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Point-affine quadrangles

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Abstract. We give an axiomatic description of point-affine quadrangles as obtained by deleting a point star from a generalized quadrangle. We define a completion and observe that isomorphisms can be extended to isomorphisms between completions.

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1. Introduction

We consider incidence structures $\mathcal{Q} = (P, L, F)$ where P and L are disjoint sets and $F \subseteq P \times L$ is the incidence relation. The elements of P and L are called points and lines respectively, the elements of F are called flags. The (incidence geometrical) *distance* will be denoted by d and we use the following notation:

$$D_i(x) := \{\, y \in P \cup L \mid d(x, y) = i \,\},$$
$$D_i := \left\{\, (x, y) \in (P \cup L)^2 \;\middle|\; d(x, y) = i \,\right\}.$$

Definition 1. An incidence structure $\mathcal{Q} = (P, L, F)$ is called *generalized quadrangle* if, and only if, the following axioms are satisfied:

(V_1) $|D_1(x)| \geq 3$ for all $x \in P \cup L$.

(V_2) There are neither binangles nor triangles in $\mathcal{Q}$.

(V_3) The diameter of $\mathcal{Q}$ is 4, that is $\sup\{\, d(x, y) \mid x, y \in P \cup L \} = 4$.

The following properties are checked easily:

($\sharp$) Take any two elements x and y of $P \cup L$. Then there exists an ordinary quadrangle containing x and y.

($\perp$) For each anti-flag $(x, g) \in (P \times L) \setminus F$ there exists exactly one flag $(y, h) \in F$ such that $(x, h) \in F$ and $(y, g) \in F$ hold. We denote y and h by $\pi(x, g)$ and $\lambda(x, g)$ respectively.

2. Affine derivation of quadrangles

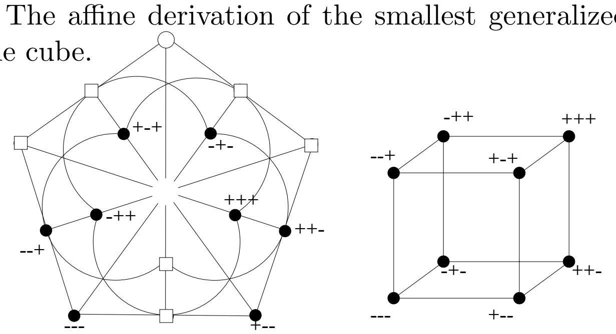
In this section we consider a generalized quadrangle $\mathcal{Q} = (P, L, F)$ with distance d and a chosen point $p \in P$. Our aim is to motivate the axioms in Section 3 and to prepare for Theorem 2.

Definition 2. The *affine derivation of* $\mathcal{Q}$ *in* p is the incidence structure $\mathcal{A}_p(\mathcal{Q}) = (A, G, I)$, where $A := D_4(p)$, $G := D_3(p)$ and I is the restriction of F to $\mathcal{A}_p(\mathcal{Q})$. We will denote the distance in $\mathcal{A}_p(\mathcal{Q})$ by δ, and put

$$\begin{aligned} \Delta_i(x) \quad &:= \{\, y \in A \cup G \mid \delta(x,y) = i \,\}\,, \\ \Delta_i := \quad &\left\{ (x,y) \in (A \cup G)^2 \,\middle|\, \delta(x,y) = i \right\}. \end{aligned}$$

We call the elements of A and G *affine points* and *affine lines* respectively.

Example. The affine derivation of the smallest generalized quadrangle is the graph of the cube.



Note that there are points at distance 6 in this example. If Q is anti-regular (for instance if Q is an orthogonal quadrangle over a field of characteristic different from 2) then $\mathcal{A}_p\mathcal{Q}$ does not contain points at distance 6.

Remarks 1.

1. For any $a \in A$ we have $\Delta_1(a) = D_1(a)$. That means, for every pair $(a, g) \in (A \times G) \setminus I$ there exists $\lambda(a, g) \in G$.
2. For $g \in G$ there exists $q \in D_2(p)$ such that $\Delta_1(g) \cup \{q\} = D_1(g)$, namely $q = \pi(p, g)$.
3. $G = \{\, a \vee b \mid a, b \in A; d(a, b) = 2\}$, where $a \vee b$ is the joining line of a and b.

(In fact, assertion 2 above gives us $\Delta_1(g) \geq 2$ for all $g \in G$. So we have $a_1, a_2 \in A$ with $g = a_1 \vee a_2$. The other direction follows from assertion 1.)

Now we consider the distances in $\mathcal{A} := \mathcal{A}_p(\mathcal{Q})$. Let $\mathcal{A}$ also denote $A \cup G$.

