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Selection principles in uniform spaces

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Abstract. We begin the investigation of selection principles in uniform spaces in a manner as it was done with selection principles theory for topological spaces. We introduced and characterized uniform versions of classical topological notions of the Menger, Hurewicz and Rothberger properties. The uniform γ-sets are also considered.

Keywords: Uniform space, selection principles, uniform Menger, uniform Hurewicz, uniform Rothberger, uniform γ-set, ω-cover, γ-cover, groupability, weak groupability.

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Introduction

We first recall the basic facts about selection principles in topological spaces.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X.

The symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n \mid n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n \mid n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{B}$ [16].

The symbol $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n \mid n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ [16].

If $\mathcal{O}$ denotes the collection of all open covers of a space X, then X is said to have the *Menger property* [13], [7], [14], [9] (resp. the *Rothberger property* [15], [14], [16] if the selection hypothesis $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ (resp. $\mathsf{S}_1(\mathcal{O},\mathcal{O})$) is true for X.

Our terminology and notation follow [4].

For a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $\mathrm{St}(A,\mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\cup\{P \in \mathcal{P} \mid A \cap P \neq \emptyset\}$; for $A = \{x\}$, $x \in X$, we write $\mathrm{St}(x,\mathcal{P})$ instead of $\mathrm{St}(\{x\},\mathcal{P})$.

Let X be a space. If α and β are families of subsets of X we denote by $\alpha \wedge \beta$ the set $\{A \cap B \mid A \in \alpha, B \in \beta\}$. $\alpha < \beta$ means that α is a refinement of β, i.e. that for each $A \in \alpha$ there is a $B \in \beta$ with $A \subset B$. If $\{\mathrm{St}(A,\alpha) \mid A \in \alpha\} < \beta$ we

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write $\alpha^* < \beta$ and say that α is a *star refinement* of β or that α is *star inscribed* in β.

Let us recall now three equivalent definitions of uniform spaces:

- using a family $\mathbb{C}$ of covers (see, for example, [17], [3]; also [4]);
- using a family $\mathbb{U}$ of entourages of the diagonal [4];
- using a family $\mathbb{D}$ of pseudometrics [5](see also [4]).

In this paper we shall use the first of these three approaches because it allows us to define uniform selection principles in the standard manner as in general topological case, and to see, after their characterizations, the right nature of them. More precisely, we shall see that uniform selection principles are actually a kind of star selection principles defined and studied in [10].

A *uniformity* on a nonempty set X is a family $\mathbb{C}$ of covers of X which satisfies the following conditions:

$(C.1)$ if $\alpha \in \mathbb{C}$ and if β is a cover of X such that $\alpha < \beta$, then $\beta \in \mathbb{C}$;

$(C.2)$ if α_1, $\alpha_2 \in \mathbb{C}$, then there exists $\beta \in \mathbb{C}$ such that $\beta^* < \alpha_1$ and $\beta^* < \alpha_2$.

The covers from $\mathbb{C}$ are called *uniform covers*, and the pair $(X, \mathbb{C})$ a uniform space.

We assume that all spaces are infinite and consider only Hausdorff uniformities satisfying also the condition

$(C.3)$ for any two distinct points x and y in X there is an $\alpha \in \mathbb{C}$ such that no member of α contains both x and y.

The topology on X generated by $\mathbb{C}$, denoted $T_{\mathbb{C}}$, is defined in such way that $\{\mathrm{St}(x, \alpha) \mid x \in X, \alpha \in \mathbb{C}\}$ is a base for $T_{\mathbb{C}}$.

Let us mention a basic fact regarding uniform spaces. For any uniformity $\mathbb{C}$ on X and each $\alpha \in \mathbb{C}$ there is a pseudometric d on X such that $\{B_d(x, 1) \mid x \in X\} < \{\mathrm{St}(x, \alpha) \mid x \in X\}$.

This fact (or the third approach for the definition of uniform structures) gives a possibility to consider also measure-like properties in uniform spaces. These properties often give characterizations of selection principles in metric spaces. We do not consider such questions in this paper.

1 The uniform Menger property

Let $(X, \mathbb{C})$ be a uniform space. We say that X has the *uniform Menger property* if for each sequence $(\alpha_n : n \in \mathbb{N})$ of elements of $\mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ β_n is a finite subset of α_n and $\bigcup_{n\in\mathbb{N}} \beta_n$ is a cover of X.

So, in notation above the uniform Menger property is the $\mathsf{S}_{fin}(\mathbb{C}, \mathcal{C})$ property.

