Note di Matematica **20,** n. 1, 2000/2001, 1–20.

The residually weakly primitive pre-geometries of the Suzuki simple groups

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Received: 26 July 1999; accepted: 21 September 1999.

Abstract. We determine all pre-geometries of rank ≥ 4, on which a Suzuki simple group $Sz(q)$, with q an odd power of two, acts residually weakly primitively (Rwpri). These pre-geometries arise from eleven constructions detailed at the end of the paper.

Keywords: coset geometries, Suzuki simple groups, residually weakly primitive

MSC 2000 classification: primary 51E24, secondary 20D08

Introduction

In 1954, Jacques Tits gave a geometric interpretation of the exceptional complex Lie groups (see [23] and [26]). Francis Buekenhout generalized in [1] and [2] certain aspects of this theory in order to achieve a combinatorial understanding of all finite simple groups. Since then, two main traces have been developed in diagram geometry. One is to try to classify geometries over a given diagram, mainly over diagrams extending buildings (see e. g. [5] chap. 22, for a survey and [25] for the theory of buildings). Another trace is to classify coset geometries for a given group under certain conditions. Rules for such classifications have been stated by Buekenhout in [3] and [4]. These guidelines led Michel Dehon to present in 1994 [13] a set of Cayley programs in order to classify all firm, residually connected and flag-transitive geometries of a given group G with an additional restriction on the subgroups forming the geometries: each stabilizer of some element is a maximal subgroup of G. Several groups were investigated as for example $U_4(2)$ [13], M_{11} [8] and a collection of projective groups G such that $PSL(2,q) \leq G \leq Aut(PSL(2,q))$ with $5 \leq q \leq 19$ [7]. This experimental work led to new rules for such classifications. In 1993, Francis Buekenhout and Michel Dehon changed the restriction of the subgroups forming the geometries, taking a residually weakly primitive condition (Rwpri). Again, experimental work was accomplished in that way. In 1994, an atlas of residually weakly primitive geometries for small groups was achieved [10]. In 1995, Harald Gottschalk determined all geometries of the group $PSL(3,4)$ is his Diplomarbeit [15]. In 1996, Dehon and Miller determined in [14] all geometries of M_{11} satisfying these new conditions, Gottschalk and Leemans classified the geometries of J_1 [16] and Leemans determined all those geometries for $Sz(8)$ in [18]. During that period, several theoretical works on the subject were also made: Buekenhout, Dehon and Leemans showed in [9] that the Mathieu group M_{12} does not have RWPRI and $(IP)_2$ geometries of rank ≥ 6. Buekenhout, Cara and Dehon described in [6] a class of inductively minimal geometries which satisfy the RWPRI condition, and Buekenhout and Leemans showed in [11] that the O'Nan sporadic simple group does not have RWPRI geometries of rank ≥ 5 (resp. six) with one of the subgroups forming the geometry isomorphic to J_1 (resp. M_{11}). All these results tended to show that the residually weakly primitive condition on the subgroups was a "good one".

The experience that we acquired led us to be more ambitious. We wanted to look at an infinite class of groups and classify all geometries satisfying some conditions for all the groups of this class. We chose to study the Suzuki simple groups. This choice was motivated by the fact that the structure of these groups is particularly easy compared to the other simple groups. We then started the classification by determining, up to isomorphism, all rank two firm, residually connected, flag-transitive geometries on which a Suzuki simple group $Sz(q)$, with q an odd power of two, acts residually weakly primitively [17]. This purely theoretical work pointed out some very surprising results, as for example the fact that a Suzuki simple group which does not have proper subgroups of Suzuki type (i. e. subgroups which are also Suzuki simple groups) gives rise to much more geometries than the others. We also showed in [17] that by adding just one condition, namely the $(2T)_1$ condition, we reduced the number of geometries arising to one for every Suzuki group except for the smallest one, i. e. $Sz(8)$ which has three such geometries. The next step was the classification of all rank three residually weakly primitive geometries of the Suzuki simple groups [19]. To do this, we first needed to know all the RWPRI geometries of the dihedral groups since some of the maximal subgroups of a Suzuki group are dihedral groups, so we classified in [20] all RWPRI geometries of the dihedral groups. In [19], we also tested the extra condition $(2T)_1$ and showed that there is no rank ≥ 4 geometry satisfying both RWPRI and $(2T)_1$ and that the only remaining rank three geometries are thin geometries which appear only in the case where $Sz(q)$ does not have subgroups of Suzuki type.

We could then have stopped our classification there. But we wanted to know if it was still possible to have a good control on the results by imposing only the RWPRI condition. In the present paper, we determine all RWPRI pre-geometries of rank ≥ 4 for any Suzuki simple group $Sz(q)$. To do that, we start from the classification theorems stated in [19] for the rank 3 case (for the rank two case, we refer to [17]). We get, as final result, eleven constructions leading to all

RWPRI pre-geometries of rank ≥ 4 for any $Sz(q)$ (see theorem 5).
We plan to determine in future work which of these pre-geometries are indeed flag-transitive geometries. The results obtained in this paper strengthen our belief that the RWPRI condition is an efficient one to impose on the subgroups forming the geometries.

Why only pre-geometries?

In order to construct a rank n geometry satisfying our properties (firmness, flag-transitivity, residual connectedness, RWPRI, and so on) we take a n-tuple $(G_0, \ldots, G_{n_1})$ of subgroups of a given group G. We then construct a pre-geometry $\Gamma(G; G_0, \ldots, G_{n-1})$ as shown in the next section. To test if Γ is a geometry, we must ensure that every flag of Γ is contained in a chamber. This is done by testing the flag-transitivity conditions given in [13]. Indeed, if Γ is flag-transitive, then it is obvious that it is a geometry. We currently do not have another way to answer the question "Is this pre-geometry a geometry?". Since in this paper we do not plan to talk about flag-transitivity, we only get pre-geometries. So the reader should keep in mind that the objects constructed here might not be geometries.

The paper is organised as follows. In section 2, we recall some basic definitions and we fix notation. In section 3, we recall some preliminary lemmas that were proved in [19], and are useful for our classification. In section 4, we determine the pre-geometries of rank 4. This section and the next one are there to show that even without imposing the $(2T)_1$ property (as suggested in [19]), it is possible to determine theoretically all the RWPRI pre-geometries of any rank. Also, nice families of pre-geometries arise, all with the same kind of diagram. In section 5, we resume the long discussion made in section 4 by giving eleven constructions that lead to all RWPRI pre-geometries of rank ≥ 4. Finally, in section 6, we give some concluding remarks about the results obtained in this paper and mention some future work we plan to do in this area.