Lemma 1. *We have $\delta \geq d$. More exactly:*

1. *For* $(x,y) \in \mathcal{A}^2 \cap (D_2 \setminus \Delta_2)$ *we have:* $(x,y) \in \Delta_6$ *and* $x, y \in G$.
2. *For* $(a,g) \in \mathcal{A}^2 \cap (D_3 \setminus \Delta_3)$ *we have:* $(a,g) \in \Delta_5$.
3. *For* $(x,y) \in \mathcal{A}^2 \cap (D_4 \setminus \Delta_4)$ *we have:* $(x,y) \in \Delta_6$ *and* $x, y \in A$.

Proof. As incidence in $\mathcal{A}$ is induced from $\mathcal{Q}$ we have $\delta(x,y) \geq d(x,y)$ for all $x, y \in \mathcal{A}$.

(1.) Let us take $(x,y) \in \mathcal{A}^2 \cap (D_2 \setminus \Delta_2)$. Then we have $x, y \in G$ by Remarks 1.3 and there exists $z \in D_1(x) \cap D_1(y) \cap D_2(p)$. With ($\sharp$) we get a chain (x, a, k, b, l, c, y) with $a \neq z \neq c$. By Remarks 1.2 we have $\{a, c\} \subseteq \mathcal{A}$. Now if $b \in \mathcal{A}$ then $k, l \in \mathcal{A}$ by Remarks 1.3. If $b \notin \mathcal{A}$ we choose $a' \in D_1(x) \setminus \{a, z\}$ such that $\pi(a', l) \in \mathcal{A}$. This is possible by Remarks 1.2. So we have a chain of length 6 from x to y in $\mathcal{A}$. Any shorter chain would give binangles or triangles in $\mathcal{Q}$.

(2.) Take $(a,g) \in \mathcal{A}^2 \cap (D_3 \setminus \Delta_3)$, say $a \in A$ and $g \in G$. So we have $\pi(a,g) \notin \mathcal{A}$. From Remarks 1.2 we get the existence of an affine point $b \in \Delta_1(h)$ for a line $h \in \Delta_1(a)$ such that $\lambda(b,g) \in \mathcal{A}$ and $\pi(b,g) \in \mathcal{A}$. Then the chain $(a, h, b, \lambda(b,g), \pi(b,g), g)$ has length 5 in $\mathcal{A}$.

(3.) Let $(x,y) \in \mathcal{A}^2 \cap (D_4 \setminus \Delta_4)$. Then we have $x, y \in A$ by Remarks 1.2 and $D_2(x) \cap D_2(y) \subseteq D_2(p)$. Choose $g \in \Delta_1(x)$ and $h \in \Delta_1(y) \setminus \{\lambda(y,g)\}$. By Remarks 1.2 there exists $a \in (\Delta_1(g) \cap A) \setminus \{x\}$ with $\pi(a,h) \in \mathcal{A}$. Thus the chain

$$(x, g, a, \lambda(a,h), \pi(a,h), h, y)$$

is a chain of length 6 in $\mathcal{A}$. $\boxed{QED}$

We want to reconstruct the generalized quadrangle from its affine derivation. To this end, we describe points of $D_2(p)$ in terms of $\mathcal{A}$.

Definition 3. Let a *pseudo-pencil* be a maximal set $H \subseteq G$ of lines with $\delta(g,h) \in \{0, 6\}$ for all $g, h \in H$.

Lemma 2. *For* $q \in D_2(p)$ *we have that* $\Pi_q := D_1(q) \cap \mathcal{A}$ *is a pseudo-pencil. Moreover, for every pseudo-pencil* H *there exists a point* $x \in D_2(p)$ *such that* $H = D_1(x) \cap \mathcal{A}$. *Thus the system of pseudo-pencils is a partition of* G.

Proof. Take two distinct lines g and h in $D_1(q) \cap \mathcal{A}$. By Lemma 1 we have $\delta(g,h) = 6$. Let $l \in G \setminus \Pi_q$, that means $d(q,l) = 3$. If $\pi(q,l) \in A$ holds, we have $\delta(\lambda(q,l), l) = 2 \neq 6$. If $\pi(q,l) \notin A$, there exists by ($\sharp$) and Remarks 1.2 a line $h \in \Pi_q \setminus \{\lambda(q,l)\}$ with $\delta(h,l) = 4 \neq 6$.

Now let H be any pseudo-pencil and $g \neq h \in H$. Lemma 1 yields $d(g,h) = 2$. So we have that all lines of H meet in $x := \pi(p,g)$ by Remarks 1.2 and thus we have $H = D_1(x) \cap \mathcal{A}$. $\boxed{QED}$

Our next aim is to describe the lines through p with δ. Let us regard the following assertions about $x, y \in D_2(p)$.

A: x and y are not collinear in $\mathcal{P}$.

B: x and y are collinear in $\mathcal{P}$.