1 Theorem. *For a uniform space $(X, \mathbb{C})$ the following are equivalent:*

(a) *X has the uniform Menger property;*

(b) *for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} St(A_n, \alpha_n)$.*

(c) *for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each n β_n is a finite subset of α_n and $X = \bigcup_{n\in\mathbb{N}} St(\cup\beta_n, \alpha_n)$.*

Proof. $(a) \Rightarrow (b)$: Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of covers from $\mathbb{C}$. By (a), for each n choose a finite subset β_n of α_n such that $\bigcup_{n\in\mathbb{N}} \beta_n$ is a cover of X. For each n and each $B \in \beta_n$ choose an element $x(B, n) \in B$ and put $A_n = \{x(B, n) \mid B \in \beta_n\}$. Then each A_n is a finite subset of X and

$$X = \bigcup_{n\in\mathbb{N}} \cup\beta_n \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \alpha_n),$$

i.e. (b) holds.

$(b) \Rightarrow (c)$: It is evident.

$(c) \Rightarrow (a)$: Let $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ be a sequence of uniform covers of X. For each n let γ_n be a uniform cover of X which is star inscribed in α_n. Apply (c) to the sequence $(\gamma_n \mid n \in \mathbb{N})$ and find a sequence $(\delta_n \mid n \in \mathbb{N})$ such that each δ_n is a finite subset of γ_n and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\delta_n, \gamma_n) = X$. For each $n \in \mathbb{N}$ and each $D \in \delta_n$ pick a member $A_D \in \alpha_n$ such that $\mathrm{St}(D, \gamma_n) \subset A_D$ and let $\mu_n = \{A_D \mid D \in \delta_n\}$. Then the sequence $(\mu_n \mid n \in \mathbb{N})$ witnesses for $(\alpha_n \mid n \in \mathbb{N})$ that X satisfies (a). QED

This theorem shows that the uniform Menger property is actually a kind of star covering properties. In [10] we introduced star-Menger and strongly star-Menger topological spaces just as spaces which satisfy conditions (c) and (b), respectively, from the previous theorem, but with open covers instead of uniform ones. In notation we used there the conditions (b) and (c) of the previous theorem can be written as $\mathsf{SS}^*_{fin}(\mathbb{C}, \mathcal{O})$ and $\mathsf{S}^*_{fin}(\mathbb{C}, \mathcal{O})$, respectively.

2 Remark. In terms of entourages of the diagonal of X the uniform Menger property is defined in this way:

A uniform space $(X, \mathbb{U})$ is uniformly Menger if for each sequence $(U_n \mid n \in \mathbb{N})$ of entourages of the diagonal there is a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} U_n[A_n] = X$, where $U_n[A_n] = \{y \in X \mid (x, y) \in U_n$ for some $x \in A_n\}$.

Recall that a uniform space $(X, \mathbb{C})$ is said to be *totally bounded* or *precompact* [4], [3] (resp. *pre-Lindelöf* or *ω-bounded* [3], [12]) if each $\alpha \in \mathbb{C}$ has a finite (resp. countable) subcover, or equivalently, if for each $\alpha \in \mathbb{C}$ there exists a finite (resp. countable) set $A \subset X$ such that $X = \mathrm{St}(A, \alpha)$.

Evidently that for a uniform space $(X, \mathbb{C})$ we have:

1. If X is totally bounded, then it is uniformly Menger;
2. If X is uniformly Menger, then it is pre-Lindelöf;
3. If $(X, T_{\mathbb{C}})$ has the Menger property, then $(X, \mathbb{C})$ is uniformly Menger.

3 Note. There is a uniform space $(X, \mathbb{C})$ which is uniformly Menger, but topological space $(X, T_{\mathbb{C}})$ has no the Menger property.

Any non-Lindelöf Tychonoff space serves as such an example. To see this we have only to observe that each (Tychonoff) space with the Menger property is Lindelöf and that, by 8.1.19 and 8.3.4 in [4], each Tychonoff space X admits a uniformity $\mathbb{C}^*$ which is totally bounded and thus uniformly Menger, and generates the original topology on X.

4 Note. A regular topological space X has the Menger property if and only if its fine uniformity has the uniform Menger property.

Let X be a regular topological space with the Menger property. Then X is Lindelóf and thus paracompact. By a result of A.H. Stone (see 5.4.H.(d) in [4]) it follows that each open cover of X is normal. So, the collection of all open covers of X is the base of a uniformity on X – the universal uniformity on X (see 8.1.C in [4]).