1. Definitions and notation

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [24] (see also [5], chapter 3).
Let I be a finite set and let G be a group together with a family of subgroups $(G_i)_{i\in I}$. We define the *pre-geometry* $\Gamma = \Gamma(G, (G_i)_{i\in I})$ as follows. The set X of *elements* of Γ consists of all cosets gG_i, $g \in G$, $i \in I$. We define an *incidence relation* $*$ on X by :

$$g_1G_i * g_2G_j \text{ iff } g_1G_i \cap g_2G_j \text{ is non-empty in } G.$$

The *type function* t on Γ is defined by $t(gG_i) = i$. The *type* of a subset Y of X is the set $t(Y)$; its *rank* is the cardinality of $t(Y)$ and we call $\mid t(X) \mid$ the *rank* of Γ.

A *flag* is a set of pairwise incident elements of X and a *chamber* of Γ is a flag of type I. An element of type i is also called an *i-element*.
The group G acts on Γ as an automorphism group, by left translation, preserving the type of each element.
As in [13], we call Γ a *geometry* provided that every flag of Γ is contained in some chamber and we call Γ *flag-transitive* (FT) provided that G acts transitively on all chambers of Γ, hence also on all flags of any type J, where J is a subset of I. It is obvious that any rank two pre-geometry $\Gamma(G; G_0, G_1)$ is a flag-transitive geometry.

Lemma 1. *Let $\Gamma(G; G_0, G_1, G_2)$ be a rank 3 pre-geometry. Then Γ is a geometry (not necessarily flag-transitive).*

This lemma permits us to talk about geometries instead of pre-geometries in the rank three case.
Let $\Gamma(G; G_0, \ldots, G_{n-1})$ be a rank n pre-geometry. We call $C = \{G_0, \ldots, G_{n-1}\}$ the *maximal parabolic chamber associated to* Γ. Assuming that F is a subset of C, the *residue* of F is the pre-geometry

$$\Gamma_F = \Gamma(\cap_{j \in t(F)} G_j, (G_i \cap (\cap_{j \in t(F)} G_j))_{i \in I \setminus t(F)})$$

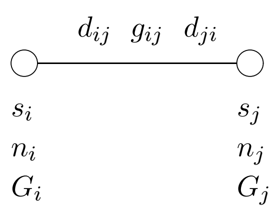
If $F = \{G_i\}$ for some $i \in I = \{0, \ldots, n-1\}$ then Γ_F is also called the G_i-*residue* of Γ and denoted Γ_i. If Γ is flag-transitive and F is any flag of Γ, of type $t(F)$, then the residue Γ_F of Γ is isomorphic to the residue of the flag $\{G_i, i \in t(F)\} \subseteq C$.

Assume Γ is a pre-geometry. We call Γ *firm* (F) (resp. *thick*, *thin*) provided that every flag of rank $\mid I \mid -1$ is contained in at least two (resp. three, exactly two) chambers. We call Γ *residually connected* (RC) provided that the incidence graph of each residue of rank ≥ 2 is connected.
Let $\Gamma(G; G_0, \ldots, G_{n-1})$ be a pre-geometry and denote $I = \{0, \ldots, n-1\}$. As in [16], for any $\emptyset \subset J \subseteq I$, we set $G_J = \bigcap_{j \in J} G_j$, and $G_\emptyset = G$. The subgroup G_I is the *Borel subgroup* of Γ. We call $\mathcal{L}(\Gamma) = \{G_J : J \subseteq I\}$ the *sublattice* (of the subgroup lattice of G) *spanned* by the collection $(G_i)_{i \in I}$. The elements of the lattice are called the *parabolic subgroups* and the subgroups G_i's are the *maximal parabolic subgroups*. When the context is clear, we write "sublattice" instead of "sublattice spanned by ...".
We call Γ *residually weakly primitive* (RWPRI) provided that for any $\emptyset \subseteq J \subset I$ there exists at least one element $i \in I \backslash J$ such that $G_{J \cup \{i\}}$ is maximal in G_J. This definition of RWPRI differs slightly from the one given in [10]. If the pre-geometry Γ is a flag-transitive geometry, then the present definition is equivalent

to the one given in [10].
The RWPRI condition implies that all subgroups of the sublattice are pairwise distinct and that $\cap_{j\in I}G_j$ is a maximal subgroup of $\cap_{j\in I\backslash\{i\}}G_j$ for all $i \in I$. Arranging the indices in suitable manner, we may also assume that $\cap_{j\in\{0,\dots,i\}}G_j$ is a maximal subgroup of $\cap_{j\in\{0,\dots,i-1\}}G_j$ for all $i = 1,\dots,n-1$.

If Γ is a geometry of rank 2 with $I = \{0,1\}$ such that each of its 0-elements is incident with each of its 1-elements, then we call Γ a *generalized digon*.
We call the pre-geometry Γ *locally 2-transitive* and we write $(2T)_1$ for this, provided that the stabilizer G_F of any flag $F \subset C$ of rank $\mid I \mid -1$ acts 2-transitively on the residue Γ_F.
Again here, if Γ is a flag-transitive geometry, then the $(2T)_1$ property is the same as the one given in [10].

Following [1] and [2], the *diagram* of a firm, residually connected, flag-transitive geometry Γ is a graph on the elements of I together with the following structure: to each vertex $i \in I$, we attach the *order* s_i which is $\mid \Gamma_F \mid -1$, where F is any flag of type $I\backslash\{i\}$, the *number* n_i *of varieties of type* i, which is the index of G_i in G, and the subgroup G_i. Elements i, j of I are not joined by an edge provided that a residue Γ_F of type $\{i,j\}$ is a generalized digon. Otherwise, i and j are joined by an edge endowed with three positive integers d_{ij}, g_{ij}, d_{ji}, where g_{ij} (the *gonality*) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i,j\}$ and d_{ij} (resp. d_{ji}), the *i-diameter* (resp. *j-diameter*) is the greatest distance from some fixed i-element (resp. j-element) to any other element in the incidence graph of Γ_F.
On a picture of the diagram, this structure will often be depicted as follows.



If $g_{ij} = d_{ij} = d_{ji} = n$, then Γ_F is called a *generalized n-gon* and we do not write d_{ij} and d_{ji} on the picture.

As to notation for groups, we follow the conventions of the Atlas [12] up to slight variations. The symbol “:” stands for split extensions, the “hat” symbol “$\hat{\cdot}$” stands for non split extensions and the symbol $\times$ stands for direct products. We write E_q for an elementary Abelian group of order q. A group is called *of Suzuki type* if it is a simple group isomorphic to $Sz(q)$ with q an odd power of two. Observe that in the present paper, we prefer not to consider the group $Sz(2) \cong AGL(1,5)$ as a group of Suzuki type. Its geometries can be found in [10].