C: For all $g \in \Pi_x$ there exists an $h \in \Pi_y$ with $\delta(g,h) = 2$.

D: For all $g \in \Pi_x$ and all $h \in \Pi_y$ we have $\delta(g,h) = 4$.

Lemma 3. *Let $x, y \in D_2(p)$ be distinct points. Then we have*

$$\mathbf{A} \Longleftrightarrow \mathbf{C} \quad \textit{and} \quad \mathbf{B} \Longleftrightarrow \mathbf{D}$$

Proof. Obviously we have implications

E: $\quad \mathbf{A} \Longleftrightarrow \neg\mathbf{B} \quad$ and $\quad \mathbf{C} \Longrightarrow \neg\mathbf{D}$.

If x and y are collinear points, assertion **D** holds because there are no triangles in $\mathcal{Q}$. So we have $\mathbf{B} \Longrightarrow \mathbf{D}$ and $\mathbf{C} \Longrightarrow \mathbf{A}$ by **E**. In order to show $\mathbf{A} \Longrightarrow \mathbf{C}$ we choose two not collinear points $x, y \in D_2(p)$, that is $d(x,y) = 4$ and we have $d(g,y) = 3$ for all lines $g \in \Pi_x$. Absence of triangles in $\mathcal{Q}$ yields $\pi(y,g) \in A$. Thus we have $\delta(g, \lambda(y,g)) = 2$ and assertion **C** holds. Now **E** implies $\mathbf{D} \Longrightarrow \mathbf{B}$. $\boxed{QED}$

Now we define an equivalence relation $\sim$ on the set of all pseudo-pencils in $\mathcal{A}$, as follows:

$$\Pi_x \sim \Pi_y \iff \delta(g,h) \neq 2 \text{ for all } g \in \Pi_x \text{ and all } h \in \Pi_y$$

We see that $\Pi_x \sim \Pi_y$ means $d(x,y) \leq 2$. Thus the equivalence classes of $\sim$ are exactly the sets $D_1(l) \setminus \{p\}$ for lines $l \in D_1(p)$.

In addition we consider the equivalence relation $\diamond$ whose equivalence classes are the pseudo-pencils of $\mathcal{A}$. Using both equivalence relations we can reconstruct the quadrangle $\mathcal{Q}$ from its affine derivation $\mathcal{A}$.

Proposition 1. *We have $\mathcal{Q} \cong (\hat{P}, \hat{L}, \hat{F})$, where*

$$\hat{P} := A \cup \tfrac{G}{\diamond} \cup \{p\} \quad \textit{and} \quad \hat{L} := G \cup \tfrac{G}{\diamond}\big/_{\sim}$$

The incidence $\hat{F}$ is described as follows:

$$\hat{F} := I \cup \{ ([g]_\diamond, g) \mid g \in G\} \cup \left\{ (x, [x]_\sim) \,\middle|\, x \in \tfrac{G}{\diamond} \right\} \cup \left\{ (p, [x]_\sim) \,\middle|\, x \in \tfrac{G}{\diamond} \right\},$$

where $I \subseteq A \times G$ is the incidence relation of $\mathcal{A}$.

For the distance γ in $(\hat{P}, \hat{L}, \hat{F})$ we obtain from 3:

Corollary 1. *If $x, y \in D_2(p)$ are two different points then we have $\gamma(\Pi_x, \Pi_y) = 6 - \min\{ \delta(g,h) \mid (g,h) \in \Pi_x \times \Pi_y \}$.*

3. Axioms for point-affine quadrangles

Our aim is to give an axiomatic description of a point-affine quadrangle $\mathcal{A}$ such that we can construct a generalized quadrangle $\mathcal{Q}$ with $\mathcal{A}$ being the affine derivation of $\mathcal{Q}$. A main motivation was the observation that several constructions of generalized quadrangles in fact start with building a point-affine quadrangle. Prominent examples are elation generalized quadrangles, see Section 5.

There is a more general notion of "affine quadrangle", denoting incidence structures obtained from a generalized quadrangle by deleting a geometric hyperplane. Pralle [6] gives an axiomatic characterization of this general class, based on the notion of totally connected pairs of lines. Our approach, in contrast, is based on distances.

Definition 4. An incidence structure $\mathcal{A} = (A, G, I)$ with distance δ is called a *point-affine quadrangle* if the following axioms hold.

(A_1) For $a \in A$ we have $|\Delta_1(a)| \geq 3$, for $g \in G$ we have $|\Delta_1(g)| \geq 2$.

(A_2) There are neither binangles nor triangles in $\mathcal{A}$.

(A_3) The diameter of $\mathcal{A}$ is 6.