5 Note. Any uniformity $\mathbb{C}$ on a Tychonoff space X such that $(X, \mathbb{C})$ is uniformly Menger is coarser or equal to the Shirota uniformity on X ([4]).

It follows from the fact that uniformly Menger uniform spaces are pre-Lindelöf and 8.1.I in [4].

Uniform spaces having the uniform Menger property have some properties which are similar to the corresponding properties of totally bounded uniform spaces.

6 Theorem. *If a uniform space Y is a uniformly continuous image of a uniformly Menger space X, then Y is also uniformly Menger.*

7 Theorem. *Every subspace of a uniformly Menger uniform space $(X, \mathbb{C})$ is uniformly Menger.*

Proof. Let $(Y, \mathbb{C}_Y)$ be a subspace of $(X, \mathbb{C})$ and let $(\mu_n \mid n \in \mathbb{N})$ be a sequence of elements of $\mathbb{C}_Y$. For each n let α_n be an element in $\mathbb{C}$ such that $\mu_n = \alpha_n \wedge Y := \{U \cap Y \mid U \in \alpha_n\}$ and let $\beta_n \in \mathbb{C}$ be such that $\beta_n^* < \alpha_n$. Apply to the sequence $(\beta_n \mid n \in \mathbb{N})$ the fact that $(X, \mathbb{C})$ is uniformly Menger and find finite sets $A_n \subset X$, $n \in \mathbb{N}$, such that $X = \bigcup_{n \in \mathbb{N}} \mathrm{St}(A_n, \beta_n)$. For each n put

$$B_n = \{a \in A_n \mid \exists\, y \in Y \text{ with } y \in \mathrm{St}(a, \beta_n)\}$$

and for each $b \in B_n$ choose an element $y_b \in Y$ with $y_b \in \mathrm{St}(b, \beta_n)$; put

$$C_n = \{y_b \mid b \in B_n\}.$$

We claim that the sequence $(C_n \mid n \in \mathbb{N})$ witnesses for $(\mu_n \mid n \in \mathbb{N})$ that $(Y, \mathbb{C}_Y)$ is uniformly Menger.

Let $y \in Y$. There are $n \in \mathbb{N}$ and $a \in A_n$ such that $y \in \mathrm{St}(a, \beta_n)$. By definition of B_n it means that $a \in B_n$ and so there is $y_a \in C_n$ satisfying $y_a \in \mathrm{St}(a, \beta_n)$, hence $a \in \mathrm{St}(y_a, \beta_n)$. Since $y \in \mathrm{St}(a, \beta_n)$ and $\beta_n^* < \alpha_n$ it follows $y \in \mathrm{St}(y_a, \alpha_n)$. This together with $y \in Y$ gives $y \in \mathrm{St}(y_a, \mu_n)$, i.e. $Y = \bigcup_{n \in \mathbb{N}} \mathrm{St}(C_n, \mu_n)$. QED

The following theorem describes a property of uniform spaces which is close to the Menger uniform property.

An open cover $\mathcal{U}$ of a space X is an *ω-cover* [6] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in an element of $\mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is said to be *weakly groupable* [2] if it is a union of countably many finite, pairwise disjoint subfamilies $\mathcal{U}_n$ such that for each finite set $F \subset X$ there is an n with $F \subset \cup \mathcal{U}_n$.

8 Theorem. *For a uniform space $(X, \mathbb{C})$ the following are equivalent:*

(1) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ of finite sets such that for each n, $\beta_n \subset \alpha_n$ and the set $\{St(\cup \beta_n, \alpha_n) \mid n \in \mathbb{N}\}$ is an ω-cover of X;*

(2) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each n β_n is a finite subset of α_n and $\{St(\cup \beta_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a weakly groupable cover of X.*

(3) *For each sequence $(\alpha_n \mid n \in \mathbb{N})$ of uniform covers of X there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\{St(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ is an ω-cover of X;*

(4) *For each sequence* $(\alpha_n \mid n \in \mathbb{N})$ *of uniform covers of* X *there is a sequence* $(F_n \mid n \in \mathbb{N})$ *of finite subsets of* X *such that the set* $\{St(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ *is a weakly groupable cover of* X*.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that each ω-cover is weakly groupable.