Structure	**Order**	**Index**	**Description**
$(E_q{:}E_q){:}(q\text{-}1)$	$q^2 \cdot (q-1)$	q^2+1	Normalizer of a 2-Sylow, stabilizer of a point of Ω.
$D_{2(q-1)}$	$2 \cdot (q-1)$	$\frac{(q^2+1)\cdot q^2}{2}$	Stabilizer of a pair of points of Ω.
$\alpha_q{:}4$	$\alpha_q{\cdot}4$	$\frac{q^2(q-1)}{4\beta_q}$	Normalizer of a cyclic group of order α_q
β_q :4	β_q ·4	$\frac{q^2(q-1)}{4\alpha_q}$	Normalizer of a cyclic group of order β_q
$Sz(2^{2f+1})$ with $2f+1 \mid_M 2e+1$	$(s^2+1)\cdot s^2 \cdot (s-1)$		

Table 1. The maximal subgroups of Sz(q)

When an integer n divides an integer m and $\frac{m}{n}$ is a prime number, we write $n \mid_M m$. When an integer n divides an integer m and $n \neq m$, we write $n \mid_P m$.

2. Preliminary lemmas

For a good introduction on the Suzuki groups, we refer to [22] (see also [17]). We remind the reader that the group $Sz(q)$ has order $q^2(q^2+1)(q-1)$. Observe that $q^2+1 = (q+\sqrt{2q}+1)(q-\sqrt{2q}+1)$. We write α_q (resp. β_q) for $q+\sqrt{2q}+1$ (resp. $q-\sqrt{2q}+1$). Let Ω be a set of q^2+1 points on which $Sz(q)$ acts doubly transitively. Table 1 is taken from [21]. It gives the list of maximal subgroups of $Sz(q)$. These subgroups are studied more deeply in [17].

Lemma 2. *Let m and n be odd positive integers. Then $(2^{2m}+1, 2^{2n}+1) = 2^{2(m,n)}+1$.*

Lemma 3. *[19] Let $G \cong Sz(q)$ and $G_0 \cong Sz(s)$ be a maximal subgroup of G. Let $G_{01} \cong D_{2(s-1)}$ be a maximal subgroup of G_0. The only subgroups of G containing G_{01} are G_0 and D_{2n}-subgroups with $s-1 \mid n \mid q-1$.*

Lemma 4. *[19] If Γ is a* Rwpri *pre-geometry of rank ≥ 4 of a Suzuki group $Sz(q)$, then each of the maximal parabolic subgroups that is maximal in $Sz(q)$ must be isomorphic to a Suzuki group $Sz(s)$ for some s.*

Since we want to construct rank four Rwpri pre-geometries, we have to find 4-tuples of subgroups of $Sz(q)$ whose sublattice satisfies the Rwpri condition.

The following classification theorems are taken from [19]. The rank four classification given in the next section rely on them.

Theorem 1. *[19] Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ and $q-1$ are primes. Then all rank 3* Rwpri *geometries of Sz(q) have a sublattice isomorphic to one of the following.*

(R_1)

G_0 $D_{2(q-1)}$	G_1 $E_q:(q-1)$	G_2 $E_q:(q-1)$
G_{01} 2	G_{02} 2	G_{12} $q-1$
	G_{012} 1	

(R_2)

G_0 $D_{2(q-1)}$	G_1 $E_q:(q-1)$	G_2 D_{2p}
G_{01} $q-1$	G_{02} 2	G_{12} 2
	G_{012} 1	

with $p \mid \alpha_q$, β_q or $q-1$, and p is a prime.

(R_3)

G_0 $D_{2(q-1)}$	G_1 D_{2p}	G_2 $D_{2p'}$
G_{01} 2	G_{02} 2	G_{12} 2
	G_{012} 1	

with $p \mid \alpha_q$, β_q, 2 or $q-1$, and $p' \mid \alpha_q$, β_q, or $q-1$, and p, p' two primes (not necessarily distinct).

Theorem 2. *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ is a prime but $q-1$ is not. Then G does not possess any* Rwpri *pre-geometry of rank ≥ 3.*

Theorem 3. *[19] Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1 = p_1^{e_1} \dots p_n^{e_n}$ with $p_i \neq p_j$, $\forall i \neq j$, and $\sum_{i=1}^n e_i \geq 2$. Suppose $q = s^{p_i}$ for some $i \in \{1 \dots n\}$. If $s-1$ is a prime (which implies that $\sum_{i=1}^n e_i = 2$) then the rank 3* Rwpri *geometries of G are the following ones.*

(R_4)

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} D_{10}	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $t = 2^{p_2}$, and $2e+1 =$

$p_1 \cdot p_2$, and $p_1 \neq p_2$ are two primes.

(R_5)

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(s')$
G_{01} $D_{2(s-1)}$	G_{02} 2^2	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $s' = 2^{p_2}$, and $2e+1 = p_1 \cdot p_2$, where $p_1 \neq p_2$ are primes.

(R_6)

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(s')$
G_{01} $D_{2(s-1)}$	G_{02} 4	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $s' = 2^{p_2}$, and $2e+1 = p_1 \cdot p_2$, where $p_1 \neq p_2$ are primes.

(R_7)

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $E_q:((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $E_s:(s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

with $p \mid \frac{q-1}{s-1}$ a prime.

(R_8)

G_0 $Sz(s)$	G_1 $(ap):4$	G_2 $Sz(t)$
G_{01} $a:4$	G_{02} 4×2	G_{12} $p:4$
	G_{012} 4	

with $a = \alpha_s$ or β_s (resp. $p = \alpha_t$ or β_t) primes, $n = 2$, $q = s^{p_1} = t^{p_2}$ and $p_1 \neq p_2$.

If $s-1$ is not a prime but $\sum_{i=1}^n e_i = 2$ then the rank 3 Rwpri *pre-geometries of G are the geometries (R_7) and (R_8) above.*
Finally, if $\sum_{i=1}^n e_i > 2$ then the rank 3 Rwpri *geometries of G are the following ones.*

(R_9)

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $> E_q:((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $E_s:(s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

with $p \mid \frac{q-1}{s-1}$ a prime.