(A_4) We require that $g \diamond h \iff \delta(g,h) \in \{0,6\}$ is an equivalence relation on G and that each equivalence class of $\diamond$ contains at least 2 elements. The equivalence class of g is denoted by $[g]_\diamond$, and the canonical map will be denoted by $\alpha : g \mapsto [g]_\diamond$.

(A_5) On $\frac{G}{\diamond}$,

$$[x]_\diamond \sim [y]_\diamond \iff \delta(g,h) \neq 2 \text{ for all } g \in [x]_\diamond \text{ and all } h \in [y]_\diamond$$

defines an equivalence relation with at least 3 equivalence classes and at least 2 elements in each class.

(A_6) For $g \in G$ we have that $\alpha|_{\Delta_2(g)}$ is a surjection onto $\{\, [x]_\diamond \mid [x]_\diamond \not\sim [g]_\diamond \,\}$.

(A_7) For $a \in A$ we have that $\alpha|_{\Delta_1(a)}$ is a surjection onto $\frac{G \setminus \Delta_3(a)}{\diamond}$.

(A_8) Let $\beta : G \to \left\{ [x]_\sim \;\middle|\; x \in \frac{G}{\diamond} \right\} : g \mapsto [[g]_\diamond]_\sim$. Then the restriction $\beta|_{\Delta_1(a)}$ is surjective for each $a \in A$.

Example. If Q is a generalized quadrangle then the affine derivation at any point of Q is a point-affine quadrangle, see Section 2. In fact, we shall see that there are no other examples, cf. Theorem 1 below.

Remark 1. Note the analogy with affine planes ("line-affine triangles"), where one would require (A_1), absence of binangles, diameter 4, transitivity of the relation $\delta(g,h) \in \{0,4\}$ (parallelism of lines) and the corresponding variant of (A_7) (existence of parallels). As, for instance, the Pappos figure shows, we have to require also that there are no pairs of points at distance 4 (existence of joining lines).

Now we want to reverse affine derivation. To this end, we construct a completion $\mathcal{C}(\mathcal{A})$. Let

$$P := A \cup \tfrac{G}{\diamond} \cup \{\infty\} \quad \text{and} \quad L := G \cup \tfrac{G}{\diamond}/_{\sim}$$

and define incidence by

$$F = I \cup \{\, ([g]_\diamond, g) \mid g \in G\} \cup \left\{\, (x, [x]_\sim) \,\middle|\, x \in \tfrac{G}{\diamond}\right\} \cup \left\{\, (\infty, [x]_\sim) \,\middle|\, x \in \tfrac{G}{\diamond}\right\}.$$

This means that affine points are only incident with affine lines and to an affine line we add exactly one new point. Moreover the point ∞ is incident only with the new lines.

Let us denote the distance on $\mathcal{C}(\mathcal{A}) = (P, L, F)$ by γ.

Definition 5. The incidence structure $\mathcal{C}(\mathcal{A}) = (P, L, F)$ described above is called the *completion* of the point-affine quadrangle $\mathcal{A} = (A, G, I)$.

Remark 2. Let $\mathcal{Q}$ be a generalized quadrangle. Then Proposition 1 says that $\mathcal{Q} \cong \mathcal{C}(\mathcal{A}_p(\mathcal{Q}))$ holds for each point $p \in \mathcal{Q}$. Thus completion reverses derivation.

Theorem 1. *The completion $\mathcal{C}(\mathcal{A}) = (P, L, F)$ of a point-affine quadrangle is a generalized quadrangle.*

Proof. (V_1) For $a \in A$ we have $|\Gamma_1(a)| \geq 3$ by (A_1). For $x := [g]_\diamond \in P \setminus A$ we have $|\Gamma_1(x)| \geq 3$ by (A_4) and $[x]_\sim \in \Gamma_1(x)$. Axiom (A_5) ensures $|\Gamma_1(\infty)| \geq 3$. Let $g \in G$. By (A_1) and $[g]_\diamond \in \Gamma_1(g)$ we have $|\Gamma_1(g)| \geq 3$. Let $l \in L \setminus G$. By (A_5) we get at least two elements in $\Delta_1(l)$, and $\infty \in \Gamma_1(l)$ yields $|\Gamma_1(l)| \geq 3$.

(V_2) We have neither binangles nor triangles in $\mathcal{A}$ by (A_2). In $\mathcal{C}(\mathcal{A}) \setminus \mathcal{A}$ the point ∞ is the only one which meets more than one line of $L \setminus G$. If there would be a triangle with two distinct collinear points $x, y \in P \setminus A$, we would have $\delta(g,h) = 4$ for all $(g,h) \in \Pi_x \times \Pi_y$ by definition of collinearity. So we cannot get any triangle. If there would be a binangle with points $x \in P \setminus A$ and $a \in A$, this would mean that $x = [a]_\diamond$. But $(a,g) \in I$ and $(a,h) \in I$ for two lines $g, h \in \Pi_x$ means that $\delta(g,h) = 0$ and thus there exists no binangle.

