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On the semifield planes of order 5^4 and dimension 2 over the kernel

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Abstract. In this article we consider the problem of determining all non-Desarguesian semifield planes of order 5^4 and kernel $\mathrm{GF}(5^2)$. We show that the class of p-primitive planes is the largest class and besides those the only other semifield planes in the class under study are the generalized twisted field planes. We conjecture that in general these two classes include all the non-Desarguesian semifield planes of order p^4 and kernel $\mathrm{GF}(p^2)$.

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1 Semifield planes of order 5^4

Let Π be a semifield plane of order p^4 and kernel containing $\mathrm{GF}(p^2)$. Let $\mathcal{S}$ be a semifield coordinatizing Π and let $\{1, \pi\}$ be a basis for $\mathcal{S}$ over $\mathrm{GF}(p^2)$. Each element in $\mathcal{S}$ is of the form $x + y\pi$ where $x, y \in \mathrm{GF}(p^2)$. In $\mathrm{GF}(p^2)$ choose an element γ such that $\gamma^2 = s$ where s is a nonsquare in $\mathrm{GF}(p)$. Then $\{1, \gamma, \pi, \gamma\pi\}$ is a basis of $\mathcal{S}$ over $\mathrm{GF}(p)$. To define the product we must determine the following:

$$\pi\gamma = a + b\pi \qquad \pi^2 = c + d\pi \qquad \pi(\gamma\pi) = e + f\pi \qquad (\mathbf{1})$$

where $a, b, c, d, e, f \in \mathrm{GF}(p^2)$.

Paralleling earlier work of Kleinfeld [13]and Boerner-Lantz [3], in [10] we constructed all the semifield planes of order 5^4 and kernel containing $\mathrm{GF}(5^2)$. With the aid of the computer, we obtained all the coefficients a, b, c, d, e, f that produce a semifield thus obtaining all the semifields of order 5^4 and kernel containing $\mathrm{GF}(5^2)$. We then grouped the corresponding semifields into isotopism classes.

In the following table we list a representative from each isotopism class:

Semifields of order 5^4

Plane	$\pi\gamma$	π^2	$\pi(\gamma\pi)$
(a)	$\gamma\pi$	γ	3
(b)	$4\gamma\pi$	γ	3
(c)	$4\gamma\pi$	γ	2
(d)	$4\gamma\pi$	$1+\gamma$	$2+4\gamma$
(e)	$4\gamma\pi$	$1+\gamma$	$3+2\gamma$
(f)	$4\gamma\pi$	$1+2\gamma$	$4+4\gamma$
(g)	$(2+\gamma)\pi$	4γ	$2+2\gamma$
(h)	$4\gamma\pi$	γ	$1+\gamma$
(i)	$4\gamma\pi$	γ	$2+\gamma$
(j)	$4\gamma\pi$	$1+\gamma$	$4+\gamma$
(k)	$4\gamma\pi$	γ	$3+2\gamma$
(L)	$\gamma+\gamma\pi$	γ	$3+\gamma+2\gamma\pi$
(m)	$\gamma+\gamma\pi$	γ	$3+4\gamma+3\gamma\pi$

Table 1

2 p-primitive semifield planes

Let Π be a semifield plane of order q^4 and kernel containing $K \simeq \mathrm{GF}(q^2)$ where q is a prime power p^r. A *p-primitive Baer collineation* of Π is a collineation σ which fixes a Baer subplane of Π pointwise and whose order is a p-primitive divisor of q^2-1, i.e. $|\sigma| | q^2-1$, but $|\sigma| \not| p^i - 1$ for $1 \leq i \leq 2r$. A semifield plane of order p^4 and kernel $\mathrm{GF}(p^2)$ where p is an odd prime is called a *p-primitive semifield plane* if it admits a p-primitive Baer collineation. This is the class of planes obtained when the construction method of Hiramine, Matsumoto and Oyama [11] is applied to the Desarguesian plane of order p^2. Equivalently (see Johnson [12], Theorem 2.1), this is the class of planes that admit a matrix spread set of the form

$$\begin{pmatrix} u & v \\ f(v) & u^p \end{pmatrix}$$

where f is an additive function in $\mathrm{GF}(p^2)$ such that $u^{p+1} \neq vf(v)$ for any $u, v \in \mathrm{GF}(p^2)$ with $(u,v) \neq (0,0)$. This class of planes was studied by Cordero in [4]-

[8]. The following was shown:

1 Theorem. *(see [6], Theorem 2.1)*
Let Π *be a* $p-$*primitive semifield plane and let* $\mathcal{N}_m, \mathcal{N}_r, \mathcal{N}_l$ *be its middle, right and left nucleus, respectively. Then exactly one of the following holds:*

(i) $\mathcal{N}_m = \mathcal{N}_l = \mathcal{N}_r \simeq GF(p^2)$

(ii) $\mathcal{N}_m = \mathcal{N}_r \simeq GF(p)$

2 Theorem. *(see [5] Corollary 2.2)*
Let Π *be a* $p-$*primitive semifield plane and let* $\mathcal{A}(\Pi)$ *be its autotopism group. Then* $\mathcal{A}(\Pi)$ *is solvable.*