(2) $\Rightarrow$ (3): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of uniform covers of X and let for each $n \in \mathbb{N}$ β_n be a uniform cover of X such that $\beta_n^* < \bigwedge_{i \leq n} \alpha_i$. Apply (2) to the sequence $(\beta_n \mid n \in \mathbb{N})$ and choose a sequence $(\gamma_n \mid n \in \mathbb{N})$ such that each γ_n is a finite subset of β_n and the family $\{\mathrm{St}(\cup\gamma_n, \beta_n) \mid n \in \mathbb{N}\}$ is a weakly groupable cover of X. Therefore, there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers such that each finite subset F of X is contained in $\cup\{\mathrm{St}(\cup\gamma_i, \beta_i) \mid n_k \leq i < n_{k+1}\}$ for some $k \in \mathbb{N}$. For each n and each $C \in \gamma_n$ let A_C be an element of α_n with $\mathrm{St}(C, \beta_n) \subset A_C$ and let $\delta_n = \{A_C \mid C \in \gamma_n\}$. For each A_C in δ_n pick a point $x(A_C) \in A_C$, denote by K_n the set $\{x(A_C) \mid C \in \gamma_n\}$ and put

$M_n = \cup\{K_i \mid i < n_1\}$, for $n < n_1$,

$M_n = \cup\{K_i \mid n_k \leq i < n_{k+1}\}$, for $n_k \leq n < n_{k+1}$.

Then each M_n is a finite subset of X. Let us show that the set $\{\mathrm{St}(M_n, \alpha_n) \mid n \in \mathbb{N}\}$ is an ω-cover of X.

Let F be a finite subset of X. There is some k such that $F \subset \cup\{\mathrm{St}(\cup\gamma_i, \beta_i) \mid n_k \leq i < n_{k+1}\}$. Then we have

$$\begin{aligned} F &\subset \cup\{\mathrm{St}(\cup\gamma_i, \beta_i) \mid n_k \leq i < n_{k+1}\} = \cup_{n_k \leq i < n_{k+1}} \cup_{C \in \gamma_i} \mathrm{St}(C, \beta_i) \\ &\subset \cup_{n_k \leq i < n_{k+1}} \cup_{C \in \gamma_i} A_C \subset \mathrm{St}(M_n, \alpha_n)(\text{ for an } n \text{ with } n_k \leq n < n_{k+1}). \end{aligned}$$

It completes the proof of (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4): As in (1) $\Rightarrow$ (2) we use the fact that each ω-cover is weakly groupable.

(4) $\Rightarrow$ (2): If $(\alpha_n \mid n \in \mathbb{N})$ is a sequence of uniform covers of X, then according to (4) choose first a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a weakly groupable cover of X and then for each point $x \in F_n$ pick a set $A(x, n) \in \alpha_n$ containing x. If for each $n \in \mathbb{N}$ we define $\beta_n = \{A(x, n) \mid n \in F_n\}$, then the sequence $(\beta_n \mid n \in \mathbb{N})$ witnesses for $(\alpha_n \mid n \in \mathbb{N})$ that X satisfies (2).

(2) $\Rightarrow$ (1): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of elements of $\mathbb{C}$ and let for each n ξ_n be an element of $\mathbb{C}$ such that $\xi_n^* < \bigwedge_{i \leq n} \alpha_i$. Choose now a sequence $(\mu_n : n \in \mathbb{N})$ such that for each n μ_n is a finite subset of ξ_n and the set $\{\mathrm{St}(\cup\mu_n, \xi_n) \mid n \in \mathbb{N}\}$ is a weakly groupable cover of X. There is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that for each finite set F in X one has $F \subset \cup\{\mathrm{St}(\cup\mu_i, \xi_i) \mid n_k \leq i < n_{k+1}\}$ for some k. Define now the sequence $(\beta_n \mid n \in \mathbb{N})$ as follows:

(i) for each $n < n_1$ let $\beta_n = \bigcup_{i<n_1} \mu_i$;
(ii) for each n with $n_k \le n < n_{k+1}$ let $\beta_n = \bigcup_{n_k \le i < n_{k+1}} \mu_i$.

Then for each n β_n is a finite subset of α_n. It is easily seen that we have: for each finite set F in X there exists an n such that $F \subset \mathrm{St}(\cup\beta_n, \alpha_n)$, i.e. X satisfies (1). QED

2 Uniform Hurewicz spaces

In 1925 in [7] (see also [8]), W. Hurewicz introduced a covering property for a topological space X, called now the *Hurewicz property*, in this way:

> For each sequence $(\mathcal{U}_n \mid n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n \mid n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{V}_n \subset \mathcal{U}_n$, and for each $x \in X$, for all but finitely many n, $x \in \cup\mathcal{V}_n$.

