a linear map and Φ is an affinor on an n-manifold M, then we have $(tr\Phi)^{(\lambda)} : F(M) \to \mathbf{R}$, where $tr\Phi : M \to \mathbf{R}$ is the trace of Φ and $(\ )^{(\lambda)}$ is the (λ)-lift of functions to F in the sense of [2].

Clearly, for a given linear map $\lambda : A \to \mathbf{R}$ the correspondence $\Phi \to (tr\Phi)^{(\lambda)}$ is a linear natural operator $T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,0)}F$ transforming affinors into functions on F in the sense of [3]. Similarly, for a given linear map $\lambda : A \to \mathbf{R}$ the correspondence $\Phi \to d(tr\Phi)^{(\lambda)}$ is a linear natural operator $T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,1)}F$ transforming affinors into 1-forms on F.

In this short note we prove

Theorem 1. *Let F and A be as above.*

1. *Every linear natural operator $T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,0)}F$ is of the form $\Phi \to (tr\Phi)^{(\lambda)}$ for a linear map $\lambda : A \to \mathbf{R}$.*

2. *Every linear natural operator $T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,1)}F$ is of the form $\Phi \to d(tr\Phi)^{(\lambda)}$ for a linear map $\lambda : A \to \mathbf{R}$.*

3. *For $p \geq 2$ every linear natural operator $T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,p)}F$ is 0.*

Problem of finding all natural operators of some type on affinors is very difficult. Classifications of base extending natural operators on affinors are unknown. The author knows only the paper of Debecki, cf. [1], where the natural operators $T^{(1,1)}_{\mathcal{M}_n} \rightsquigarrow T^{(p,q)}$ for $p = q = 0, 1, 2$ and $(p, q) = (0, 1)$ are classified. Recently Debecki obtained a classification for $p = q = 3$. It seems that classifications of natural operators on affinors would be very useful because affinors play important role in differential geometry.

Throughout this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \ldots, x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$.

All manifolds and maps are assumed to be of class C^∞.

1. A reducibility lemma

The crucial point in our consideration is the following general lemma.

Lemma 1. *Let $\mathcal{L} : T^{(1,1)}|_{\mathcal{M}_n} \rightsquigarrow HF$ be a linear natural operator, where $F : \mathcal{M}_n \to \mathcal{FM}$ is a natural bundle and $H : \mathcal{M}_{\dim(F(\mathbf{R}^n))} \to \mathcal{VB} \subset \mathcal{FM}$ is a natural vector bundle. If $\mathcal{L}(x^1\partial_1 \otimes dx^1) = 0$, then $\mathcal{L} = 0$.*

Proof. At first we prove that

$$\mathcal{L}((x^1)^p\partial_1 \otimes dx^1) = 0 \qquad \text{over } 0 \in \mathbf{R}^n \tag{1}$$

for $p = 0, 1, 2, \ldots$.

We consider three cases:

1. $p = 0$. Applying the invariance of $\mathcal{L}$ with respect to the translation $(x^1 - 1, x^2, \ldots, x^n)$ from the assumption $\mathcal{L}(x^1\partial_1 \otimes dx^1) = 0$ it follows that $\mathcal{L}((x^1 - 1)\partial_1 \otimes dx^1) = 0$. Then $\mathcal{L}(\partial_1 \otimes dx^1) = 0$ because of the linearity of $\mathcal{L}$.

2. $p = 1$. The equality (1) for $p = 1$ is the assumption.

3. $p \geq 2$. Applying the invariance of $\mathcal{L}$ with respect to the local diffeomorphism $(x^1 + (x^1)^p, x^2, \ldots, x^n)^{-1}$ from the assumption it follows that $\mathcal{L}((x^1 + (x^1)^p)\partial_1 \otimes dx^1) = 0$ over $0 \in \mathbf{R}^n$. Then we have (1) because of the same reasons as in case 1.

Next we prove that if $n \geq 2$, then

$$\mathcal{L}((x^1)^p x^2\partial_1 \otimes dx^1) = 0 \qquad \text{over } 0 \in \mathbf{R}^n \tag{2}$$

for $p = 0, 1, 2, \ldots$.

Let $p \in \{0, 1, 2, \ldots\}$.

We shall use (1). We have $\mathcal{L}(\partial_1 \otimes dx^1) = 0$ over $0 \in \mathbf{R}^n$. Then by the invariance of $\mathcal{L}$ with respect to the diffeomorphism $(x^1 - x^2, x^2, \dots, x^n)$ we derive that $\mathcal{L}(\partial_1 \otimes (dx^1 + dx^2)) = 0$ over $0 \in \mathbf{R}^n$. Then

$$\mathcal{L}(\partial_1 \otimes dx^2) = 0 \qquad \text{over } 0 \in \mathbf{R}^n. \tag{3}$$