(V_3) First we show $\gamma(x,y) \leq 4$ for all $x, y \in \mathcal{C}(\mathcal{A})$.
(i) Let us consider the point ∞. For $x \in A$ the chain $(x, g, [g]_\diamond, [[g]_\diamond]_\sim, \infty)$ has length 4 and $\gamma(x,\infty) \leq 4$ holds. For $x \in \frac{G}{\diamond}$ we have $\gamma(x,\infty) = 2$ by definition

of F. For $x \in G$ we have the chain $(x, [x]_\diamond, [[x]_\diamond]_\sim, \infty)$ and so $\gamma(x,\infty) \le 3$. For $x \in L \setminus G$ we have $\gamma(x,\infty) = 1$ again by definition of F.
(ii) Let $p \in \frac{G}{\diamond}$. For $x \in \frac{G}{\diamond}$ we have the chain $(x, [x]_\sim, \infty, [p]_\sim, p)$, that is $\gamma(x,p) \le 4$. For $x \in L \setminus G$ we have the chain $(x, \infty, [p]_\sim, p)$, that is $\gamma(x,p) \le 3$. For $x \in G$ either we have $[x]_\diamond \sim p$ and $\gamma(x,p) \le 3$ or there exists a line in $\Delta_2(x)$ incident with p by (A_6). So we have $\gamma(x,p) \le 3$ again. For $x \in A$ choose a line $g \in \Delta_1(x)$ and get $\gamma(x,p) \le 4$ with the same argument as above.
(iii) Let $a \in A$. For $x \in \Delta_6(a)$ choose $g \in \Delta_1(x)$ and get $\delta(a,g) \le 5$. If $\delta(a,g) = 5$ holds, we get a line $l \in \Delta_1(a)$ with $[l]_\diamond = [g]_\diamond$ by (A_7). Thus we have $\gamma(x,a) \le 4$. For $x \in G$ and $\delta(a,x) > 4$ we have again by (A_7) a line $l \in \Delta_1(a)$ with $[l]_\diamond = [x]_\diamond$ and so $\gamma(x,a) \le 3$ holds. For $x \in L \setminus G$ the axiom (A_8) yields a line in $\Delta_1(a)$ which meets x, and we have $\gamma(x,a) \le 3$.
(iv) It remains to look at distances between lines. Let $g \in G$. For $x \in \Delta_6(g)$ we have $[x]_\diamond = [g]_\diamond$, that is $\gamma(x,g) \le 2$. For $x \in L \setminus G$ we have the chain $(x, \infty, [[g]_\diamond]_\sim, [g]_\diamond, g)$ and so $\gamma(x,g) \le 4$ holds. For two new lines $l, h \in L \setminus G$ we have $\gamma(l,h) \le 2$ because of $(\infty, x) \in F$ for all $x \in L \setminus G$.

Finally, we observe that $(A \times \Gamma_1(\infty)) \cap F = \emptyset$ and so $\gamma(x,\infty) > 2$ for all $x \in A$. Thus we have $\Gamma_4(\infty) \neq \emptyset$ and (V_3) is proved. $\boxed{QED}$

4. Isomorphisms

Definition 6. Let $\mathcal{Q} = (P, L, F)$ and $\mathcal{Q}' = (P', L', F')$ be incidence structures with distance functions d and d', respectively. A bijection $\varphi : P \cup L \to P' \cup L'$ is called an *isomorphism* (from Q onto Q') if we have $P^\varphi = P'$ and $F^\varphi = F'$.

Remark 3. A surjection $\varphi : \mathcal{Q} \to \mathcal{Q}'$ is an isomorphism if, and only if, we have $P^\varphi \subseteq P'$ and $d(x,y) = d'(x^\varphi, y^\varphi)$ for all $x, y \in Q$.

From Definition 3 and Corollary 1 we infer:

Lemma 4. *Each isomorphism $\varphi : \mathcal{A} \mapsto \mathcal{A}'$ between point-affine quadrangles maps pseudo-pencils to pseudo-pencils, and for $\Pi_x, \Pi_y \in \mathcal{C}(\mathcal{A})$ we have: $\gamma'((\Pi_x)^\varphi, (\Pi_y)^\varphi) = \gamma(\Pi_x, \Pi_y)$, where γ and γ' are the distances in $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{A}')$ respectively.*

Now let $\mathcal{A}$ and $\mathcal{A}'$ be point-affine quadrangles and let φ be an isomorphism from $\mathcal{A}$ onto $\mathcal{A}'$. Then we can define an extension Φ from $\mathcal{C}(\mathcal{A})$ onto $\mathcal{C}(\mathcal{A}')$ of φ as follows. Let $\Phi|_{\mathcal{A}} = \varphi$ and $([g]_\diamond)^\Phi := [g^\varphi]_\diamond$, $([x]_\sim)^\Phi := \left[x^\Phi\right]_\sim$, $\infty^\Phi := \infty'$. Then Φ is an isomorphism from $\mathcal{C}(\mathcal{A})$ onto $\mathcal{C}(\mathcal{A}')$.