We define now the natural uniform analogue of this property.

A uniform space $(X, \mathbb{C})$ is said to have the *uniform Hurewicz property* if for each sequence $(\alpha_n \mid n \in \mathbb{N})$ of uniform covers of X there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that each β_n is a finite subset of α_n and for each $x \in X$ we have $x \in \cup\beta_n$ for all but finitely many n.

The following theorem gives a characterization of the uniform Hurewicz property.

9 Theorem. *For a uniform space $(X, \mathbb{C})$ the following statements are equivalent:*

(1) *X is uniformly Hurewicz;*

(2) *For each sequence $(\alpha_n \mid n \in \mathbb{N})$ of uniform covers of X there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $St(F_n, \alpha_n)$;*

(3) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each n β_n is a finite subset of α_n and each $x \in X$ is contained in all but finitely many sets $St(\cup\beta_n, \alpha_n)$.*

Proof. (1) $\Rightarrow$ (2): It is evident.

(2) $\Rightarrow$ (3): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of covers from $\mathbb{C}$. By (2) there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ is an element of $\mathrm{St}(F_n, \alpha_n)$ for all but finitely many n. For each n and each $x \in F_n$ take an element $A(x, n) \in \alpha_n$ containing x and put $\beta_n = \{A(x, n) \mid n \in F_n\}$. It is easy to see that the sequence $(\beta_n \mid n \in \mathbb{N})$ guarantees that (3) is true for X.

(3) $\Rightarrow$ (1): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of uniform covers of X. For each $n \in \mathbb{N}$ choose a $\beta_n \in \mathbb{C}$ with $\beta_n^* < \alpha_n$. Applying (3) to the sequence $(\beta_n \mid n \in \mathbb{N})$ one may find a sequence $(\mu_n \mid n \in \mathbb{N})$ such that each μ_n is a finite subset of β_n and each $x \in X$ is a member of $\mathrm{St}(\cup\mu_n, \beta_n)$ for all but finitely many n. For each $M \in \mu_n$ choose an element $U(M) \in \alpha_n$ such that $\mathrm{St}(M, \mu_n) \subset U(M)$ and put $\nu_n = \{U(M) \mid M \in \mu_n\}$. Then for each n ν_n is a finite subset of α_n and it is routine to check that the sequence $(\nu_n \mid n \in \mathbb{N})$ shows for $(\alpha_n \mid n \in \mathbb{N})$ that X is uniformly Hurewicz. *QED*

10 Remark. When studying a uniform spaces using entourages of the diagonal we shall use the condition (2) of the previous theorem as an official definition of the uniform Hurewicz property:

A uniform space$(X, \mathbb{U})$ is uniformly Hurewicz if for each sequence $(U_n \mid n \in \mathbb{N})$ of entourages of the diagonal there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $U_n[F_n]$.

Notice the following simple facts. For a uniform space $(X, \mathbb{C})$ we have:

1. If X is totally bounded, then it is uniformly Hurewicz;
2. If X is uniformly Hurewicz, then it is uniformly Menger;
3. If $(X, T_{\mathbb{C}})$ has the Hurewicz property, then $(X, \mathbb{C})$ is uniformly Hurewicz.

Arguments similar to those in 3 and 4 give:

11 Note. There is a uniform space $(X, \mathbb{C})$ which is uniformly Hurewicz, but topological space $(X, T_{\mathbb{C}})$ has no the Hurewicz property.

12 Note. A regular topological space X has the Hurewicz property if and only if its fine uniformity has the uniform Hurewicz property.

From 4 and 12 we conclude that the classes of uniformly Menger and uniformly Hurewicz spaces are different.

The following two results are similar to the corresponding results in Section 1.

13 Theorem. *If a uniform space Y is a uniformly continuous image of a uniformly Hurewicz space X, then Y is also uniformly Hurewicz.*

14 Theorem. *Every subspace of a uniformly Hurewicz uniform space $(X, \mathbb{C})$ is uniformly Hurewicz.*

The next three results concerning the uniform Hurewicz property are variations on some properties of totally bounded uniform spaces.