(R_{10})

G_0 $Sz(s)$	G_1 $D_{2(t-1)pp'}$	G_2 $Sz(s')$
G_{01} $D_{2(t-1)p}$	G_{02} $Sz(t)$	G_{12} $D_{2(t-1)p'}$
	G_{012} $D_{2(t-1)}$	

with $p \mid \frac{s-1}{t-1}$, $p' \mid \frac{s'-1}{t-1}$ *two distinct primes,* $s = t^{p_i}$ *and* $s' = t^{p_j}$, *with* $i \neq j$.

(R_{11})

G_0 $Sz(s)$	G_1 $(mpp') : 4$	G_2 $Sz(s')$
G_{01} $mp : 4$	G_{02} $Sz(t)$	G_{12} $mp' : 4$
	G_{012} $m : 4$	

with $m = \alpha_t$ *or* β_t, $mp \mid \alpha_s$ *or* β_s, $mp' \mid \alpha_{s'}$ *or* $\beta_{s'}$, *and* p, p' *two primes.*

(R_{12})

G_0 $Sz(s)$	G_1 $E_q{:}E_t : (t-1)$	G_2 $Sz(s')$
G_{01} $E_s{:}E_t : (t-1)$	G_{02} $Sz(t)$	G_{12} $E_{s'}{:}E_t : (t-1)$
	G_{012} $E_t{:}E_t : (t-1)$	

with $q = s^{p_i} = s'^{p_j} = t^{p_i p_j}$, *and* $i \neq j$.

(R_{13})

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $Sz(s'')$
G_{01} $Sz(t)$	G_{02} $Sz(t')$	G_{12} $Sz(t'')$
	G_{012} $Sz(u)$	

with $\sum_{i=1}^{n} e_i \geq 4$, $n \geq 3$, $q = s^{p_i} = s'^{p_j} = s''^{p_k} = t^{p_i p_j} = t'^{p_i p_k} = t''^{p_j p_k} = u^{p_i p_j p_k}$, *and* i, j, k *three pairwise distinct numbers.*

We remind the reader that lemma 1 permits us to rely on this classification when we determine all rank ≥ 4 Rwpri pre-geometries since in the rank three case, every pre-geometry is a geometry.

Theorem 4. *[20] Let* $G = D_{2n}$ *be a dihedral group with* $n = p_1^{e_1} \dots p_m^{e_m} > 2$. *Up to isomorphism, the group* G *has* $\binom{m}{\alpha-1} + \binom{m}{\alpha} + m \cdot \binom{m-1}{\alpha-2}$ *geometries of rank* α, *that satisfy F, RC, FT and* Rwpri *(*$\alpha = 2 \dots m+1$*). They are given below.*

- $\binom{m}{\alpha-1}$ *geometries with the following diagram:*
  ($\{i_1, \dots, i_{\alpha-1}\} \subseteq \{1, \dots, m\}$ *and* $i_j \neq i_k \forall j \neq k$*)*

$$\begin{array}{cccccc} \bigcirc & & \bigcirc & \cdots & \bigcirc & \\ 1 & & p_{i_1}-1 & & p_{i_{\alpha-1}}-1 & \\ 2 & & p_{i_1} & & p_{i_{\alpha-1}} & B = Z_{n_{p_{i_1}\cdots p_{i_{\alpha-1}}}} \\ Z_n & & D_{2n_{p_{i_1}}} & & D_{2n_{p_{i_{\alpha-1}}}} & \end{array}$$

- $\binom{m}{\alpha}$ *geometries with the following diagram:*
  ($\{i_1,\ldots,i_\alpha\} \subseteq \{1,\ldots,m\}$ *and* $i_j \neq i_k \forall j \neq k$*)*

$$\begin{array}{cccccc} \bigcirc & & \bigcirc & \cdots & \bigcirc & \\ p_{i_1}-1 & & p_{i_2}-1 & & p_{i_\alpha}-1 & \\ p_{i_1} & & p_{i_2} & & p_{i_\alpha} & B = D_{2n_{p_{i_1}\cdots p_{i_\alpha}}} \\ D_{2n_{p_{i_1}}} & & D_{2n_{p_{i_2}}} & & D_{2n_{p_{i_\alpha}}} & \end{array}$$

- $m \cdot \binom{m-1}{\alpha-2}$ *geometries with the following diagram:*
  ($\{i_1,\ldots,i_{\alpha-1}\} \subseteq \{1,\ldots,m\}$ *and* $i_j \neq i_k \forall j \neq k$*)*

$$\begin{array}{ccccccc} \bigcirc & \overset{p_{i_1}}{\rule[0.5ex]{4em}{0.4pt}} & \bigcirc & & \bigcirc & \cdots & \bigcirc \\ 1 & & 1 & & p_{i_2}-1 & & p_{i_{\alpha-1}}-1 \\ p_{i_1} & & p_{i_1} & & p_{i_2} & & p_{i_{\alpha-1}} \\ D_{2n_{p_{i_1}}} & & D_{2n_{p_{i_1}}} & & D_{2n_{p_{i_2}}} & & D_{2n_{p_{i_{\alpha-1}}}} \end{array}$$

$$B = Z_{n_{p_{i_1}\cdots p_{i_{\alpha-1}}}}.$$

3. The rank ≥ 4 pre-geometries of $Sz(q)$

We now concentrate on the rank 4. The higher ranks will appear clearly as generalizations of the extension process we use to construct rank 4 pre-geometries from rank 3 residues. These generalizations are summarized in the next section in eleven constructions of Rwpri pre-geometries. By lemma 4, we may assume $G_0 = Sz(s)$ with $q = s^p$ and p a prime. Throughout this section we assume $s = 2^{2f+1}$ and $q = s^p = 2^{2e+1}$. Since the G_0-residue must be a Rwpri pre-geometry, theorems 1, 2 and 3 give the possible sublattices of this residue. In section 3.1. (resp. 3.2., 3.3.), we take every possible residue of theorem 1 (resp. 2, 3) and try to construct rank four pre-geometries having it as G_0-residue. Remark that, since we want residually weakly primitive pre-geometries, we may assume that all subgroups of the sublattice of a pre-geometry are pairwise distinct. This fact is often used in the next subsections.

3.1. $s-1$ is a prime

If $s-1$ is a prime, the possible G_0-residues are (R_1), (R_2) and (R_3) (see theorem 1). For each of these three residues, we examine whether or not it can be extended to a rank four residually weakly primitive geometry.

