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On sewing neighbourly polytopes

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Abstract. In 1982, I. Shemer introduced the sewing construction for neighbourly 2m-polytopes. We extend the sewing to simplicial neighbourly d-polytopes via a verification that is not dependent on the parity of the dimension. We present also descibable classes of 4-polyopes and 5-polytopes generated by the construction.

Keywords: d-polytope, neighbourly, sewing, desciption

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Introduction

The increased use of polytopes, as models for problems in areas such as economics (see, for example, the correspondence between polytope pairs and equilibria of bimatrix games in [5]), operations research and theoretical chemistry, emphasises the importance of well-understood examples for which all the facets are explicitly described.

We examine the elegant sewing construction of Shemer from this point of view, and show that it is a practical tool for generating describable non-cyclic simplicial neighbourly polytopes in both even and odd dimensions. In a future paper, we consider these describable polytopes from the point of view of Hadwiger's Covering conjecture; cf. [1].

Our notation closely follows the ones in [2] and [3].

Let Y be a set of points in $\mathbb{R}^d$. Then conv Y and aff Y denote, respectively, the convex hull and the affine hull of Y. For sets $Y_1, Y_2, \ldots, Y_k$ let

$$[Y_1, Y_2, \ldots, Y_k] = \text{conv}\,(Y_1 \cup Y_2 \cup \cdots \cup Y_k)$$

and

$$\langle Y_1, Y_2, \ldots, Y_k \rangle = \text{aff}\,(Y_1 \cup Y_2 \cup \cdots \cup Y_k).$$

For a point $y \in \mathbb{R}^d$, let $[y] = [\{y\}]$ and $\langle y \rangle = \langle \{y\} \rangle$.

Let $P \subset \mathbb{R}^d$ denote a (convex) d-polytope with $\mathcal{V}(P)$, $\mathcal{F}(P)$ and $\mathcal{L}(P)$ denoting, respectively, the set of vertices, the set of facets and the *face lattice* of P. We recall that $\mathcal{L}(P)$ is the collection of all faces of P ordered by inclusion. Let $\mathcal{B}(P) = \mathcal{L}(P)\backslash\{P\}$. For $G \in \mathcal{B}(P)$, let $\mathcal{F}(G, P) = \{F \in \mathcal{F}(P) \mid G \subseteq F\}$.

Let $F \in \mathcal{F}(P)$ and $y \in \mathbb{R}^d \backslash \langle F \rangle$. Then y is *beneath* (*beyond*) F, with respect to P, if y and P are (are not) on the same side of the hyperplane $\langle F \rangle$.

Let $y \notin P$ and $P^* = [P, y]$. We recall from [2] the following relation between $\mathcal{B}(P)$ and $\mathcal{B}(P^*)$.

Lemma 1. *Let $G \in B(P)$. Then*

1. *$G \in \mathcal{B}(P^*)$ if, and only if, y is beneath some $F \in \mathcal{F}(G, P)$, and*
2. *$G^* = [G, y] \in \mathcal{B}(P^*)$ if, and only if, either $y \in \langle G \rangle$ or y is beneath some F_1 and beyond some F_2 in $\mathcal{F}(G, P)$.*

Moreover, each face of P^* is obtained in this manner.

Let $\{G_1, G_2, \ldots, G_k\} \subset \mathcal{B}(P)$ such that $G_1 \subset G_2 \subset \cdots \subset G_k$ and $\emptyset \neq G_1$. We set $\mathcal{T} = \{G_i\}_{i=1}^k$, and call it a *tower* in P. For the sake of convenience, let $\mathcal{F}_i = \mathcal{F}(G_i, P)$. Then $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_k$, and we set

$$\mathcal{C}(\mathcal{T}, P) = (\mathcal{F}_1 \backslash \mathcal{F}_2 \backslash (\cdots \backslash \mathcal{F}_k) \cdots);$$

that is,

$$\mathcal{C}(\mathcal{T}, P) = \begin{cases} (\mathcal{F}_1 \backslash \mathcal{F}_2) \cup (\mathcal{F}_3 \backslash \mathcal{F}_4) \cup \cdots \cup (\mathcal{F}_{k-1} \backslash \mathcal{F}_k) & \text{if} \quad k \text{ is even} \\ (\mathcal{F}_1 \backslash \mathcal{F}_2) \cup \cdots \cup (\mathcal{F}_{k-2} \backslash \mathcal{F}_{k-1}) \cup \mathcal{F}_k & \phantom{\text{if}} \quad k \text{ is odd.} \end{cases}$$

