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$\mathcal{Q}_\alpha^{\#}$-bounded composition maps on normal classes

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Abstract. This note gives a function-theoretic characterization of composition maps sending (meromorphic) normal classes to their Möbius invariant subclasses.

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1. Introduction and results

Let Δ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the extended complex plane $\hat{\mathbb{C}}$ (i. e. the Riemann sphere), and let $\mathcal{M}(\Delta)$ be the class of all meromorphic functions $\Delta \to \hat{\mathbb{C}}$. Any holomorphic function $\phi : \Delta \to \Delta$ gives rise to a map $C_\phi : \mathcal{M}(\Delta) \to \mathcal{M}(\Delta)$ defined by $C_\phi f := f \circ \phi$, the *composition map* induced by ϕ.

The study of composition maps, or composition operators, acting on spaces of analytic functions has engaged many analysts for many years, where the basic problems are to relate the mapping properties of C_ϕ to function theoretic or geometric properties of the function ϕ. Among the mapping properties often considered are boundedness, compactness or weak compactness of C_ϕ, and a computation of the norm of C_ϕ. A good description of the classical problems in this area can be found in [12]. Relevant to the investigation here, we can cite a result of Rubel and Timoney which says that if X is a "reasonable" space of analytic functions on Δ, and if X is mapped onto itself by composition with any Möbius function, in a uniformly bounded way, then X must be a subset of the Bloch space B (see [11] for a more precise formulation). Arazy, Fisher and Peetre [1, Theorem 12, p. 125] proved that if X is any one of the spaces H^∞, the disk algebra, $BMOA$, $VMOA$, B (the Bloch space), or B_0 (the "little" Bloch space), then C_ϕ is a composition operator on X if and only if $\phi \in X$. (Of course, all of the spaces mentioned here are subspaces of B). Recent work has expanded on these results, see, for example, [8, 9, 10, 13, 14].

The normal class $\mathcal{N}$ is defined by all functions $f \in \mathcal{M}(\Delta)$ obeying

$$\|f\|_{\mathcal{N}} := \sup_{z\in\Delta}(1-|z|^2)f^{\#}(z) < \infty.$$

The little normal class $\mathcal{N}_0$ is given by all functions $f \in \mathcal{M}(\Delta)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z) = 0.$$

Here, $f^{\#}(z) := |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of $f \in \mathcal{M}(\Delta)$.

The problem addressed here deals with the boundedness of composition maps acting on the normal classes N and N_0. These are classes of meromorphic functions, so that the statements mentioned above, where the spaces under consideration are spaces of analytic functions, are relevant only by analogy. However, there are many ways in which the spaces N and N_0 resemble the spaces B and B_0, respectively, so it seems reasonable that there will be results similar to those involving B and B_0. It is this idea that we pursue here.

Note that $\mathcal{N}$ and $\mathcal{N}_0$ are not linear spaces. In fact, there are two functions in $\mathcal{N}$ whose sum is outside $\mathcal{N}$ [7].

Given $w \in \Delta$, let

$$\varphi_w(z) := \frac{w-z}{1-\bar{w}z}$$

be the Möbius transformation which exchanges w and 0. For $\alpha \in (-1,\infty)$, let $\mathcal{Q}_\alpha^{\#}$ be the set of all functions $f \in \mathcal{M}(\Delta)$ satisfying

$$\|f\|^2_{\mathcal{Q}_\alpha^{\#}} := \sup_{w\in\Delta}\int_\Delta [f^{\#}(z)]^2\Big[-\log|\varphi_w(z)|\Big]^\alpha dm(z) < \infty.$$