Proposition 1 gives isomorphisms from $\mathcal{Q}_1$ onto $\mathcal{C}(\mathcal{A}_{p_1}(\mathcal{Q}_1))$ and from $\mathcal{C}(\mathcal{A}_{p_2}(\mathcal{Q}_2))$ onto $\mathcal{Q}_2$ that induce the identity on $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Combining these with Φ, we obtain:

Theorem 2. *Let $\mathcal{A}_1 = \mathcal{A}_{p_1}(\mathcal{Q}_1)$ and $\mathcal{A}_2 = \mathcal{A}_{p_2}(\mathcal{Q}_2)$ be affine derivations of generalized quadrangles and let $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ be an isomorphism. Then there exists an isomorphism $\hat{\varphi} : \mathcal{Q}_1 \to \mathcal{Q}_2$ with $\hat{\varphi}|_{\mathcal{A}_1} = \varphi$ and $p_1^{\hat{\varphi}} = p_2$.*

Corollary 2. *For each generalized quadrangle $\mathcal{Q}$ and any two points $p, q \in \mathcal{Q}$ we have $\mathcal{A}_p(\mathcal{Q}) \cong \mathcal{A}_q(\mathcal{Q})$ if, and only if, there exists $\alpha \in \operatorname{Aut} \mathcal{Q}$ with $p^\alpha = q$.*

Proof. Apply Theorem 2 to find α and use Remark 3 to conclude $D_4(p)^\alpha = D_4(q)$ and $D_3(p)^\alpha = D_3(q)$ from $p^\alpha = q$. $\boxed{QED}$

5. Elation generalized quadrangles

In [4] Kantor constructs finite elation quadrangles as group coset geometries. This construction was extended to the infinite case in [2], [5]. See Remarks 2 below for more remarks on the literature. We give an alternative proof of these results, which serves as an example how the axioms for point-affine quadrangles can be verified. Note also the analogies with the treatment of affine translation planes by André [1].

Proposition 2. *Let E be a group and $\mathcal{F}$ a family of subgroups. To each $T \in \mathcal{F}$, let a subgroup T^* be associated such that the following conditions are satisfied: For all $T, S, V \in \mathcal{F}$ we have*

(i) $\{1\} < T < T^* < E$,

(ii) *$T \neq S$ implies $E = T^* S (= S T^*)$ and $T^* \cap S = \{1\}$,*

(iii) *$T \neq S \neq V \neq T$ implies $TS \cap V = \{1\}$,*

(iv) $E = T^* \cup \bigcup_{S \in \mathcal{F}} ST$.

Then $\mathcal{A} := (E, \bigcup_{T \in \mathcal{F}} E/T, \in)$ is a point-affine quadrangle and $\mathcal{C}(\mathcal{A})$ is an elation quadrangle with elation group E and elation center ∞.

Before we start with the proof of Proposition 2, we note the following

Lemma 5. *Under the assumptions of Proposition 2, one also has for all $T, V \in \mathcal{F}$:*

(v) *$T \neq V$ implies $E = \bigcup_{S \in \mathcal{F}} TSV$.*

Proof. Consider $T, V \in \mathcal{F}$. For any $x \in E = V^* \cup \bigcup_{S \in \mathcal{F}} SV$ we then have $x \in V^*$ or $x \in \bigcup_{S \in \mathcal{F}} SV$. For $x \in V^*$ we pick $t \in T \setminus \{1\}$, then (ii) yields $tx \notin V^*$, and $tx \in \bigcup_{S \in \mathcal{F}} SV$ follows. Thus $x \in \bigcup_{S \in \mathcal{F}} TSV$ holds for all $x \in E$. $\boxed{QED}$

Remark 4. Assertions (i)–(v) hold in each elation quadrangle. However, assertion (iv) is independent of (i), (ii), (iii), (v), see [3].