For $y \in \mathbb{R}^d$, we say that y *lies exactly beyond* $\mathcal{C}(\mathcal{T}, P)$, with respect to P, if y is beyond (beneath) each facet in $\mathcal{C}(\mathcal{T}, P)$ $(\mathcal{F}(P) \backslash \mathcal{C}(\mathcal{T}, P))$. Recalling that $\mathcal{F}(P) = \mathcal{F}(\emptyset, P)$, it is convenient to let $G_0 = \emptyset$, $G_{k+1} = P$, $\mathcal{F}_0 = \mathcal{F}(P)$ and $\mathcal{F}_{k+1} = \emptyset$. Then, for suitable i, the following are equivalent:

$$\left.\begin{array}{l} \bullet \ \ y \text{ lies exactly beyond } \mathcal{C}(\mathcal{T}, P). \\ \bullet \ \ y \text{ is beyond (beneath) each } \mathcal{F} \in \mathcal{F}_{2i+1} \backslash \mathcal{F}_{2i+2} (\mathcal{F}_{2i} \backslash \mathcal{F}_{2i+1}). \end{array}\right\} \qquad (1)$$

We note from [4] that, given P and $\mathcal{T}$, there is a point in $\mathbb{R}^d$ that lies exactly beyond $\mathcal{C}(\mathcal{T}, P)$.

Let $G \in \mathcal{B}(P)$. Then G is a *universal face* of P if $[G, S] \in \mathcal{B}(P)$ for every $S \subset \mathcal{V}(P)$ with $|S| \leq \left[\frac{1}{2}(d - 1 - \dim G)\right]$. Thus, each $(d-2)$-face and each facet of P is a universal face of P. We remark also that if the empty set $\emptyset$ is a universal face of P then

$$[S] \in \mathcal{B}(P) \quad \text{for every} \quad S \subset \mathcal{V}(P) \quad \text{with} \quad |S| \leq \left[\tfrac{d}{2}\right];$$

that is, P is a *neighbourly* d-polytope.

Let $Q \subset \mathbb{R}^d$ denote a simplicial neighbourly d-polytope and $m = \left[\frac{d}{2}\right]$. Then $d \in \{2m, 2m+1\}$ and for $0 \leq j \leq m$, the following are equivalent for a $(2j-1)$-face G of Q:

- G is a universal $(2j-1)$-face of Q.
- $[G, S] \in \mathcal{B}(Q)$ for every $S \subset \mathcal{V}(Q)$ with $|S| \leq m - j$.
- $[X] \in \mathcal{B}(Q)$ for every $X \subset \mathcal{V}(Q)$ such that $\mathcal{V}(G) \subset X$ and $|X| = m + j$.

(2)

Finally, let $\mathcal{T} \subset \mathcal{F}(Q)$ be a tower. Then $\mathcal{T}$ is a *universal tower* if $\mathcal{T} = \{G_j\}_{j=1}^m$, each G_j is a universal face of Q and $|\mathcal{V}(G_j)| = 2j$. Now if $\mathcal{T}$ is a universal tower in Q, $x^* \in \mathbb{R}^d$ lies exactly beyond $\mathcal{C}(\mathcal{T}, Q)$ and $Q^* = [Q, x^*]$ then we say that Q^* is obtained by *sewing* x^* *onto* Q.