15 Theorem. *If a uniformly Hurewicz space $(X, \mathbb{C}_X)$ is dense in a uniform space $(Y, \mathbb{C})$, then Y is also uniformly Hurewicz.*

PROOF. Let $(\mu_n \mid n \in \mathbb{N})$ be a sequence in $\mathbb{C}$ and let for each n $\alpha_n = \mu_n \wedge X := \{V \cap X \mid V \in \mu_n\}$. Then $(\alpha_n \mid n \in \mathbb{N})$ is a sequence in $\mathbb{C}_X$. Choose for each n a member $\nu_n \in \mathbb{C}$ for which $\nu_n^* < \mu_n$ and let $\beta_n = \nu_n \wedge X$. Since X is uniformly Hurewicz, there are finite sets $A_n \subset X$, $n \in \mathbb{N}$, such that each x in X belongs to all but finitely many $\mathrm{St}(A_n, \beta_n)$'s. Let us show that $(A_n \mid n \in \mathbb{N})$ witnesses for $(\mu_n \mid n \in \mathbb{N})$ that Y is uniformly Hurewicz.

Let $y \in Y$. Let n be big enough. Then there is $x \in X \cap \mathrm{St}(y, \nu_n)$. Since X is uniformly Hurewicz, x belongs to all but finitely many sets $\mathrm{St}(A_k, \beta_k)$, i.e. there is k_0 such that for each $k > k_0$ there exists $a_k \in A_k$ with $x \in \mathrm{St}(a_k, \beta_k) \subset \mathrm{St}(a_k, \nu_k)$. But $\nu_k^* < \mu_k$ and so for all $k > \max\{n, k_0\}$ we have $y \in \mathrm{St}(a_k, \mu_k)$. This means that $(Y, \mathbb{C}_Y)$ has indeed the uniform Hurewicz property. QED

16 Corollary. *A uniform space X is uniformly Hurewicz if and only if its completion $\tilde{X}$ is uniformly Hurewicz.*

17 Theorem. *The product $(X \times Y, \mathbb{C}_X \times \mathbb{C}_Y)$ of uniformly Hurewicz uniform spaces $(X, \mathbb{C}_X)$ and $(Y, \mathbb{C}_Y)$ is uniformly Hurewicz.*

PROOF. Let $(\gamma_n \mid n \in \mathbb{N})$ be a sequence in $\mathbb{C}_X \times \mathbb{C}_Y$; one may suppose that all elements in each γ_n are of the form $U_n \times V_n$, with $U_n \in \alpha_n \in \mathbb{C}_X$, $V_n \in \beta_n \in \mathbb{C}_Y$. For each n let $\mu_n \in \mathbb{C}_X$ and $\nu_n \in \mathbb{C}_Y$ be such that $\mu_n^* < \alpha_n$ and $\nu_n^* < \beta_n$. There is a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of X and a sequence $(B_n \mid n \in \mathbb{N})$ of finite subsets of Y such that each $x \in X$ and each $y \in Y$ one has $x \in \mathrm{St}(A_n, \mu_n)$ and $y \in \mathrm{St}(B_n, \nu_n)$ for all but finitely many n. We prove that the sets $C_n = A_n \times B_n$, $n \in \mathbb{N}$, show that $X \times Y$ uniformly Hurewicz.

Let $(x, y) \in X \times Y$. There is $n_0 \in \mathbb{N}$ such that for each $n > n_0$ we have $x \in \mathrm{St}(A_n, \mu_n)$ and $y \in \mathrm{St}(B_n, \nu_n)$, i.e for some $a_n \in A_n$ and some $b_n \in B_n$ we have $x \in \mathrm{St}(a_n, \mu_n)$, $y \in \mathrm{St}(b_n, \nu_n)$. Further, there is k_0 such that for all $k > k_0$ we have $a_n \in \mathrm{St}(A_k, \mu_k)$ and $b_n \in \mathrm{St}(B_k, \nu_k)$. Since $\mu_n^* < \alpha_n$ and $\nu_n^* < \beta_n$ it follows that for $n > \max\{n_0, k_0\}$ we have $x \in \mathrm{St}(A_n, \alpha_n)$ and $y \in \mathrm{St}(B_n, \beta_n)$, i.e. $(x, y) \in \mathrm{St}(A_n \times B_n, \gamma_n)$. QED

Let us remark that in a similar way one can prove the following result.

18 Theorem. *The product of a uniformly Menger space X and a uniformly Hurewicz space Y is uniformly Menger.*

We end this section showing a result related to the uniform Hurewicz property. Before that we need the following notion [11].

An open cover $\mathcal{U}$ of a space X is called *groupable* if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each $x \in X$ belongs to $\cup \mathcal{U}_n$ for all but finitely many n.