There is a diffeomorphism $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi \times id_{\mathbf{R}^{n-1}}$ sends the germ of ∂_1 at 0 into the germ of $\partial_1 + (x^1)^p \partial_1$ at 0. Then using the invariance of $\mathcal{L}$ with respect to $\varphi \times id_{\mathbf{R}^{n-1}}$ from (3) we obtain that $\mathcal{L}((\partial_1 + (x^1)^p\partial_1) \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$. Then

$$\mathcal{L}((x^1)^p \partial_1 \otimes dx^2) = 0 \qquad \text{over } 0 \in \mathbf{R}^n. \tag{4}$$

On the other hand by the invariance of $\mathcal{L}$ with respect to the diffeomorphisms $(x^1 - \tau x^2, x^2, \dots, x^n)$, $\tau \neq 0$ from (1) for $p+1$ instead of p we have $\mathcal{L}((x^1 + \tau x^2)^{p+1}\partial_1 \otimes (dx^1 + \tau dx^2)) = 0$ over $0 \in \mathbf{R}^n$. The left hand side of this equality is a polynomial in τ. Considering the coefficients at τ^1 of this polynomial we get

$$(p+1)\mathcal{L}((x^1)^p x^2 \partial_1 \otimes dx^1) + \mathcal{L}((x^1)^{p+1}\partial_1 \otimes dx^2) = 0 \qquad \text{over } 0 \in \mathbf{R}^n.$$

Then we have (2) because of (4) for $p+1$ instead of p.

We continue the proof of the lemma. By the linearity of $\mathcal{L}$ and the base-extending version of Peetre theorem (see Th. 19.9 in [3]) it is sufficient to verify that

$$\mathcal{L}(x^\alpha \partial_i \otimes dx^j) = 0 \qquad \text{over } 0 \in \mathbf{R}^n \tag{5}$$

for any $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ and $i, j = 1, \dots, n$.

Because of (1) we can assume that $n \geq 2$. Using the invariance of $\mathcal{L}$ with respect to the diffeomorphisms permuting the coordinates we can assume that either $i = j = 1$ or $i = 1$ and $j = 2$.

Consider two cases:

1. $i = j = 1$. If $\alpha_2 = \dots = \alpha_n = 0$, then by (1) for $p = \alpha_1$ we get $\mathcal{L}(x^\alpha \partial_1 \otimes dx^1) = 0$ over $0 \in \mathbf{R}^n$. So, we can assume that $(\alpha_2, \dots, \alpha_n) \neq 0$. Then by the invariance of $\mathcal{L}$ with respect to the local diffeomorphisms $(x^1, x^2 + (x^2)^{\alpha_2} \cdots (x^n)^{\alpha_n}, x^3, \dots, x^n)^{-1}$ from (2) for $p = \alpha_1$ we derive that

$$\mathcal{L}((x^1)^{\alpha_1}(x^2 + (x^2)^{\alpha_2} \cdots (x^n)^{\alpha_n})\partial_1 \otimes dx^1) = 0 \qquad \text{over } 0 \in \mathbf{R}^n.$$

   Then $\mathcal{L}(x^\alpha \partial_1 \otimes dx^1) = 0$ over $0 \in \mathbf{R}^n$.

2. $i = 1$, $j = 2$. We consider two subcases:

a. Assume $n \geq 3$ and $(\alpha_3, \ldots, \alpha_n) \neq 0$. Then from the case 1 we have (in particular) that $\mathcal{L}(x^3\partial_1 \otimes dx^1) = 0$ over $0 \in \mathbf{R}^n$. Then using the invariance of $\mathcal{L}$ with respect to the diffeomorphisms $(x^1 - x^2, x^2, \ldots, x^n)$ we obtain $\mathcal{L}(x^3\partial_1 \otimes (dx^1 + dx^2)) = 0$ over $0 \in \mathbf{R}^n$. Consequently $\mathcal{L}(x^3\partial_1 \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$.

There is a diffeomorphism $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi \times id_{\mathbf{R}^{n-1}}$ sends the germ of ∂_1 at 0 into the germ of $\partial_1 + (x^1)^{\alpha_1}\partial_1$ at 0. Using the invariance of $\mathcal{L}$ with respect to $\varphi \times id_{\mathbf{R}^{n-1}}$ from $\mathcal{L}(x^3\partial_1 \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$ we derive that $\mathcal{L}(x^3(\partial_1 + (x^1)^{\alpha_1}\partial_1) \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$. Then

$$\mathcal{L}((x^1)^{\alpha_1}x^3\partial_1 \otimes dx^2) = 0 \qquad \text{over } 0 \in \mathbf{R}^n. \tag{6}$$

There is a diffeomorphism $\psi \colon \mathbf{R} \to \mathbf{R}$ such that $id_{\mathbf{R}} \times \psi \times id_{\mathbf{R}^{n-2}}$ sends the germ of dx^2 at 0 into the germ of $dx^2 + (x^2)^{\alpha_2}dx^2$ at 0. Using the invariance of $\mathcal{L}$ with respect to $id_{\mathbf{R}} \times \psi \times id_{\mathbf{R}^{n-2}}$ from (6) we deduce that $\mathcal{L}((x^1)^{\alpha_1}x^3\partial_1 \otimes (dx^2 + (x^2)^{\alpha_2}dx^2)) = 0$ over $0 \in \mathbf{R}^n$. Then

$$\mathcal{L}((x^1)^{\alpha_1}(x^2)^{\alpha_2}x^3\partial_1 \otimes dx^2) = 0 \qquad \text{over } 0 \in \mathbf{R}^n. \tag{7}$$