Residue (R_1)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong E_s : (s-1)$, $G_{03} \cong E_s : (s-1)$, $G_{012} \cong 2$, $G_{013} \cong 2$, $G_{023} \cong s-1$ and $G_{0123} \cong 1$. By lemma 3, we have $G_1 \cong D_{2(s-1)n}$ with $n \mid \frac{q-1}{s-1}$ a prime. Theorem 4 gives $G_{12} \cong D_{2n}$, $G_{13} \cong D_{2n}$ and $G_{123} \cong n$. Now, $G_2 =< E_s : (s-1), D_{2n}, G_{23} >$ implies that G_2 is a subgroup of Suzuki type, let's say $G_2 = Sz(s')$, with $q = s'^{p'}$ and $s'-1$, p' two primes. Thus the rank 3 residue of G_2 must be given by theorem 1. Since all pre-geometries appearing in this theorem have at least two subgroups isomorphic to Z_2 in their sublattice, and since n cannot be 2 (n is a divisor of $q-1$), we have a contradiction with the residually weakly primitive condition.

Residue (R_2)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong E_s : (s-1)$, $G_{03} \cong D_{2n}$, $G_{012} \cong s-1$, $G_{013} \cong 2$, $G_{023} \cong 2$ and $G_{0123} \cong 1$. By lemma 3, we have $G_1 \cong D_{2(s-1)n'}$ with $n' \mid \frac{q-1}{s-1}$ a prime. Theorem 4 gives two possible residues for G_1.
The first one is with $G_{12} = D_{2(s-1)}$. It implies that $G_2 =< E_s : (s-1)$, $D_{2(s-1)}$, $G_{23} >$ must be a subgroup of Suzuki type. Thus $G_2 = Sz(s')$ with $q = s'^{p'}$, and $s'-1$ and p' two primes. Then $s-1 = s'-1$. Since $E_s : (s-1)$ is self-normalizing in $Sz(q)$, we have $G_0 = G_2$, a contradiction.
The second residue for G_1 gives $G_{12} \cong (s-1)n'$, $G_{13} \cong D_{2n'}$ and $G_{123} \cong n'$. Suppose $n \mid q-1$. Then $n = s-1$ and G_3 is a dihedral group. By theorem 4, we have $G_{23} = D_{2n'}$ and hence $G_2 =< E_s : (s-1), (s-1)n', D_{2n'} >$ must be a subgroup of Suzuki type. Then, $G_2 = Sz(s)$ and since $E_s : (s-1)$ is self-normalizing in $Sz(q)$, we get $G_0 = G_1$, a contradiction.
Suppose then that $n \mid s^2+1$. Then $G_3 =< D_{2n}, D_{2n'}, G_{23} >$ must be a subgroup of Suzuki type, say $Sz(s')$ with $q = s'^{p'}$ and $s'-1$ and p' two primes. Again, $p' \neq p$ otherwise $s = s'$ and then $G_0 = G_2$ as in the preceding case. This yields $G_{23} = E_{s'} : (s'-1)$ and $G_2 = E_q : ((s-1)(s'-1))$. Finally, by lemma 2, we have that $n = 5$. The sublattice is then fully determined. We don't know whether the group $Sz(q)$ acts flag-transitively on this pre-geometry, but if it does, the diagram of this pre-geometry looks as in figure 1. The generalization to higher ranks is obvious. It is described in construction 1 of the next section.

Residue (R_3)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong D_{2(s-1)}$, $G_{02} \cong D_{2n}$, $G_{03} \cong D_{2n'}$, $G_{012} \cong 2$, $G_{013} \cong 2$, $G_{023} \cong 2$ and $G_{0123} \cong 1$. By lemma 3, we have $G_1 \cong D_{2(s-1)n''}$ with $n'' \mid \frac{q-1}{s-1}$ a prime. Theorem 4 gives $G_{123} \cong Z_{n''}$ and $G_{12} \cong G_{13} \cong D_{2n''}$.
Suppose $n = s-1$. Then $G_2 \cong G_1$ and since $N_{Sz(q)}(D_{2n''}) = D_{2n''}$, we get $G_1 = G_2$, a contradiction.
Suppose then that $n \neq s-1 \neq n'$. Then $G_2 \cong Sz(s')$ (resp. $G_3 \cong Sz(s'')$), with $q = s'^{p'}$ (resp. $q = s''^{p''}$) and $s'-1$ and p' (resp. $s''-1$ and p'') two primes. This

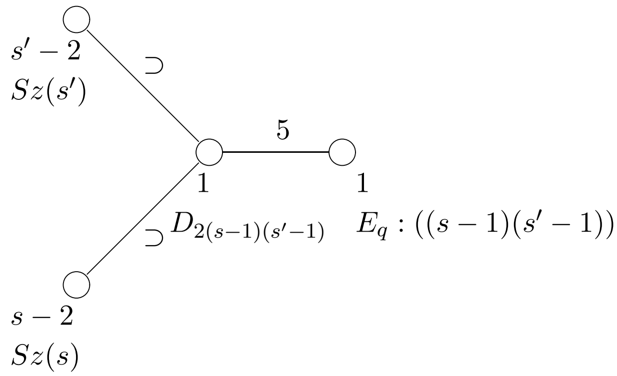


Figure 1. The diagram obtained for residue (R_2).

yields $s' = s''$ and thus $G_2 = G_3$, a contradiction.

3.2. $s-1$ is not a prime and $2f+1$ is

In this case, theorem 2 yields that there is no Rwpri pre-geometry of rank greater than 3.

3.3. $2f+1$ is not a prime

This is the most complicated case to analyze. The rank 3 residues of G_0 are those given in theorem 3. They are subdivided into three cases depending of the primality of $t-1$ and $2g+1$, where $s = t^{p'}$ with p' a prime and $2f+1 = (2g+1)p'$.

3.3.1. The case where $t-1$ is a prime

In this case, the G_0-residues are numbers (R_4) to (R_8) of theorem 3. We analyze them one by one.

Residue (R_4)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong D_{2(t-1)(t'-1)}$, $G_{03} \cong Sz(t')$, $G_{012} \cong D_{2(t-1)}$, $G_{013} \cong D_{10}$, $G_{023} \cong D_{2(t'-1)}$ and $G_{0123} \cong 2$ where $2f+1 = p_1 \cdot p_2$ with $p_1 \neq p_2$, $s = t^{p_1} = t'^{p_2}$, and $t-1$, $t'-1$ two primes. Since $Sz(s)$ is a maximal subgroup of $Sz(q)$, we have $2e+1 = p_1p_2p_3$ with p_i a prime for $i = 1$, 2, and 3. Also, G_1 (resp. G_3) contains a subgroup of Suzuki type. Thus G_1 (resp. G_3) is of Suzuki type. This yields $G_1 \cong Sz(s')$ (resp. $G_3 \cong Sz(s''))$ with $s \neq s'$ (resp. $s \neq s''$) and hence $p_3 \neq p_1$ (resp. $p_3 \neq p_2$). They both have a residue of type (R_4) as G_0. Now, G_2 must be a dihedral group and by theorem 4, we have $G_2 \cong D_{2(t-1)(t'-1)(t''-1)}$, $G_{12} \cong D_{2(t-1)(t''-1)}$, $G_{13} \cong Sz(t'')$, $G_{23} \cong D_{2(t'-1)(t''-1)}$ and $G_{123} \cong D_{2(t''-1)}$. The generalization to higher ranks is obvious. It is described in construction 2 of the next section.