Proof of Proposition. We remark that the line Tx determines $T \in \mathcal{F}$, in view of

$$Tx \ = \ Sy \iff T = S \text{ and } yx^{-1} \in T.$$

Multiplication (from the right) with $a \in E$ induces an automorphism of the incidence structure. Therefore, it suffices to describe distances from 1. For $x \in E$ we have:

$$\delta(1,x)\left\{\begin{array}{c} =0 \\ \leq 2 \\ \leq 4 \\ \leq 6 \end{array}\right\} \iff \left\{\begin{array}{l} x=1 \\ x \in \bigcup_{V\in\mathcal{F}} V \\ x \in \bigcup_{S,V\in\mathcal{F}} SV \\ x \in \bigcup_{T,S,V\in\mathcal{F}} TSV \end{array}\right.$$

$$\delta(1,Tx)\left\{\begin{array}{c} =1 \\ \leq 3 \\ \leq 5 \end{array}\right\} \iff \left\{\begin{array}{l} x \in T \\ x \in \bigcup_{S\in\mathcal{F}} TS \\ x \in \bigcup_{S,V\in\mathcal{F}} TSV \end{array}\right.$$

(A_1) We have $\Delta_1(1) = \mathcal{F}$. From (iv) and (i) we deduce $|\mathcal{F}| \geq 2$. If we have $|\mathcal{F}| = 2$, then (v) and (ii) yield $E = TV = T^*V$ and so $T = T^*$, which contradicts (i). For $T \in \mathcal{F}$ the inequality $|\Delta_1(T)| = |T| \geq 2$ is a consequence of $T \neq \{1\}$.

(A_2) If there is a 2-gon with lines T and S, then there exists $x \in E\setminus\{1\}$ with $x \in T\cap S \subseteq T^*\cap S$, contradicting (ii). If there is a 3-gon with lines S, T, Vx, we have $Vt = Vx = Vs$ for some $t \in T$ and $s \in Vt\cap S \subseteq VT\cap S$, which contradicts (iii).

(A_3) For any $x \in E$ we have $x \in \bigcup_{S\in\mathcal{F}} TSV$ by (v) and therefore $\delta(1,x) \leq 6$. The same argument shows that $\delta(1,Tx) \leq 5$ for all $T \in \mathcal{F}$. The proof of (A_4) will also show that there exist lines at distance 6.

(A_4) Here we will show

$$\delta(Tx,Sy) \in \{0,6\} \iff T = S \text{ and } xy^{-1} \in T^*.$$

First we choose $x = 1$, so we must prove

$$\delta(T,Sy) \in \{0,6\} \iff T = S \text{ and } y \in T^*.$$

Starting from $T = S$ and $y \in T^*$, we obtain $\delta(T,Sy) = \delta(T,Ty) = 0$ if $y \in T$. For $y \in T^*\setminus T$ we have $T\cap Ty = \emptyset$ and therefore $\delta(T,Ty) \geq 4$. But $\delta(T,Ty) = 4$ is equivalent to the existence of some $z \in Ty \subseteq T^*$ and some $V \in \mathcal{F}\setminus\{T\}$ such that $Vz\cap T \neq \emptyset$. So we take $u \in Vz\cap T$ and get $uz^{-1} \in V\cap Tz^{-1} \subseteq V\cap T^*$. This yields $z = u \in T$ and $Ty = Tz = T$, a contradiction. We have thus proved that $y \in T^*$ implies $\delta(T,Ty) \in \{0,6\}$.

Now we assume $\delta(T,Sy) = 6$. For $T \neq S$ we would have $E = \bigcup_{V\in\mathcal{F}} SVT$ by (v) and for each $y \in E$ there exist $s \in S, t \in T, v \in V$ such that $y = s^{-1}vt$.

Thus we have $vt \in Sy$ and (Sy, vt, Vt, t, T) is a chain of length 4 from Sy to T. So we have $S = T$. Moreover $\delta(T, Ty) \geq 6$ implies $\delta(1, Ty) \geq 5$, that is $y \in E \setminus \bigcup_{S\in\mathcal{F}} TS \subseteq T^*$ by (iv).

To show transitivity we take $Tx \diamond Sy$ and $Sy \diamond Vz$ and obtain first $T = S = V$. Without loss of generality we take $x = 1$ and get $y \in T^*$ and there exists $w \in E$ such that $y, z \in T^*w$, that is $T^*w = T^*y = T^*$ and therefore we have $z \in T^*$. We have shown $[T]_\diamond = T^*/T$ and $[Tx]_\diamond = (T^*/T)x = \{Ttx \mid t \in T^*\}$. The inequality $[T]_\diamond \neq \{T\}$ follows from $T^* > T$.

(A_5) We claim

$$(T^*/T)x \sim (S^*/S)y \iff T = S$$

By definition $(T^*/T)x \sim (S^*/S)y$ means

$$\forall t^* \in T^* \forall s^* \in S^* : \delta(Tt^*x, Ss^*y) \neq 2 \tag{1}$$

that is, we have $Tt^*x = Ss^*y$ or $Tt^*x \cap Ss^*y = \emptyset$.

