Note di Matematica **20,** n. 1, 2000/2001, 37–42.

On determinants and permanents of minimally 1-factorable cubic bipartite graphs

D. Labbate*
Dipartimento di Matematica, Università della Basilicata, via N. Sauro 85, 85100 Potenza, Italy, ld487sci@pzuniv.unibas.it

Received: 12 November 1999; accepted: 2 December 1999.

Abstract. A minimally 1-factorable cubic bigraph is a graph in which every 1-factor lies in precisely one 1-factorization. The author investigates determinants and permanents of such graphs and, in particular, proves that the determinant of any minimally 1-factorable cubic bigraph of girth 4 is 0.

Keywords: determinant, permanent, 1-factorization, hamiltonian circuit, bipartite

MSC 2000 classification: primary 05c70, secondary 05c45, 05c50

1. Preliminaries

We assume that every graph is simple unless differently stated.

Every r-regular bigraph G with bipartition sets $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_n\}$ gives rise to a square matrix A of order n in the following way:

$$(A)_{ij} := \begin{cases} 1 & \text{if} \quad a_i b_j \quad \text{is an edge of } G\,; \\ 0 & \qquad\qquad \text{otherwise} \end{cases}$$

The matrix A is referred to as the **adjacency matrix** of G. Since the bigraph G is r-regular, A has precisely r 1's in each row and column. Conversely, every square $\{0,1\}$-matrix A of order n with precisely r 1's in each row and column represents an r-regular bigraph of order $2n$.

The adjacency matrix A of an r-regular bigraph G depends on the choice of the labelling of the vertices in V_1 and V_2. Every matrix obtained from A by permuting independently rows and columns still represents the same graph G. Therefore,

$$\det(G) := |\det(A)| \quad \text{and} \quad \operatorname{per}(G) := \operatorname{per}(A)$$

*The author was supported by GNSAGA of the Italian CNR (project “Calcolo Simbolico”) and by the Italian Ministry MURST.

are important algebraic invariants for the classes of adjacency matrices representing one and the same r-regular bigraph. Recall that the **permanent** per(A) of a matrix A is defined by

$$\mathrm{per}(A) \;:=\; \sum_{\pi \in S_n} a_{1,\pi(1)} \ldots a_{n,\pi(n)}.$$

In general, one has $0 \leq \det(G) \leq \mathrm{per}(G)$. Therefore, an r-regular bigraph with $\det(G) = \mathrm{per}(G)$ is called **det-extremal**, cf. [2].

An **r-factor** of a graph G is an r-regular spanning subgraph. Hence, a **1-factor** or a **perfect matching** is any set of edges in which each vertex appears precisely once. A **1-factorization** of G is a partition of the edge set of G into 1-factors.

It is well known that:

(i) Every 1-factorization of a κ-regular bigraph corresponds to a decomposition of the adjacency matrix in a direct sum of κ distinct permutation matrices.

(ii) Each 1-factor F of a graph G corresponds to a permutation submatrix of the (reduced) adjacency matrix of G.

(iii) The permanent of an r-regular bigraph G equals the number of the 1-factors of G.

A Corollary to the famous P. Hall Marriage Theorem [3] says that:

Corollary 1. *Every r-regular bigraph G has a 1-factor and every 1-factor can be completed to a 1-factorization of G.*

This, in turn, gives rise to the following:

Definition 1. An r-regular bigraph G is called **minimally 1-factorable** if every 1-factor of G lies in precisely one 1-factorization of G.

All 1-regular and 2-regular bigraphs are minimally 1-factorable, whereas this is no longer true for r-regular bigraphs with $r \geq 3$. The Heawood graph H (i. e. the flag graph of the Fano Plane $\Gamma(PG(2,2))$) is an instance of a minimally 1-factorable cubic bigraph (cf. [2]). In [2] has been proved that *minimally 1-factorable r-regular bigraphs exist only for $r \leq 3$.*

Definition 2. The **valency** of a 1-factor F of an r-regular bigraph G is the number of distinct one-factorizations of G that contain F.

2. Hamiltonian Circuits

The following Lemma has been proved in [2]:

Lemma 1. *Let G be an r-regular bigraph and F a 1-factor of G having valency 1. Then, the removal $G \backslash F$ is an $(r-1)$-regular bigraph with precisely one 1-factorization.*

The above Lemma can be sharpened into:

Corollary 2. *Let G be a cubic bigraph and F a 1-factor of G.*
(i) The removal $G\backslash F$ is a 2-factor of G.
(ii) If F has valency 1, the removal $G\backslash F$ is a Hamiltonian circuit of G.
(iii) If the removal $G\backslash F$ splits into at least two distinct circuits, then F has valency ≥ 2. In this case, G is not minimally 1-factorable.

Proof. (i) Complete F to a 1-factorization F, F_2, F_3 of G. The edges of the removal $G\backslash F$ are just the edges in $F_2 \cup F_3$. So $F_2 \cup F_3$ induce a 2-regular spanning subgraph of G.