Residue (R_5)**:** We can do the same analysis as in the previous case, just by changing D_{10} by 2^2. The generalization to higher ranks is obvious. It is described in construction 3 of the next section.

Residue (R_6)**:** We can do the same analysis as in the previous case, just by changing 2^2 by Z_4. The generalization to higher ranks is obvious. It is described in construction 4 of the next section.

Residue (R_7)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong D_{2(t-1)n}$, $G_{03} \cong E_s : ((t-1)n)$, $G_{012} \cong D_{2(t-1)}$, $G_{013} \cong E_t : (t-1)$, $G_{023} \cong (t-1)n$ and $G_{0123} \cong t-1$ where $n \mid \frac{s-1}{t-1}$ is a prime, and $s = 2^{p_1p_2}$ with p_1 and p_2 two primes (not necessarily distinct). Since G_1 contains a subgroup of Suzuki type, it must be a subgroup of Suzuki type. Thus $G_1 = Sz(s')$ with $s' \neq s$ and $q = 2^{p_1p_2p_3}$ with $p_2 \neq p_3$, and $s' = 2^{p_1p_3}$. The Borel subgroup of Γ_1 being $t-1$, the residue of G_1 is as the residue of G_0. Hence $G_{12} \cong D_{2(t-1)n'}$, $G_{13} \cong E_{s'} : ((t-1)n')$, $G_{23} \cong (t-1)nn'$ and $G_{123} \cong (t-1)n'$. Again here, $n' \mid \frac{q-1}{s'-1}$ is a prime. Now, $G_2 =< D_{2(t-1)n}, D_{2(t-1)n'} >$ must be a dihedral group. This yields $G_{23} \cong (t-1)nn'$ and $G_2 = D_{2(t-1)nn'}$. Finally, $G_4 =< E_s : ((t-1)n), E_{s'} : ((t-1) : n'), (t-1)nn' >= E_q : ((t-1)nn')$. Thus the sublattice is fully known. The generalization to higher ranks is obvious. It is described in construction 5 of the next section.

Residue (R_8)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong (\gamma_t\gamma_{t'}) : 4$, $G_{03} \cong Sz(t')$, $G_{012} \cong \gamma_t : 4$, $G_{013} \cong 4 \times 2$, $G_{023} \cong \gamma_{t'} : 4$ and $G_{0123} \cong 4$ where $\gamma_t = \alpha_t$ or β_t is a prime and $\gamma_{t'} = \alpha_{t'}$ or $\beta_{t'}$ is also a prime. Since G_{01} and G_{03} are Suzuki groups, G_1 and G_3 must also be Suzuki groups. Thus $q = 2^{p_1p_2p_3}$, $s = 2^{p_1p_2}$, $t = 2^{p_1}$, $t' = 2^{p_2}$, and $G_1 \cong Sz(s')$, $G_3 \cong Sz(s'')$ with $s' = 2^{p_1p_3}$ and $s'' = 2^{p_2p_3}$. The G_1 and G_3 residues are the same as the one of G_0. This yields $G_{13} \cong Sz(t'')$ with $t'' = 2^{p_3}$, $G_{123} \cong \gamma_{t''} : 4$ with $\gamma_{t''} = \alpha_{t''}$ or $\beta_{t''}$ a prime number, and $G_{23} \cong \gamma_{t'}\gamma_{t''} : 4$. Finally, $G_2 =< \gamma_t\gamma_{t'} : 4, \gamma_t\gamma_{t''} : 4 >$ yields $G_2 \cong \gamma_t\gamma_{t'}\gamma_{t''} : 4$. The sublattice is thus completely determined. The generalization to higher ranks is obvious. It is described in construction 6 of the next section.

3.3.2. The case where $t-1$ is not a prime and $2g+1$ is

Theorem 3 says that residues (R_7) and (R_8) are the only possible G_0-residues. In the preceding discussion for these residues, the primality of $t-1$ is not important. So these two cases are exactly the same, whether $t-1$ is a prime or not.

3.3.3. The case where $2g+1$ is not a prime

In this case, the residues are numbers (R_9) to (R_{13}) of theorem 3. We analyze them one by one.

Residue (R_9)**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong D_{2(t-1)n}$, $G_{03} \cong E_s : ((t-1)n)$, $G_{012} \cong D_{2(t-1)}$, $G_{013} \cong E_t : (t-1)$, $G_{023} \cong (t-1)n$ and $G_{0123} \cong t-1$ where $n \mid \frac{s-1}{t-1}$ is a prime. Because of G_{01}, the subgroup G_1 must be a subgroup of Suzuki type. Its residue can only be of type (R_9). Also, $G_1 \cong Sz(s')$ with $s' \neq s$. We thus have $G_{12} \cong D_{2(t-1)n'}$, $G_{13} \cong E_{s'} : ((t-1)n')$ and $G_{123} \cong (t-1)n'$, where $n' \mid \frac{s'-1}{t-1}$ is a prime. Since $G_2 =< G_{02}, G_{12}, G_{23} >$, we know that G_2 must be a dihedral group. By theorem 4, we then have $G_{23} \cong (t-1)nn'$. Thus $G_3 \cong E_q : ((t-1)nn')$ and the sublattice is completely determined. The generalization to higher ranks is obvious. It is described in construction 7 of the next section.

Residue (R_{10})**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong D_{2(u-1)nn'}$, $G_{03} \cong Sz(t')$, $G_{012} \cong D_{2(u-1)n}$, $G_{013} \cong Sz(u)$, $G_{023} \cong D_{2(u-1)n'}$ and $G_{0123} \cong D_{2(u-1)}$ with $n \mid \frac{t-1}{u-1}$ and $n' \mid \frac{t'-1}{u-1}$ two primes. Because of G_{01} and G_{03}, we know that subgroups G_1 and G_3 are of Suzuki type. Their residue has the same form as that of G_0. Thus we have $G_1 \cong Sz(s')$, $G_2 \cong Sz(s'')$, $G_{12} \cong D_{2(u-1)nn''}$, $G_{13} \cong Sz(t'')$ and $G_{123} \cong D_{2(u-1)n''}$. There we have $t = u^{p_i}$, $t' = u^{p_j}$ and $t'' = u^{p_k}$ with i, j, k three pairwise distinct integers in $\{1, \ldots, m\}$. Also, $n'' \mid \frac{t''-1}{u-1}$ is a prime. The subgroup $G_3 =< G_{03}, G_{13}, G_{23} >\cong D_{2(u-1)nn'n''}$ by theorem 4. The generalization to higher ranks is obvious. It is described in construction 8 of the next section.