For $T \neq S$ we have $TS^* = E \ni xy^{-1}$, that is $xy^{-1} \in Ts^*$ for some $s^* \in S^*$, which implies $s^*y \in Tx \cap Ss^*y$. This means that (1) is false, and $(T^*/T)x \not\sim (S^*/S)y$. For $T = S$ we have $Tu \cap Tv = \emptyset$ whenever $Tu \neq Tv$ and so the corresponding pseudo-pencils are collinear. This establishes the claim. It follows that $\sim$ is an equivalence relation with $|\mathcal{F}| \geq 3$ equivalence classes and $|E/T^*| \geq 2$ elements in each equivalence class.

(A_6) Let Sx be a given line and $(T^*/T)y \not\sim (S^*/S)x$, that is $T \neq S$. Then we have $yx^{-1} \in E = T^*S$. Take $t^* \in T^*, s \in S$ such that $sx = t^*y \in Sx \cap Tt^*y$. Thus we have $Tt^*y \in \Delta_2(Sx)$ and $Tt^*y \in [Ty]_\diamond$.

(A_7) Let $a \in E$ be arbitrary. Consider lines Ty with $\delta(a, Ty) \neq 3$.

If $\delta(a, Ty) = 1$ holds, we see that $Ta = Ty \in \Delta_1(a)$ and $Ta \diamond Ty$. If $\delta(a, Ty) = 5$, we have $\forall S \in \mathcal{F} : Sa \cap Ty = \emptyset$. So we see that $ay^{-1} \notin \bigcup_{S\in\mathcal{F}} ST$ and therefore $ay^{-1} \in T^*$ by (iv). This is equivalent to $a \in T^*y$ and so Ta is the line with $Ta \diamond Ty$ we searched for.

(A_8) If $[T^*/T]_\sim$ and $x \in E$ are given, then Tx is the pre-image we need. $\boxed{QED}$

Remarks 2. In addition to conditions (i)–(iii) as in Proposition 2, Kantor [4] assumes finiteness of the groups and equality of certain cardinalities. Bader and Payne [2] and also Löwe [5] generalize this construction to infinite elation generalized quadrangles. In both papers, cardinality conditions are formulated, but it turns out that these are superfluous. Bader and Payne add an axiom (K_4) that says (in the notation from Proposition 2): $\forall T \in \mathcal{F} : T^* = B(T)$, where

$$B(T) := T \cup \bigcup \left\{ Tx \,\middle|\, x \in E \text{ and } Tx \cap S = \emptyset \text{ for all } S \in \mathcal{F} \setminus \{T\} \right\}.$$

Löwe assumes Proposition 2 (iv) and characterizes T^* as complement $L(T)$ of the set $\bigcup_{S\in\mathcal{F}\setminus\{T\}} T(S\setminus\{1\})$.

Lemma 6. *Let E be a group and $\mathcal{F}$ a family of subgroups which satisfies (i) and (ii) of Proposition 2. Then we have $B(T) = L(T)$ for each $T \in \mathcal{F}$.*

Proof. Let $z \in L(T)$. Then either $z \in T \subseteq B(T)$ holds, or $z \in L(T)\setminus T = E \setminus \bigl(T \cup \bigcup_{S\neq T} T(S\setminus\{1\}\bigr) = E \setminus \bigcup_{S\in\mathcal{F}} TS$. So we have for all $S \in \mathcal{F}$ that $z \notin TS$. This means $z \notin Ts$ and $s \notin Tz$ for all $s \in S$. Thus $Tz \cap S = \emptyset$ holds for all $S \in \mathcal{F}\setminus\{T\}$. This conclusion implies $z \in Tz \subseteq B(T)$ and thus we have $L(T) \subseteq B(T)$.

Now consider $z \in B(T)$. Again we have two possibilities. First assume $z \in T \setminus L(T)$. Then there would exist $S \neq T$ such that $z \in T(S\setminus\{1\})$, say $z = ts$. But then we have $t^{-1}z = s \in S\setminus\{1\}$ and $t^{-1}z \in T^*$ which contradicts (ii). Now we take $z \in B(T)\setminus T$. Then there exists $x \in E$ such that $z \in Tx$ and for all $S \neq T$ we have $Tx \cap S = \emptyset$. Let us suppose $z = tx \in T(S\setminus\{1\})$. Then $tx = t_0 s$ for some $t_0 \in T, s \in S\setminus\{1\}$. This yields $s = t_0^{-1}tx \in Tx \cap S$ which contradicts $z \in B(T)$. Thus we have for all $S \in \mathcal{F}\setminus\{T\}$ that $z \notin T(S\setminus\{1\})$ holds which implies $z \in L(T)$. $\boxed{QED}$

Lemma 6 implies that the additional axioms of [2] and [5] are equivalent. In Proposition 2 we include Löwe's additional axiom as assumption (iv). Note that, in the infinite case, some axiom beyond Kantor's original conditions is needed, as the example in [3] shows.

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