An open cover $\mathcal{U}$ of a space X is a *γ-cover* [6] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

19 Theorem. *For a uniform space $(X, \mathbb{C})$ the following statements are equivalent:*

(1) *For each sequence $(\alpha_n \mid n \in \mathbb{N})$ of uniform covers of X there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\{St(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a γ-cover of X;*

(2) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each n β_n is a finite subset of α_n and the set $\{St(\cup\beta_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a γ-cover of X;*

(3) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each n β_n is a finite subset of α_n and $\{St(\cup\beta_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a groupable cover of X;*

(4) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\{St(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a groupable cover of X.*

Proof. (1) $\Rightarrow$ (2): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of covers from $\mathbb{C}$. By (2) there is a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a γ-cover X. For each n and each $x \in F_n$ take an element $A(x, n) \in \alpha_n$ containing x and then put $\beta_n = \{A(x, n) \mid n \in F_n\}$. It is easy to see that the sequence $(\beta_n \mid n \in \mathbb{N})$ guarantees that (3) is true for X.

(2) $\Rightarrow$ (3): Because each γ-cover is groupable it follows that this implication is true.

(3) $\Rightarrow$ (4): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of uniform covers of X. For each $n \in \mathbb{N}$ choose a uniform cover β_n which is star inscribed in α_n. Applying (3) to the sequence $(\beta_n \mid n \in \mathbb{N})$ one finds a sequence $(\mu_n \mid n \in \mathbb{N})$ such that each μ_n is a finite subset of β_n and the family $\{\mathrm{St}(\cup\mu_n, \beta_n) \mid n \in \mathbb{N}\}$ is a groupable cover of X. This means that there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ in $\mathbb{N}$ such that each $x \in X$ belongs to $\cup\{\mathrm{St}(\cup\mu_i, \beta_i) \mid n_k \leq i < n_{k+1}\}$ for all but finitely many k. For each $n \in \mathbb{N}$ and each $M \in \mu_n$ pick a set A_M in α_n containing $\mathrm{St}(M, \beta_n)$ and then a point $x_M \in A_M$. If we put $F_n = \{x_M \mid M \in \mu_n\}$ we obtain a sequence $(F_n \mid n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(F_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a groupable cover for X; the sequence $n_1 < n_2 < \cdots < n_k < \cdots$ shows this fact.

(4) $\Rightarrow$ (1): Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of covers from $\mathbb{C}$. For each $n \in \mathbb{N}$ choose a $\beta_n \in \mathbb{C}$ such that $\beta_n^* < \bigwedge_{i \leq n} \alpha_i$. Apply now (4) to the sequence $(\beta_n \mid n \in \mathbb{N})$ and for each $n \in \mathbb{N}$ choose a finite subset F_n of X such that $\{\mathrm{St}(F_n, \beta_n) \mid n \in \mathbb{N}\}$ is a groupable cover of X, i.e. there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of positive integers such that each $x \in X$ belongs to $\cup\{\mathrm{St}(F_i, \beta_i) \mid n_k \leq i < n_{k+1}\}$ for all but finitely many k. Further, we define

$\Phi_n = \bigcup_{i<n_1} F_i$, for each $n < n_1$,
$\Phi_n = \bigcup_{n_k \leq i < n_{k+1}} F_i$, for each n with $n_k \leq n < n_{k+1}$.

For each n Φ_n is a finite subset of X. It is easily verified that the set $\{\mathrm{St}(\Phi_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a γ-cover of X, i.e. that (1) holds. *QED*

3 Rothberger's property in uniform spaces

By analogy with the definition of the Rothberger property in topological spaces we introduce the next notion.

A uniform space $(X, \mathbb{C})$ has the *uniform Rothberger property* if for each sequence $(\alpha_n \mid n \in \mathbb{N})$ of elements of $\mathbb{C}$ there is a sequence $(U_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ $U_n \in \alpha_n$ and $\bigcup_{n\in\mathbb{N}} U_n = X$.

20 Theorem. *For a uniform space $(X, \mathbb{C})$ the following are equivalent:*

- (a) *X has the uniform Rothberger property;*
- (b) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(x_n \mid n \in \mathbb{N})$ of elements of X such that $X = \bigcup_{n\in\mathbb{N}} St(x_n, \alpha_n)$.*
- (c) *For each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(A_n \mid n \in \mathbb{N})$ such that for each n $A_n \in \alpha_n$ and $X = \bigcup_{n\in\mathbb{N}} St(A_n, \alpha_n)$.*

Proof. $(a) \Rightarrow (b)$: Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of covers from $\mathbb{C}$. By (a), for each n choose a member U_n in α_n such that $\{U_n \mid n \in \mathbb{N}\}$ is a cover of X. For each n choose an element $x_n \in U_n$. Then

$$X = \bigcup_{n\in\mathbb{N}} U_n \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(x_n, \alpha_n),$$

i.e. (b) holds.