Here, dm means the two-dimensional Lebesgue measure on Δ. If the integral above tends to 0 as $|w|\to 1$ then we say that $f \in \mathcal{Q}_{\alpha,0}^{\#}$. It is easy to see that $\mathcal{Q}_\alpha^{\#}$ and $\mathcal{Q}_{\alpha,0}^{\#}$ are Möbius invariant in sense that $\|f\circ\psi\|_{\mathcal{Q}_\alpha^{\#}} = \|f\|_{\mathcal{Q}_\alpha^{\#}}$, where ψ is any Möbius self-map of Δ. $\mathcal{Q}_\alpha^{\#}$ and $\mathcal{Q}_{\alpha,0}^{\#}$ are nondecreasing with α [5]. On the one hand, if $\alpha \in (-1,0)$ then for any $r \in (0,1)$, one can choose a suitable $w \in \Delta$ (for instance, $|w| > (1+2r)/(2+r)$) such that

$$\begin{aligned}\|f\|^2_{\mathcal{Q}_\alpha^{\#}} &\geq (-\log r)^\alpha \int_{|z|>r} [(f\circ\varphi_w)^{\#}(z)]^2 dm(z)\\ &\geq (-\log r)^\alpha \int_{|z|<1/2} [f^{\#}(z)]^2 dm(z).\end{aligned}$$

These inequalities imply (by letting $r\to 1$) that if $f \in \mathcal{Q}_\alpha^{\#}$, $\alpha \in (-1,0)$ then f must be a constant. On the other hand, as shown in [3], if $\alpha > 1$, then $\mathcal{Q}_\alpha^{\#}$ and

$\mathcal{Q}^{\#}_{\alpha,0}$ are equal to $\mathcal{N}$ and $\mathcal{N}_0$, respectively. Hence $\mathcal{N}$ and $\mathcal{N}_0$ are the maximal Möbius invariant classes in this context.

We are going to work with composition maps acting on $\mathcal{N}$ and $\mathcal{N}_0$. Although both classes are not linear spaces (certainly, not topological vector spaces), we may still define a bounded composition map. As usually, for a holomorphic function $\phi : \Delta \to \Delta$, we say that $C_\phi : \mathcal{N}(\mathcal{N}_0) \to \mathcal{Q}^{\#}_{\alpha}$ is bounded if

$$\|C_\phi\| := \inf\{M : \|C_\phi f\|_{\mathcal{Q}^{\#}_{\alpha}} \leq M\|f\|_{\mathcal{N}}, \quad f \in \mathcal{N}(\mathcal{N}_0)\} < \infty.$$

Moreover, we say that $C_\phi : \mathcal{N}(\mathcal{N}_0) \to \mathcal{Q}^{\#}_{\alpha,0}$ is bounded if $C_\phi \mathcal{N}(\mathcal{N}_0) \subset \mathcal{Q}^{\#}_{\alpha,0}$ and $C_\phi : \mathcal{N}(\mathcal{N}_0) \to \mathcal{Q}^{\#}_{\alpha}$ is bounded.

With these definitions and notations, we obtain the following result.

Theorem 1. *Let $\alpha \in (0,\infty)$ and let $\phi : \Delta \to \Delta$ be a holomorphic function. Then the following are equivalent:*

(i) $C_\phi : \mathcal{N} \to \mathcal{Q}^{\#}_{\alpha}$ *is bounded .*

(ii) $C_\phi : \mathcal{N}_0 \to \mathcal{Q}^{\#}_{\alpha}$ *is bounded .*

(iii)
$$\sup_{w\in\Delta} \int_\Delta \Big[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2 \Big[-\log|\varphi_w(z)|\Big]^\alpha dm(z) < \infty. \tag{1}$$

An interesting $\mathcal{Q}^{\#}_{\alpha,0}$-version of the Theorem can be given as follows.