Then using the invariance of $\mathcal{L}$ with respect to the local diffeomorphism $(x^1, x^2, x^3 + (x^3)^{\alpha_3} \cdots (x^n)^{\alpha_n}, x^4, \ldots, x^n)^{-1}$ from (7) we deduce that $\mathcal{L}((x^1)^{\alpha_1}(x^2)^{\alpha_2}(x^3 + (x^3)^{\alpha_3} \cdots (x^n)^{\alpha_n})\partial_1 \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$. Hence $\mathcal{L}(x^\alpha\partial_1 \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$.

b. $n = 2$ or $\alpha_3 = \cdots = \alpha_n = 0$. By (4), $\mathcal{L}((x^1)^{\alpha_1}\partial_1 \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$. Now, using the invariance of $\mathcal{L}$ with respect to $id_{\mathbf{R}} \times \psi \times id_{\mathbf{R}^{n-2}}$ (as above) we deduce that $\mathcal{L}((x^1)^{\alpha_1}\partial_1 \otimes (dx^2 + (x^2)^{\alpha_2}dx^2)) = 0$ over $0 \in \mathbf{R}^n$. Then $\mathcal{L}((x^1)^{\alpha_1}(x^2)^{\alpha_2}\partial_1 \otimes dx^2) = 0$ over $0 \in \mathbf{R}^n$.

QED

2. The proof of Theorem 1

We are now in position to prove the theorem. Let F and A be as in the Introduction. Let $a_1, \ldots, a_k \in A$ be a basis of A, and let $a_1^*, \ldots, a_k^*$ be the dual basis.

1. Consider a linear natural operator $\mathcal{L} : T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,0)}F$. Since the $(x^i)^{(a_\nu^*)}$ for $i = 1, \ldots, n$ and $\nu = 1, \ldots, k$ form a coordinate system on $F(\mathbf{R}^n)$ (see [2]), we can write $\mathcal{L}(x^1\partial_1 \otimes dx^1) = f((x^i)^{(a_\nu^*)})$ for some $f : \mathbf{R}^N \to \mathbf{R}$, with $N = \{1, \ldots, n\} \times \{1, \ldots, k\}$. By the invariance of $\mathcal{L}$ with respect to

the diffeomorphisms $(x^1, tx^2, \dots, tx^n)$, $t \neq 0$, we deduce that $\mathcal{L}(x^1\partial_1 \otimes dx^1) = f((x^1)^{(a^*_\nu)})$ for some $f : \mathbf{R}^{\{1,\dots,k\}} \to \mathbf{R}$. Now, by the linearity and the invariance of $\mathcal{L}$ with respect to the diffeomorphisms $(tx^1, x^2, \dots, x^n)$, $t \neq 0$, f is homogeneous of weight 1. Then by the homogeneous function theorem, cf. [3], f is linear. Hence $\mathcal{L}(x^1\partial_1 \otimes dx^1) = (x^1)^{(\lambda)} = (tr(x^1\partial_1 \otimes dx^1))^{(\lambda)}$ for some linear $\lambda : A \to \mathbf{R}$. Applying Lemma we end the proof of part 1.

2. Consider a linear natural operator $\mathcal{L} : T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,1)}F$. We can write $\mathcal{L}(x^1\partial_1 \otimes dx^1) = \sum_{j=1}^n \sum_{\mu=1}^k f_{j\mu}((x^i)^{(a^*_\nu)})d(x^j)^{(a^*_\mu)}$ for some functions $f_{j\mu} : \mathbf{R}^{\{1,\dots,n\}\times\{1,\dots,k\}} \to \mathbf{R}$. By the linearity and the invariance of $\mathcal{L}$ with respect to the homotheties $(tx^1, tx^2, \dots, tx^n)$, $t \neq 0$, we deduce that the functions $f_{j\mu}$ are constants. Now, by the invariance of $\mathcal{L}$ with respect to the diffeomorphisms $(x^1, tx^2, \dots, tx^n)$, $t \neq 0$, we deduce that $f_{j\mu} = 0$ for $j = 2, \dots, n$. Hence $\mathcal{L}(x^1\partial_1 \otimes dx^1) = d(x^1)^{(\lambda)} = d(tr(x^1\partial_1 \otimes dx^1))^{(\lambda)}$ for some linear $\lambda : A \to \mathbf{R}$. Applying Lemma we end the proof of part 2.

3. Consider a linear natural operator $\mathcal{L} : T^{(1,1)}_{|\mathcal{M}_n} \rightsquigarrow T^{(0,p)}F$, where $p \geq 2$. Similarly as above, from the linearity and the invariance of $\mathcal{L}$ with respect to the homotheties it follows that $\mathcal{L}(x^1\partial_1 \otimes dx^1) = 0$. Applying Lemma we finish the proof.

References

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