With the preceding notation, we cite from [2] the Sewing Theorem of Shemer:

Theorem 1. *Let Q be a neighbourly $2m$-polytope and $Q^* = [Q, x^*]$ be obtained by sewing x^* onto Q through the universal tower $\{G_j\}_{j=1}^m$, $m \geq 2$.*

1. *Q^* is a neighbourly $2m$-polytope with $\mathcal{V}(Q^*) = \mathcal{V}(Q) \cup \{x^*\}$.*
2. *If $0 \leq j \leq m$ is even then G_j is a universal face of Q^*.*
3. *If $x \in \mathcal{V}(G_j) \backslash \mathcal{V}(G_{j-1})$ for some $1 \leq j \leq m$ then $[G_{j-1}, x, x^*]$ is a universal face of Q^*.*

1. Extension and application

Let $Q \subset \mathbb{R}^d$ denote a simplicial neighbourly d-polytope with $\mathcal{V}(Q) = \{x_1, x_2, \ldots, x_{n-1}\}$, $n \geq d + 3$ and $m = \left[\frac{d}{2}\right] \geq 2$. Let $\mathcal{T} = \{G_j\}_{j=1}^m$ be a universal tower in Q with

$$G_j = \{x_1, x_2, \ldots, x_{2j}\} \quad \text{for} \quad j = 1, \ldots, m.$$

Let $G_0 = \emptyset$, $G_{m+1} = Q$ and $\mathcal{F}_j = \mathcal{F}(G_j, Q)$. Then $\mathcal{F}_0 = \mathcal{F}(Q)$, $\mathcal{F}_{m+1} = \emptyset$ and as Q is neighbourly, G_0 is a universal face of Q. Let

$$\mathcal{C} = \mathcal{C}(\mathcal{T}, Q) = \mathcal{F}_1 \backslash (\mathcal{F}_2 \backslash (\ldots \mathcal{F}_m) \ldots),$$

$x_n \in \mathbb{R}^d$ lie exactly beyond $\mathcal{C}$ with respect to Q, and set $Q_n = [Q, x_n] = [x_1, x_2, \ldots, x_{n-1}, x_n]$.

In the extension of Theorem 1, we use only 1, (1) and (2). To start: we have from (1) that $x_n \notin \langle \tilde{F} \rangle$ for any $\tilde{F} \in \mathcal{F}_0$, and x_n is beneath each $F \in \mathcal{F}_0 \backslash \mathcal{F}_1$. Since each vertex of Q is contained in some such F, it follows from 1 that $\mathcal{V}(Q_n) = \{x_1, \ldots, x_{n-1}, x_n\}$ and Q_n is simplicial.

Theorem 2 (The Sewing Theorem). *Let $Q \subset \mathbb{R}^d$ be a simplicial neighbourly d-polytope with $V(Q) = \{x_1, x_2, \ldots, x_{n-1}\}$ and the universal tower $\mathcal{T} = \{G_j\}_{j=1}^m$ as described above, $n \geq d+3$ and $m = [d/2] \geq 2$. Let $Q_n = [Q, x_n]$ be obtained by sewing x_n onto Q through $\mathcal{T}$.*

1. *Q_n is a simplicial neighbourly d-polytope with $\mathcal{V}(Q_n) = \mathcal{V}(Q) \cup \{x_n\}$.*
2. *Let $0 \leq j \leq m$ be even. Then G_j is a universal face of Q_n.*
3. *Let $G'_j = [G_{j-1}, x, x_n]$ for some $x \in \{x_{2j-1}, x_{2j}\}$ and $1 \leq j \leq m$. Then G'_j is a universal face of Q_n.*

Proof. (1) Let $X \subset \mathcal{V}(Q_n)$, $|X| = m$. We need to show that $[X] \in \mathcal{B}(Q_n)$. We apply (1) if $[X] \in \mathcal{B}(Q)$, and (2) if $[X] = [X', x_n]$ and $[X'] \in \mathcal{B}(Q)$.

Case 1. $x_n \notin X$.

Then $[X] \in \mathcal{B}(Q)$ by (2). Let $u = \left[\frac{m-1}{2}\right]$ and

$$Y = \{x_1, x_2, x_5, x_6, \ldots, x_{4u+1}, x_{4u+2}\}.$$

Then $|Y| = 2u+2$, $Y \subset G_{2u+1} \subseteq G_m$ and either $Y = X$ and m is even or $Y \neq X$ and there is a smallest integer i such that $0 \leq i \leq u$ and $\{x_{4i+1}, x_{4i+2}\} \not\subset X$.