Corollary 1. *Let $\alpha \in (0,\infty)$ and let $\phi : \Delta \to \Delta$ be a holomorphic function. Then the following are true:*

(i) $C_\phi : \mathcal{N} \to \mathcal{Q}^{\#}_{\alpha,0}$ *is bounded if and only if*

$$\lim_{|w|\to 1} \int_\Delta \Big[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2 \Big[-\log|\varphi_w(z)|\Big]^\alpha dm(z) = 0. \tag{2}$$

(ii) $C_\phi : \mathcal{N}_0 \to \mathcal{Q}^{\#}_{\alpha,0}$ *is bounded if and only if* $\phi \in \mathcal{Q}^{\#}_{\alpha,0}$ *and* (1) *holds.*

Remark that the forms of (1) and (2) are not new and actually define the hyperbolic classes Q^h_α and $Q^h_{\alpha,0}$, respectively, in the notation used in [13]. However, we found it surprising that these conditions also characterize bounded composition maps on the normal classes. Observe that if $\alpha > 1$ then (iii) of the Theorem is automatically valid and hence C_ϕ is always bounded on $\mathcal{N}$. Meanwhile, it is not hard to verify that the conditions in (i) and (ii) of the Corollary are equivalent to

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} = 0$$

and $\phi \in \mathcal{N}_0$ which describes boundedness of $C_\phi : \mathcal{N} \to \mathcal{N}_0$ and $C_\phi : \mathcal{N}_0 \to \mathcal{N}_0$. In fact, this looks like a kind of "compactness" of the composition maps on $\mathcal{N}$ and $\mathcal{N}_0$ (see [8] for the holomorphic case).

Proof and comments. Our proof is based on the following lemma:

Lemma 1. *There are two functions $f_1, f_2 \in \mathcal{N}$ such that*

$$M_0 := \inf_{z\in\Delta}(1-|z|^2)[f_1^{\#}(z) + f_2^{\#}(z)] > 0.$$

Proof. We will consider two Schwarz triangle functions (see [6]). Take a hyperbolic triangle in the unit disk, with appropriate vertex angles which divide π, say, $\pi/m, \pi/n, \pi/p$ where m, n, p are natural numbers satisfying the inequality $1/m + 1/n + 1/p < 1$. Then map the interior of the triangle conformally onto the upper half plane so that the vertices map to 0, 1, and ∞. Then use analytic continuation to get a function f_1 meromorphic in Δ with poles at all the points corresponding to ∞. Such a function f_1 is a normal function, and the expression $(1-|z|^2)f_1^{\#}(z)$ is invariant under the Fuchsian group Γ_1 of alternate reflections of the triangle and is zero only at the vertices of the original triangle and their reflected images. Note that f_1 is a normal function because of the invariance of $(1-|z|^2)f_1^{\#}(z)$ under Γ_1, since this expression is bounded on any compact subset of the unit disk. Now take a second hyperbolic triangle congruent to the first, but with vertices at points disjoint from the vertices of the original triangle and the images of vertices of the original triangle under Γ_1. Then create another function f_2 for this second triangle by the same method as before, where f_2 and $(1-|z|^2)f_2^{\#}(z)$ are both invariant under the Fuchsian group Γ_2 of alternate reflections of the second triangle. Then $(1-|z|^2)f_1^{\#}(z)$ and $(1-|z|^2)f_2^{\#}(z)$ do not have a common zero, and by the invariance of $(1-|z|^2)f_j^{\#}(z)$ under Γ_j, $j = 1, 2$, each is bounded away from zero near where the other is zero. These bounds are uniform, since they can be taken on a sufficiently large fixed compact set. Thus, these two functions satisfy the conclusion of the lemma. $\boxed{QED}$

It is worth mentioning that our lemma is a meromorphic counterpart of Proposition 5.4 in [10]. Nevertheless, our method and construction are very different from that of [10] where they used Hadamard gap series to construct appropriate Bloch functions.