Residue (R_{11})**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong (\gamma_u nn') : 4$, $G_{03} \cong Sz(t')$, $G_{012} \cong (\gamma_u n) : 4$, $G_{013} \cong Sz(u)$, $G_{023} \cong (\gamma_u n') : 4$ and $G_{0123} \cong \gamma_u : 4$ where γ_u is either α_u or β_u and n (resp. n') is a prime dividing $\frac{\alpha_t}{\gamma_u}$ or $\frac{\beta_t}{\gamma_u}$ (resp. $\frac{\alpha_{t'}}{\gamma_u}$ or $\frac{\beta_{t'}}{\gamma_u}$) depending whether γ_u divides α_t (resp. $\alpha_{t'}$) or β_t (resp. $\beta_{t'}$).

Because of G_{01} and G_{03}, we know that G_1 and G_3 must be of Suzuki type. So we may assume $G_1 \cong Sz(s')$ and $G_3 \cong Sz(s'')$ with $q = s^p = s'^{p'} = s''^{p''}$ and p, p', p'' three pairwise distinct primes. These two subgroups have residues of the same kind as G_0. Thus $G_{13} \cong Sz(t'')$ with $t'' = q^{\frac{1}{p'p''}}$, $G_{123} \cong (\gamma_u n'') : 4$ with n'' a prime dividing $\frac{\alpha''_t}{\gamma_u}$ or $\frac{\beta''_t}{\gamma_u}$. Also, $G_{12} \cong (\gamma_u nn'') : 4$ and $G_{23} \cong (\gamma_u n'n'') : 4$. Finally, since $G_2 =< G_{02}, G_{12}, G_{23} >$ we have $G_2 \cong (\gamma_u nn'n'') : 4$. The generalization to higher ranks is obvious. It is described in construction 9 of the next section.

Residue (R_{12})**:** In this case, we have $G_0 \cong Sz(s)$, $G_{01} \cong Sz(t)$, $G_{02} \cong E_s \hat{\cdot} E_u : (u-1)$, $G_{03} \cong Sz(t')$, $G_{012} \cong E_t \hat{\cdot} E_u : (u-1)$, $G_{013} \cong Sz(u)$, $G_{023} \cong E_{t'} \hat{\cdot} E_u : (u-1)$ and $G_{0123} \cong E_u \hat{\cdot} E_u : (u-1)$. Because of G_{01} (resp. G_{03}), the subgroup G_1 (resp. G_3) must be of Suzuki type. So $G_1 \cong Sz(s')$ (resp. $G_3 \cong Sz(s''))$ with $q = s'^{p'} = s''^{p''}$ and $p' \neq p''$ two primes. Because of G_{0123}, we know that their residues are as the one of G_0. This yields $G_{12} \cong E_{s'} \hat{:} E_u : (u-1)$, $G_{13} \cong Sz(t'')$ with $q = t''^{p'p''}$, $G_{23} \cong E_{s''} \hat{:} E_u : (u-1)$, and $G_{123} \cong E_{t''} \hat{:} E_u : (u-1)$. It is then obvious that $G_2 \cong E_q \hat{:} E_u : (u-1)$. Thus the sublattice is fully determined. The generalization to higher ranks is obvious. It is described in construction 10 of the next section.

Residue (R_{13})**:** This is the easiest pre-geometry to complete. It is obvious that all subgroups appearing in the sublattice must be Suzuki groups. The generalization to higher ranks is obvious. It is described in construction 11 of the next section.

This ends the determination of all the RWPRI pre-geometries for a Suzuki simple group $Sz(q)$. We now resume the long discussion of this section in the next one.

4. Constructions and classification theorem

We give now the eleven constructions mentioned in the previous discussion. These are all generalizations of constructions of rank 4 pre-geometries described above. Of course, in order to use these constructions to build a RWPRI pre-geometry, the reader has to take care of the whole sublattice which is not completely detailed here, but which can be easily guessed by looking at the corresponding rank 4 pre-geometry from which it arises. After these constructions, we state a classification theorem for the RWPRI pre-geometries of rank greater than 3 for a Suzuki simple group $Sz(q)$.

Construction 1. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$.
Suppose $n_i = 2^{p_i} - 1$ is a prime for all $i = 1, \dots, m$.

Take
$$G_{i-1} \cong Sz(q^{\frac{1}{p_i}}) \text{ for all } i = 1, \dots, m;$$
$$G_m \cong D_{2n_1 \dots n_m};$$
$$G_{m+1} \cong q : (n_1 \dots n_m),$$

with $\cap_{j=0}^{m-1} G_j = D_{10}$. Then $\Gamma(G; G_0, \dots, G_{m+1})$ is a rank $m+2$ RWPRI pre-geometry.

Construction 2. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$.
Suppose $n_i = 2^{p_i} - 1$ is a prime for all $i = 1, \dots, m$.

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ for all $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$,

with $\cap_{j=0}^{m} G_j = D_{10}$. Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI pre-geometry.

Remark that the pre-geometries obtained in construction 2 are truncations of those obtained in construction 1.

Construction 3. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$.
Suppose $n_i = 2^{p_i} - 1$ is a prime for all $i = 1, \dots, m$.

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ for all $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$,

with $\cap_{j=0}^{m} G_j = D_4$. Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI pre-geometry.

Construction 4. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$.
Suppose $n_i = 2^{p_i} - 1$ is a prime for all $i = 1, \dots, m$.

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ for all $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$,

with $\cap_{j=0}^{m} G_j = Z_4$. Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI pre-geometry.

Remark that the only difference between constructions 2, 3 and 4 is the intersection of the Suzuki subgroups.

Construction 5. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1 \dots p_{m-1} \cdot p_m^{\epsilon}$ with $p_i \neq p_j$ for all $i \neq j$ and $\epsilon = 1$ or 2.
We distinguish the case $\epsilon = 1$ and the case $\epsilon = 2$.

Case 1: $\epsilon = 1$

Suppose $t - 1 = 2^{p_i} - 1$ is a prime for some $i \in \{1, \dots, m\}$. We construct a pre-geometry as follows.