(ii) The removal $G' := G\backslash F$ is a 2-factor, i. e. a union of a number t of disjoint circuits of even length. If one had $t \geq 2$, the graph G' would have more than one 1-factorization; in fact, a first 1-factorization of G' is obtained by colouring the edges of every circuit alternatively, say red and blue; a distinct second 1-factorization is obtained by exchanging the colours red and blue in one, and only one, circuit and leaving the colours invariant in all the other circuits. This contradicts Lemma 1. Hence, $t = 1$, i.e. G' is a single circuit passing through all vertices of G; thus G' is a Hamiltonian circuit of G.

(iii) The first statement of (iii) is logically equivalent to (ii). Clearly, a 1-factor of valency ≥ 2 implies that G is no longer minimally 1-factorable. $\boxed{QED}$

3. Signatures of 1-factorizations

Definition 3. Let G be a cubic bigraph of order $2n$.
(i) For a fixed labelling of the vertices in $V(G)$, every 1-factor F of G can uniquely be represented by a permutation $\pi \in S_n$. We define the **sign of the 1-factor F** to be the sign of the permutation π.
(ii) Let F_1, F_2, F_3 be a 1-factorization of G. For a fixed labelling, represent the 1-factor F_i by the permutation $\pi_i \in S_n$, $i = 1, 2, 3$. Let s_i be the sign of the permutation π_i. Define the **signature** of this 1-factorization to be the number $|s_1 + s_2 + s_3|$.

Remark 1. The sign of a 1-factor depends on the choice of the labelling for the vertices, whereas the signature of a 1-factorization does not depend on this choice. The range for the signature is the set $\{1, 3\}$.

Proposition 1. *Let G be a cubic bigraph. Then, every two 1-factors of valency 1 belonging to the same 1-factorization have the same sign.*

Proof. Let F_1, F_2, F_3 be a 1-factorization of G and suppose that both F_2 and F_3 have valency 1. By Corollary 2, since F_3 has valency 1, the 2-factor $F_{12} := F_1 \cup F_2$ is a Hamiltonian circuit of G. We may choose a suitable labelling for the vertices of G in such a way that F_{12} turns out to be the sequence of vertices

$$a_0 b_0 a_1 b_1 \dots a_{n-1} b_{n-1}$$

with $a_i b_i \in F_1$ and $a_{i+1} b_i \in F_2$ (indices taken modulo n). This labelling gives rise to a reduced adjacency matrix $A = (a_{ij})$ of order n with $a_{ij} = 1$ if, and only if, $a_{i+1} b_{j+1}$ is an edge of G. The 1-factor F_1 corresponds to the entries 1 in the main diagonal of A and it is represented by the identical permutation. The 1-factor F_2 corresponds to the entries 1 in positions $(i, i-1)$, $i = 2, \dots, n$ as well as $(1, n)$. Hence F_2 is represented by the permutation ρ with orbit $(n\ n-1\ \dots\ 2\ 1)$. The 1-factor F_3 corresponds to the remaining n entries 1 in position $(i, \pi(i))$ for some permutation $\pi \in S_n$ acting on the index set $\{1, \dots, n\}$. Since the entries 1 in the main diagonal make up the first 1-factor, π has no fixed points in $\{1, \dots, n\}$. Should π decompose $\{1, \dots, n\}$ into more than one orbit, say

$$(1\ \pi(1)\ \pi^2(1)\ \dots\ \pi^{r-1}(1))\ \ (a\ \pi(a)\ \dots)$$

for some $r < n-1$ with $a \notin \{1\ , \pi(1)\ , \pi^2(1)\ , \dots\ , \pi^{r-1}(1)\}$, then

$$a_1,\ b_{\pi(1)},\ a_{\pi(1)},\ b_{\pi^2(1)},\ \dots\ ,\ a_{\pi^{r-2}(1)},\ b_{\pi^{r-1}(1)},\ a_{\pi^{r-1}(1)},\ b_1$$

make up a circuit of length $2r < 2n$ (in fact the edge $a_1 b_1 \in F_1$ closes the circuit). This, however, contradicts the fact that the 2-factor $F_{13} := F_1 \cup F_3$ is a Hamiltonian circuit (F_2 having valency 1). Therefore, π has a single orbit $(1\ \pi(1)\ \pi^2(1)\ \dots\ \pi^{n-1}(1))$. Hence, π and ρ have the same sign. $\boxed{QED}$

Note that the sign of π (and hence ρ) is positive or negative if n is odd or even, respectively, whereas the sign of the identical permutation is always positive. Hence, one has the following:

Corollary 3. *Let G be a cubic bigraph of order κ. If G admits a 1-factorization all of whose 1-factors have valency 1 then $\kappa \equiv 2 \bmod 4$, or, equivalently, the (reduced) adjacency matrix A of G has odd order.*

This, in turn, can be sharpened to the following:

Corollary 4. *Let G be a cubic bigraph of order κ. If G is minimally 1-factorable then $\kappa \equiv 2 \bmod 4$.*

A further consequence of the Proposition 1 is the following:

Corollary 5. *Every 1-factorization of a minimally 1-factorable cubic bigraph has signature 3, i. e. all its 1-factors have either sign $+1$ or -1.*

Theorem 1. *Let G be a minimally 1-factorable cubic bigraph of girth 4. Then* $\det(G) = 0$.