In case of the former, there is an $F \in \mathcal{F}_m$ such that $X \subset F$. Since m is even and $F \in \mathcal{F}_m \backslash \mathcal{F}_{m+1}$, x_n is beneath F by (1). In case of the latter, let $U = X \cup \mathcal{V}(G_{2i})$. Then

$$\begin{aligned} |U| &= |X| + |\mathcal{V}(G_{2i})| - |X \cap \mathcal{V}(G_{2i}) \\ &\leq m + 4i - \left| \bigcup_{k=0}^{i-1} \{x_{4k+1}, x_{4k+2}\} \right| = m + 2i. \end{aligned}$$

Since G_{2i} is a universal face of Q, it follows by (2) that $[U] \in \mathcal{B}(Q)$. Thus, there is an $F \in \mathcal{F}(Q)$ such that $X \cup G_{2i} \subset F$ and $G_{2i+1} \not\subset F$. Then $F \in \mathcal{F}_{2i} \backslash \mathcal{F}_{2i+1}$, and x_n is beneath F by (1).

Case 2. $x_n \in X$.

Let $X' = X \backslash \{x_n\}$. Then $[X] = [X', x_n]$, $[X'] \in \mathcal{B}(Q) \cap \mathcal{B}(Q_n)$ from above, and there is an $F \in \mathcal{F}(Q)$ such that $X' \subset F$ and x_n is beneath F.

Let $w = \left[\frac{m-2}{2}\right]$ and

$$Z = \{x_3, x_4, x_7, x_8, \ldots, x_{2w+3}, x_{2w+4}\}.$$

Then $|Z| = 2w + 2$, $Z \subset G_{2w+2} \subseteq G_m$ and either $Z = X'$ and m is odd or $Z \neq X'$ and there is a smallest i such that $0 \leq i \leq w$ and $\{x_{2i+3}, x_{2i+4}\} \not\subset X'$.

In case of the former, there is an $F' \in \mathcal{F}_m \backslash \mathcal{F}_{m+1}$ such that $X' \subset F'$. Since m is odd, x_n is beyond F' by (1). In case of the latter, let $W = X' \cup \mathcal{V}(G_{2i+1})$. Then

$$|W| \leq (m-1) + (4i+2) - 2i = m + (2i+1),$$

$[W] \in \mathcal{B}(Q)$ by (2), and there is an $F' \in \mathcal{F}_{2i+1} \backslash \mathcal{F}_{2i+2}$ such that $X' \subset F$. Again, x_n is beyond F' by (1).

(2)Since Q_n is neighbourly; G_0 is a universal of Q_n, and we may assume that the assertion is true for $j-2$. Let $j \geq 2$, $V(G_j) \subset X \subset V(Q_n)$ and $|X| = m+j$. By (2), we need to show that $[X] \in B(Q_n)$.

Case 1. $x_n \notin X$.

Let $X' = X \backslash \{x_{2j-1}, x_{2j}\}$. Then $|X'| = m+j-2$, $\mathcal{V}(G_{j-2}) \subset \mathcal{V}(G_{j-1}) \subset X'$ and $[X'] \in \mathcal{B}(Q) \cap \mathcal{B}(Q_n)$ by (2) and the induction. By 1, there is an $F' \in \mathcal{F}(Q)$ such that $X' \subset F'$ and x_n is beneath F'. Since $F' \in \mathcal{F}_{j-1}$ and x_n is beyond each facet in $\mathcal{F}_{j-1} \backslash \mathcal{F}_j$ when j is even, we have that $F' \in \mathcal{F}_j$; that is $X \subset F'$.

Case 2. $x_n \in X$.

From above, $[X \backslash \{x_n\}] \in \mathcal{B}(Q) \cap \mathcal{B}(Q_{n-1})$ and there is an $F \in \mathcal{F}(Q)$ such that $X \backslash \{x_n\} \subset F$ and x_n is beneath F.

Let $\tilde{X} = X \backslash \{x_{2j-3}, x_{2j-2}, x_n\}$. Then $[\tilde{X}] \in \mathcal{B}(Q) \cap \mathcal{B}(Q_n)$, $[\tilde{X}, x_n] \in \mathcal{B}(Q_n)$ by the induction and there is an $\tilde{F} \in \mathcal{F}_{j-2}$ such that $\tilde{X} \subset \tilde{F}$ and x_n is beyond $\tilde{F}$. Now (1) and j even imply that $\tilde{F} \in \mathcal{F}_{j-1}$; that is $X \backslash \{x_n\} \subset \tilde{F}$.