Take $G_{j-1} \cong Sz(q^{\frac{1}{p_j}})$ for all $j = 1, \dots, m$, $j \neq i$;
$G_i \cong D_{2\Pi_{j \in \{1,\dots,m\} \setminus \{i\}} n_j}$
where $n_j \mid \frac{2^{p_i p_j} - 1}{t-1}$ are pairwise distinct primes;
$G_m \cong q : ((t-1)\Pi_{j \in \{1,\dots,m\} \setminus \{i\}} n_j)$.

Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI pre-geometry. The choice of i might lead to m pairwise distinct pre-geometries.

Case 2: $\epsilon = 2$

Suppose $t-1 = 2^{p_m}-1$ is a prime. We construct a pre-geometry as follows.

Take $G_{j-1} \cong Sz(q^{\frac{1}{p_j}})$ for all $j = 1, \dots, m$;
$G_m \cong D_{2n_1\dots n_m}$ where $n_j \mid \frac{2^{p_j p_m}-1}{t-1}$ are pairwise distinct primes;
$G_{m+1} \cong q : ((t-1)\Pi_{j\in\{1,\dots,m\}\setminus\{i\}} n_j)$.

Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+2$ RWPRI pre-geometry.

Construction 6. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$.
Suppose that for all $i = 1, \dots, m$, at least one of $\alpha_{2^{p_i}}$ or $\beta_{2^{p_i}}$ is a prime and denote that number by γ_i.

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ for all $i = 1, \dots, m$;
$G_m \cong (\gamma_1 \dots \gamma_m) : 4$.

Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI pre-geometry.

Construction 7. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$.
Let $\alpha \in \{3, \dots \min(m+2, \sum_{i=1}^m e_i)\}$.

Take $\alpha - 2$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ ($i = 1, \dots, \alpha-2$) and put $a_i = \frac{\log_2(q)}{\log_2(s_i)}$. Let $t = q^{\frac{1}{a_1 \cdots a_{\alpha-2}}}$ and take $n_i \mid \frac{t^{a_i}-1}{t-1}$ pairwise distinct primes ($i = 1, \dots, \alpha-2$).
Take $G_{\alpha-2} \cong D_{2(t-1)n_1\dots n_{\alpha-2}}$ and $G_{\alpha-1} \cong q : ((t-1)n_1 \dots n_{\alpha-2})$.

Then $\Gamma(G; G_0, \dots, G_{\alpha-1})$ is a rank α RWPRI pre-geometry.

Construction 8. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$.
Let $\alpha \in \{3, \dots \min(m+1, \sum_{i=1}^m e_i)\}$.

Take $\alpha - 1$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ with $i = 1, \dots, \alpha-1$. Put $a_i = \frac{\log_2(q)}{\log_2(s_i)}$. Let $t = q^{\frac{1}{a_1 \cdots a_{\alpha-1}}}$ and take $n_i \mid \frac{t^{a_i}-1}{t-1}$ pairwise distinct primes ($i = 1, \dots, \alpha-1$).
Take $G_{\alpha-1} \cong D_{2(t-1)n_1\dots n_{\alpha-1}}$.

Then $\Gamma(G; G_0, \dots, G_{\alpha-1})$ is a rank α RWPRI pre-geometry.

Construction 9. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$.
Let $\alpha \in \{3, \dots \min(m+1, \sum_{i=1}^m e_i)\}$.

Take $\alpha - 1$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ with $i = 1, \dots, \alpha-1$. Put $a_i = \frac{\log_2(q)}{\log_2(s_i)}$. Let $t = q^{\frac{1}{a_1 \cdots a_{\alpha-1}}}$.

Take $\gamma = \alpha_t$ or β_t and take $n_i \mid \frac{\alpha_t a_i}{\gamma}$ or $\frac{\beta_t a_i}{\gamma}$ pairwise distinct primes ($i = 1, \ldots, \alpha - 1$) such that $\gamma n_1 \ldots n_{\alpha-1}$ divides either α_q of β_q.
Take $G_{\alpha-1} \cong (\gamma n_1 \ldots n_{\alpha-1}) : 4$.

Then $\Gamma(G; G_0, \ldots, G_{\alpha-1})$ is a rank α Rwpri pre-geometry.

Construction 10. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e + 1 = p_1^{e_1} \ldots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$.
Let $\alpha \in \{3, \ldots \min(m + 1, \sum_{i=1}^{m} e_i)\}$.

Take $\alpha - 1$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ with $i = 1, \ldots, \alpha - 1$. Put $a_i = \frac{\log_2(q)}{\log_2(s_i)}$. Let $t = q^{\overline{\frac{1}{a_1 \cdots a_{\alpha-1}}}}$.
Take $G_{\alpha-1} \cong q{:}t : (t - 1)$.

Then $\Gamma(G; G_0, \ldots, G_{\alpha-1})$ is a rank α Rwpri pre-geometry.

Construction 11. Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer.
Let $2e + 1 = p_1^{e_1} \ldots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$.
Let $\alpha \in \{2, \ldots m - 1\}$.

Take α pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ with $i = 1, \ldots, \alpha$.

Then $\Gamma(G; G_0, \ldots, G_{\alpha-1})$ is a rank α Rwpri pre-geometry.

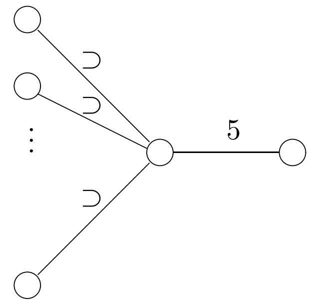
Theorem 5. *The* Rwpri *pre-geometries of rank ≥ 4 are those given by constructions 1 to 11.*

Proof. This is an immediate consequence of the discussion given in section 3. and the eleven constructions described in this section. $\boxed{QED}$

5. Observations and future work

A first observation is that for almost all Rwpri pre-geometries of rank greater than 3 of a Suzuki simple group $Sz(q)$, if they are flag-transitive geometries, then, their diagram is a star-diagram. Indeed, only those obtained by construction 11 do not have a star-diagram.

The pre-geometries given by construction 1 look very attractive. Indeed, if they are flag-transitive geometries, their diagram is fully known. It is a star diagram with one rank 2 residue being a pentagon and all other non-trivial rank 2 residues being circles. It looks as follows.



As future work, we plan to study the flag-transitivity of these RWPRI pre-geometries, and also of all those given by theorem 5.

Acknowledgements. We would like to thank Francis Buekenhout and Michel Dehon for fruitful comments on a preliminary version of this paper.

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