PROOF. Let the bipartition of G be given by $V(G) = V_1 \cup V_2$ and choose a labelling of the vertices of G such that Q has vertices $a_1, a_2 \in V_1$ and $b_1, b_2 \in V_2$ and edges $a_i b_j$ with $i, j = 1, 2$. Let A be the (reduced) adjacency matrix corresponding to this labelling.

By Corollary 2, the edges of Q are 3-coloured with respect to every 1-factorization of Q; in fact, should Q be 2-coloured with respect to a 1-factorization F_1, F_2, F_3, say $E(Q) \subseteq F_1 \cup F_2$, then $F_1 \cup F_2$ would fall into at least two distinct circuits and G would be no longer minimally 1-factorable.

Since G is minimally 1-factorable, we can write down a complete list of all pairwise distinct 1-factorizations, say $F_1^{(i)}$, $F_2^{(i)}$, $F_3^{(i)}$ with $i = 1, 2, \ldots, f := \operatorname{per}(G)/3$; for each i, we may choose the indices j of $F_j^{(i)}$ such that $F_1^{(i)}$ is the 1-factor containing two edges of Q, whereas $F_2^{(i)}$ is the 1-factor containing the edge $a_1 b_k$ of Q for some $k \in \{1, 2\}$.

We consider a generic 1-factor $F_1^{(i)}$: This 1-factor is represented by a permutation submatrix of A with entries 1 in positions $(1, \sigma(1))$, $(2, \sigma(2))$, $(3, \rho(3))$, …, $(n, \rho(n))$; σ is either the identity or the permutation $(1, 2)$, whereas ρ is a permutation on the set $\{3, \ldots, n\}$. Thus, the 1-factor $F_1^{(i)}$ is characterized by the permutation $\pi = \sigma\rho = \rho\sigma$.

Clearly, the permutation $\pi' := (12)\ \pi = \pi\ (12)$ characterizes a distinct 1-factor F of G which still contains two edges of Q, i.e. $F = F_1^{(k)}$ for some $k \in \{1, \ldots, n\} \backslash \{i\}$. This gives rise to a fixed-point-free involution $\iota := i \longleftrightarrow k(i)$ on the index set $\{1, \ldots, n\}$. Since $\operatorname{sign}(\pi\ (12)) = -\operatorname{sign}(\pi)$, the list of the 1-factors $F_1^{(i)}$ splits up into two subsets of equal size, one half with positive sign, the other one with negative sign. Since G is minimally 1-factorable, Corollary 5 implies that the two lists of 1-factors $F_2^{(i)}$ and $F_3^{(i)}$ split up accordingly.

Therefore $\operatorname{per}(G)/2$ 1-factors of G give a contribution $+1$ to $\det(G)$, whereas the remaining $\operatorname{per}(G)/2$ 1-factors give a contribution -1. Hence $\det(G) = 0$. $\boxed{QED}$

In general the converse is not true. In fact, there are cubic bigraphs with determinant 0 which are either not minimally 1-factorable (e. g. the R^2-Hexagon-R^2 graph) or minimally 1-factorable with girth ≥ 6 (cf. [2, 6]).

A computer check on all known small examples of minimally 1-factorable cubic bigraph leads us to conjecture that *a minimally 1-factorable cubic bigraph G has either* $\det(G) = 0$ *or* $\det(G) = \operatorname{per}(G)$.

In particular, a det-extremal cubic bigraph is not necessarily minimally 1-factorable. In fact, the graph G' by H. L. Dorwart and B. Grünbaum on 44 vertices [1], has $\det(G') = \operatorname{per}(G') = 6144$ but it is not minimally 1-factorable, since its order is not $\equiv 2 \mod 4$

Acknowledgements. The author would like to thank M. Funk, P. Plaumann, and K. Strambach for some useful discussions.

References

[1] H.L. Dorwart, B. Grünbaum: *Are these figures oximora?* Math. Magazine, **65** (3), (1992), 158–169.

[2] M. Funk, D. Labbate: *On Minimally One-Factorable r-Regular Bipartite Graphs*, Discrete Math., **216** (2000), 121–137.

[3] P. Hall: *On representatives of subsets*, J. London Math. Soc., **10**, (1935), 26–30.

[4] F. Harary: Graph Theory, Addison Wesley, Reading, Mass. (1969).

[5] D.A. Holton, J. Sheehan: The Petersen Graph, Cambridge University Press, Cambridge, (1993).

[6] D. Labbate: Charakterisierungen minimal 1–faktorisierbarer r–regulärer Bigraphen, dissertation, Erlangen, (1998).

[7] L. Lovász, M.D. Plummer: *Matching Theory*, Annals of Discrete Mathematics, **29**, North–Holland, Amsterdam, (1986).