(3) Let $\mathcal{V}(G'_j) = \mathcal{V}(G_{j-1}) \cup \{x, x_n\} \subset X \subset \mathcal{V}(Q_n)$, $|X| = m+j$ and $X' = X \backslash \{x_n\}$.

Case 1. j is odd.

Let $X'' = X' \backslash \{x\}$ and note that G_{j-1} is a universal face of both Q and Q_n. Thus,

$$[X''] \subset [X'] \in \mathcal{B}(Q) \cap \mathcal{B}(Q_n),\ [X'', x_n] \in \mathcal{B}(Q_n)$$

and there is an F' (F'') in $\mathcal{F}_{j-1}$ such that $X' \subset F'$ ($X'' \subset F''$) and x_n is beneath F' (beyond F''). Now (1) and j odd imply that $F'' \in \mathcal{F}_j$. Then $X' \subset X'' \cup \{x_{2j-1}, x_{2j}\} \subset F''$, and $[X] = [X', x_n] \in \mathcal{B}(Q_n)$ by 1.

Case 2. j is even.

Let $\tilde{X} = X' \backslash \{x_{2j-3}, x_{2j-2}\}$ and $\{x_{2j-1}, x_{2j}\} = \{x, \bar{x}\}$. Then $\mathcal{V}(G_{j-2}) \subset \tilde{X} \cup \{x_n\}$, $\mathcal{V}(G_j) \subset X' \cup \{\bar{x}\}$, $|\tilde{X} \cup \{x_n\}| = m+j-2$, $|X' \cup \{\bar{x}\}| = m+j$, and it follows by (2) and 2 that

$$[\tilde{X}] \subset [X'] \subset [X', \bar{x}] \in \mathcal{B}(Q) \cap \mathcal{B}(Q_n)$$

and $[\tilde{X}, x_n] \in B(Q_n)$.

Thus, there is an F' $(\tilde{F})$ in $\mathcal{F}(Q)$ such that $X' \subset F'$ $(\tilde{X} \subset \tilde{F})$ and x_n is beneath F' (beyond $\tilde{F}$). Now $\tilde{F} \in \mathcal{F}_{j-2}$, (1) and j even imply that $\tilde{F} \in \mathcal{F}_{j-1}$; that is, $X' \subset \tilde{F}$. QED

In order to complete the verification of the sewing construction in $\mathbb{R}^d$, we need to demonstrate a simplicial neighbourly d-polytope with a universal tower.

Let $m = \left[\frac{d}{2}\right] \geq 2$, $v = 2m+3$ and $Q_v(d) \subset \mathbb{R}^d$ denote a cyclic d-polytope with the ordered vertices $x_1 < x_2 < \cdots < x_v$. Then Gale's Evenness Condition yields explicitly the facets of $Q_v(d)$. From the explicit list of facets, it is easy to check that $Q_v(d)$ is neighbourly with

$$\{[x_1, x_2, \ldots, x_{2j}]\}_{j=1}^{m}$$

as a universal tower.

Let us now use Theorem 2 to generate a describable class of d-polytopes.

With the preceding $Q_v(d)$ and the reverse ordering on the vertices, we note that

$$\mathcal{T} = \{[x_{v+1-2j}, \ldots, x_{v-1}, x_v]\}_{j=1}^{m}$$

is also a universal tower. Let $x_{v+1} \in \mathbb{R}^d$ lie exactly beyond $\mathcal{C}(\mathcal{T}, Q_v(d))$. Then $Q_{v+1}(d) = [Q_v(d), x_{v+1}]$ is a simplicial neighbourly d-polytope, and with $x = x_{v+2-2j}$ in 3,

$$\{[x_{v+2-2j}, \ldots, x_v, x_{v+1}]\}_{j=1}^{m}$$

is a universal tower.

Repeating this particular sewing, we obtain a class of simplicial non-cyclic neighbourly d-polytopes $\{Q_n(d)\}_{n \geq 2m+4}$ such that

$$Q_n(d) = [x_1, x_2, \ldots, x_n]$$

with a universal tower $\{[x_{n+1-2j}, \ldots, x_{n-1}, x_n]\}_{j=1}^{m}$.