$(b) \Rightarrow (c)$: It is clear.

$(c) \Rightarrow (a)$: Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of uniform covers of X. Choose a sequence $(\gamma_n \mid n \in \mathbb{N}) \subset \mathbb{C}$ such that for each n γ_n is star inscribed in $\alpha_n \mid \gamma_n^* < \alpha_n$. Apply (c) to the sequence $(\gamma_n \mid n \in \mathbb{N})$; there is a sequence $(G_n \mid n \in \mathbb{N})$ such that for each n $G_n \in \gamma_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n, \gamma_n) = X$. Let for each n A_n be an element of α_n such that $\mathrm{St}(G_n, \gamma_n) \subset A_n$. Then $X = \bigcup_{n\in\mathbb{N}} A_n$. *QED*

Notice that each uniformly Rothberger uniform space is uniformly Menger but these two classes of spaces are distinct. It follows from the fact (that can be verified in a similar manner as in 4 and 12) that a regular topological space X has the Rothberger property if and only if its fine uniformity has the uniform Rothberger property.

4 Uniform γ-sets

A topological space X is a γ-set [6] if for each sequence $(\mathcal{U}_n \mid n \in \mathbb{N})$ of ω-covers of X there is a sequence $(U_n \mid n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and the set $\{U_n \mid n \in \mathbb{N}\}$ is a γ-cover of X.

We define uniform γ-sets in the following way.

A uniform space $(X, \mathbb{C})$ is said to be a *uniform γ-set* if for each sequence $(\alpha_n \mid n \in \mathbb{N})$ of uniform covers of X there is a sequence $(x_n \mid n \in \mathbb{N})$ of elements of X such that each x is contained in $\mathrm{St}(x_n, \alpha_n)$ for all but finitely many n.

21 Theorem. *For a uniform space $(X, \mathbb{C})$ the following assertions are equivalent:*

(1) *X is a uniform γ-set;*

(2) *For each sequence $(\alpha_n \mid n \in \mathbb{N})$ of uniform covers of X there is a sequence $(A_n \mid n \in \mathbb{N})$ such that for each n $A_n \in \alpha_n$ and the set $\{St(A_n, \alpha_n) \mid n \in \mathbb{N}\}$ is a γ-cover of X.*

Proof. To prove (1) implies (2) we need to argue as in the proof of (2) implies (3) of Theorem 9. So, we have to prove only (2) $\Rightarrow$ (1).

Let $(\alpha_n \mid n \in \mathbb{N})$ be a sequence of uniform covers of X. For each n choose a $\beta_n \in \mathbb{C}$ such that $\beta_n^* < \alpha_n$. We apply (2) to the sequence $(\beta_n \mid n \in \mathbb{N})$ and select for each n an element $B_n \in \beta_n$ such that the set $\{\mathrm{St}(B_n, \beta_n) \mid n \in \mathbb{N}\}$ is a γ-cover of X. For each $n \in \mathbb{N}$ let A_n be a member of α_n satisfying $\mathrm{St}(B_n, \beta_n) \subset A_n$. For each n take a point $x_n \in A_n$. Then we have the sequence $(x_n \mid n \in \mathbb{N})$ witnessing for $(\alpha_n \mid n \in \mathbb{N})$ that X is a uniform γ-set. QED

5 Closing remarks

To each selection principle for topological spaces it is naturally associated the corresponding game and often selection principles can be characterized game-theoretically (see [16]).

In uniform case to each uniform selection principle one can assign also the corresponding game. For example, the game associated to the uniform Menger property is defined in the following way. Two players, ONE and TWO, play a round for each positive integer. In the n-th round ONE chooses a uniform cover α_n and TWO responds by choosing a finite set $F_n \subset X$. TWO wins a play $\alpha_1, F_1; \alpha_2, F_2, \cdots$ if $X = \bigcup_{n \in \mathbb{N}} \mathrm{St}(F_n, \alpha_n)$; otherwise ONE wins. The other games corresponding to the uniform Hurewicz property and the uniform Rothberger property are defined similarly. Evidently, if ONE does not have a winning strategy in these games, then the corresponding uniform selection principle holds.

Find examples that show that no converse is true in each of these three situations.

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