Proof of Theorem. That $(i) \Longrightarrow (ii)$ is clear. That (iii) implies (i) comes from the following inequality, where $w \in \Delta$:

$$\int_\Delta [(f\circ\phi)^{\#}(z)]^2 \big[-\log|\varphi_w(z)|\big]^\alpha dm(z) \le \|C_\phi f\|^2_{\mathcal{Q}_\alpha^{\#}}$$
$$\le \|f\|^2_{\mathcal{N}} \sup_{w\in\Delta}\int_\Delta \Big[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2 \big[-\log|\varphi_w(z)|\big]^\alpha dm(z).$$

So, it remains to check $(ii) \Longrightarrow (iii)$. First, note that if $f \in \mathcal{N}$ and $f_r(z) = f(rz)$ for $r \in (0,1)$, then $f_r \in \mathcal{N}_0$ with $\|f_r\|_{\mathcal{N}} \leq \|f\|_{\mathcal{N}}$. Secondly, for the two functions f_1 and f_2 given by the Lemma, we have that for $k = 1, 2$, and $w \in \Delta$,

$$\begin{aligned}\|C_\phi\|^2\|f_k\|^2_{\mathcal{N}} \geq \|C_\phi\|^2\|(f_k)_r\|^2_{\mathcal{N}} \geq \|C_\phi(f_k)_r\|^2_{\mathcal{Q}_\alpha^\#}\\ \geq \int_\Delta [f_k^\#(r\phi(z))|r\phi'(z)|]^2[-\log|\varphi_w(z)|]^\alpha dm(z).\end{aligned}$$

Because of the Lemma,

$$0 < M_0 \leq (1-|r\phi(z)|^2)[f_1^\#(r\phi(z)) + f_2^\#(r\phi(z))].$$

Thus,

$$\int_\Delta \Big[\frac{|r\phi'(z)|}{1-|r\phi(z)|^2}\Big]^2\big[-\log|\varphi_w(z)|\big]^\alpha dm(z) \leq \|C_\phi\|^2(\|f_1\|^2_{\mathcal{N}} + \|f_2\|^2_{\mathcal{N}})/M_0^2.$$

This inequality and Fatou's lemma imply (iii), and the proof is complete. $\boxed{QED}$

Proof of Corollary. Indeed, (i) can be verified by an argument very similar to that used in the proof of the Theorem. We need only to give a proof for (ii). On the one hand, if $C_\phi : \mathcal{N}_0 \to \mathcal{Q}^\#_{\alpha,0}$ is a bounded map, then it follows from the fact that the identity function is in $\mathcal{N}_0$ that $\phi \in \mathcal{Q}^\#_{\alpha,0}$. Also since $\mathcal{Q}^\#_{\alpha,0}$ is a subset of $\mathcal{Q}^\#_\alpha$, the Theorem says that ϕ satisfies the condition (1). On the other hand, suppose that $\phi \in \mathcal{Q}^\#_{\alpha,0}$ and it obeys (1). Let now $f \in \mathcal{N}_0$. Then for any $\epsilon > 0$ there is a $\delta \in (0,1)$ such that $(1-|z|^2)f^\#(z) < \epsilon$ whenever $|z| > \delta$. Hence

$$\begin{aligned}I(w) :=& \int_\Delta [(C_\phi f)^\#(z)]^2\big[-\log|\varphi_w(z)|\big]^\alpha dm(z)\\ \leq& \int_{|\phi(z)|\leq\delta}[(C_\phi f)^\#(z)]^2\big[-\log|\varphi_w(z)|\big]^\alpha dm(z)\\ &+ \epsilon^2\int_{|\phi(z)|>\delta}\Big[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2\big[-\log|\varphi_w(z)|\big]^\alpha dm(z)\\ :=& A(w) + \epsilon^2 B(w).\end{aligned}$$

Note that $f^\#$ is continuous in Δ and bounded in $\{z : |\phi(z)| \leq \delta\}$, $\phi \in \mathcal{Q}^\#_{\alpha,0}$, and

$$(C_\phi f)^\#(z) = f^\#(\phi(z))|\phi'(z)| \leq 2f^\#(\phi(z))\phi^\#(z),$$

so that $\lim_{|w|\to1} A(w) = 0$ follows from $\phi \in \mathcal{Q}^\#_{\alpha,0}$. Since $\sup_{w\in\Delta} B(w) < \infty$ (by (1)), it turns out that $\lim_{|w|\to1} I(w) = 0$. $\boxed{QED}$

Finally, we give some remarks. The conditions on ϕ of the Theorem and the Corollary can be described in terms of α-Carleson measures, as in [4], and can also be related to α-Nevanlinna counting functions (cf. [13]). In addition, we would like to use the Lemma above to produce bounded composition maps from the normal classes to the meromorphic spherical Besov classes, which can be viewed as Möbius invariant subsets (other than $\mathcal{Q}_\alpha^\#$) of $\mathcal{N}$ [2].