In the case $m = 2$ and $n \geq 8$, $Q_n(4)$ and $Q_n(5)$ are particularly easy to describe:

- $\mathcal{F}(Q_n(4)) = A \cup \left(\bigcup_{j=7}^{n} B_j\right) \cup \left(\bigcup_{j=8}^{n} C_j\right) \cup \left(\bigcup_{j=9}^{n} D_j\right) \cup Y_n \cup Z_n$

where

$$A = \{[x_1, x_2, x_3, x_4],\ [x_1, x_2, x_4, x_5],\ [x_1, x_2, x_5, x_6],\ [x_2, x_3, x_4, x_5],\\ [x_2, x_3, x_5, x_6],\ [x_3, x_4, x_5, x_6],\ [x_1, x_2, x_3, x_7],\ [x_1, x_3, x_4, x_7],\ [x_1, x_4, x_5, x_7]\},$$

$$B_j = \{[x_{j-3}, x_{j-2}, x_{j-1}, x_j]\},$$

$$C_j = \{[x_1, x_2, x_{j-2}, x_j],\ [x_2, x_3, x_{j-2}, x_j],\ [x_3, x_4, x_{j-2}, x_j],\ [x_1, x_5, x_{j-2}, x_j]\},$$

$$D_j = \{[x_i, x_{i+2}, x_{j-2}, x_j] \mid i = 4, \ldots, j-5\},$$

$$Y_n = \{[x_i, x_{i+2}, x_{n-1}, x_n] \mid i = 4, \ldots, n-4\},$$

and

$$\mathbb{Z}_n = \{[x_1, x_2, x_{n-1}, x_n],\ [x_2, x_3, x_{n-1}, x_n],\ [x_3, x_4, x_{n-1}, x_n],\ [x_1, x_5, x_{n-1}, x_n]\}.$$

- $\mathcal{F}(Q_n(5)) = A \cup \left(\bigcup_{j=7}^{n} B_j\right) \cup \left(\bigcup_{j=8}^{n} C_j\right) \cup \left(\bigcup_{j=9}^{n} D_j\right) \cup Y_n \cup Z_n$

where

$$A = \{[x_1, x_2, x_3, x_4, x_5],\ [x_1, x_2, x_3, x_5, x_6],\ [x_1, x_3, x_4, x_5, x_6],\\ [x_1, x_2, x_3, x_4, x_7],\ [x_1, x_2, x_4, x_5, x_7],\ [x_2, x_3, x_4, x_5, x_7]\},$$

$$B_j = \{[x_1, x_{j-3}, x_{j-2}, x_{j-1}, x_j],\ [x_3, x_{j-3}, x_{j-2}, x_{j-1}, x_j]\},$$

$$C_j = \{[x_1, x_2, x_3, x_{j-2}, x_j],\ [x_1, x_3, x_4, x_{j-2}, x_j],\ [x_1, x_2, x_5, x_{j-2}, x_j],\\ [x_2, x_3, x_5, x_{j-2}, x_j]\},$$

$$D_j = \{[x_1, x_i, x_{i+2}, x_{j-2}, x_j],\ [x_3, x_i, x_{i+2}, x_{j-2}, x_j] \mid i = 4, \ldots, j-5\}$$

$$Y_n = \{[x_1, x_i, x_{i+2}, x_{n-1}, x_n],\ [x_3, x_i, x_{i+2}, x_{n-1}, x_n] \mid i = 4, \ldots, n-4\}$$

and

$$Z_n = \{[x_1, x_2, x_3, x_{n-1}, x_n],\ [x_1, x_3, x_4, x_{n-1}, x_n],\ [x_1, x_2, x_5, x_{n-1}, x_n],\\ [x_2, x_3, x_5, x_{n-1}, x_n]\}.$$

Finally, we remark that $|\mathcal{F}(Q_n(4))| = \frac{n(n-3)}{2}$, $|\mathcal{F}(Q_n(5))| = (n-3)(n-4)$ and, with $Y_7 = \emptyset$ in the case $n = 7$, the preceding formulae also yield the set of facets of the cyclic polytopes $Q_7(4)$ and $Q_7(5)$.

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