Recall that $B_p^\#, p \in (1, \infty)$ is the set of all $f \in \mathcal{M}(\Delta)$ for which

$$\|f\|_{B_p^\#}^p := \int_\Delta [f^\#(z)]^p (1-|z|^2)^{p-2} dm(z) < \infty.$$

It is well-known that $B_2^\#$ is the spherical Dirichlet space, and

$$B_p^\# \subset \cap_{\{\alpha: \frac{p-2}{p} < \alpha < 1\}} \mathcal{Q}_{\alpha,0}^\#$$

for $p \in (2, \infty)$ (see [2]).

We say that $C_\phi : \mathcal{N}(\mathcal{N}_0) \to B_p^\#$ is bounded if

$$\inf\{M : \|C_\phi f\|_{B_p^\#} \le M\|f\|_{\mathcal{N}}, \quad f \in \mathcal{N}(\mathcal{N}_0)\} < \infty.$$

From these definitions and notations, we conclude

Remark 1. Let $p \in (1, \infty)$ and let $\phi : \Delta \to \Delta$ be a holomorphic function. Then the following are equivalent:

(i) $C_\phi : \mathcal{N} \to B_p^\#$ is bounded .

(ii) $C_\phi : \mathcal{N}_0 \to B_p^\#$ is bounded .

(iii)
$$\int_\Delta \Big[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\Big]^p (1-|z|^2)^{p-2} dm(z) < \infty. \tag{3}$$

The proof is very similar to that of the Theorem.

Observe that $(iii) \implies (i)$ is just Theorem 4 of [9] and the condition (3) defines the hyperbolic Besov spaces.

We should point out that $\mathcal{Q}_\alpha^\#$ and $\mathcal{Q}_{\alpha,0}^\#$ are produced by composition between the most typical Möbius self-map φ_w of the unit disk and the meromorphic weighted Dirichlet classes, denoted by $D_\alpha^\#$, which are the sets of all $f \in \mathcal{M}(\Delta)$ with

$$\|f\|_{D_\alpha^\#}^2 := \int_\Delta [f^\#(z)]^2 (-\log|z|)^\alpha dm(z) < \infty.$$

However, the classes $D_\alpha^\#$ are not Möbius invariant classes except for $D_0^\#$. As before, we define $C_\phi : \mathcal{N}(\mathcal{N}_0) \to D_\alpha^\#$ to be a bounded map provided

$$\inf\{M : \|C_\phi f\|_{D_\alpha^\#} \le M\|f\|_{\mathcal{N}}, \quad f \in \mathcal{N}(\mathcal{N}_0)\} < \infty.$$

With no more than trivial changes of the argument for the Theorem, we have

Proposition 1. *Let $\alpha \in (-1,\infty)$ and let $\phi : \Delta \to \Delta$ be a holomorphic function. Then the following are equivalent:*

(i) $C_\phi : \mathcal{N} \to D_\alpha^\#$ *is bounded .*

(ii) $C_\phi : \mathcal{N}_0 \to D_\alpha^\#$ *is bounded .*

(iii)
$$\int_\Delta \Big[\frac{|\phi'(z)|}{1-|\phi(z)|^2}\Big]^2 (-\log|z|)^\alpha dm(z) < \infty. \tag{4}$$

As a fact of matter, (4) introduces the hyperbolic Dirichlet spaces of the unit disk which especially include the hyperbolic Bloch spaces [13].

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