3 Theorem. *(see [4] Theorem 4.2)*
For any odd prime p*, there are* $\left(\frac{p+1}{2}\right)^2$ *nonisomorphic* $p-$*primitive semifield planes of order* p^4*.*

In [8] a matrix spread set of the p-primitive planes for $p = 3, 5, 7, 11$ was given. A representative from each isotopism class of semifields of order 5^4 is given in the table below:

p-primitive planes of order 5^4

Name	$\pi\gamma$	π^2	$\pi(\gamma\pi)$
Primi 1	$4\gamma\pi$	3γ	1
Primi 2	$4\gamma\pi$	$1+4\gamma$	$1+\gamma$
Primi 3	$4\gamma\pi$	$2+4\gamma$	1
Primi 4	$4\gamma\pi$	$3+4\gamma$	$1+4\gamma$
Primi 5	$4\gamma\pi$	$2+\gamma$	3
Primi 6	$4\gamma\pi$	$3+\gamma$	$3+4\gamma$
Primi 7	$4\gamma\pi$	$1+\gamma$	$3+\gamma$
Primi 8	$4\gamma\pi$	$1+3\gamma$	$1+4\gamma$
Primi 9	$4\gamma\pi$	$2+3\gamma$	$1+3\gamma$

Table 2

3 Determinants

Let $\mathcal{S}$ be a semifield of order p^4 and kernel containing GF(p^2) and let $u, v, x, y \in$ GF(p^2). Then the product $(u + v\pi)(x + y\pi) \in S$ may be represented by:

$$(u, v)\begin{pmatrix} x & y \\ f(x, y) & g(x, y) \end{pmatrix}$$

where the functions $f(x, y)$ and $g(x, y)$ are given by:

$$f(x, y) = p_{1,1}x + p_{1,2}x^p + p_{2,1}y + p_{2,2}y^p, \qquad g(x, y) = q_{1,1}x + q_{1,2}x^p + q_{2,1}y + q_{2,2}y^p$$

where $p_{i,j}, q_{i,j}$ $\in$GF(p^2) for $i, j \in 0, 1$. For the basis $\{1, \gamma, \pi, \gamma\pi\}$ the associated matrices are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ a & b \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & \gamma \\ e & f \end{pmatrix} \qquad (\mathbf{2})$$

Let $\mathcal{M}$ be a matrix spread set for $\mathcal{S}$. Then each matrix $M \in \mathcal{M}$ is a linear combination of the matrices in (2) above where the coefficients belong to GF(p).

For the case when $p = 5$, we calculated the determinants of a matrix spread set of each semifield with product given in Table 1 above. In the table below we give the "pattern" of determinants; that is, the ordered p^2-tuple consisting of the frequency of each element in GF(p^2) as a determinant of a matrix in the matrix spread set.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
(a)	1	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26
(b)	1	6	6	6	6	26	26	36	36	26	26	36	26	26	36	26	36	26	26	36	26	26	36	36	26
(c)	1	6	6	6	6	6	36	36	36	36	6	36	36	36	36	6	36	36	36	36	6	36	36	36	36
(d)	1	6	6	6	6	36	6	36	36	36	36	36	6	36	36	36	36	36	6	36	36	36	36	36	6
(e)	1	6	6	6	6	36	16	36	26	36	36	36	36	16	26	36	26	16	36	36	36	36	26	36	16
(f)	1	6	6	6	6	36	36	36	6	36	36	6	36	36	36	36	36	36	36	6	36	36	6	36	36
(g)	1	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26
(h)	1	6	6	6	6	26	36	36	26	26	26	26	36	26	36	26	36	26	36	26	26	26	26	36	36
(i)	1	6	6	6	6	26	36	16	36	36	36	36	26	16	36	36	36	16	26	36	26	36	36	16	36
(j)	1	6	6	6	6	36	26	16	36	36	36	16	36	26	36	36	36	26	36	16	36	36	36	16	26
(k)	1	6	6	6	6	26	36	26	26	36	26	26	36	36	26	26	26	36	36	26	26	36	26	26	36
(L)	1	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26
(m)	1	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26

Table 3

Note that (a) corresponds to the Desarguesian plane and the planes (g), (L) amd (m) have the same determinant pattern as the Desarguesian plane.

4 Classification of the semifields of order 5^4 and kernel containing GF(5^2)

Let Π_1 and Π_2 be two semifield planes of order p^4 with matrix spread sets S_1 and S_2, respectively. If the planes are isomorphic, then there exist 2×2 non-singular matrices A and B in GF(p^2) and $\sigma \in$ Aut(GF(p^2)) such that $A^{-1}M^\sigma B = N \in S_2$ for every $M \in S_1$. Moreover, $det(A^{-1}M^\sigma B) = detN$. Letting $k = \frac{detB}{detA}$ we have $(detM)^\sigma k = detN$. Now letting $M_{\sigma,k}$ be the pattern of determinants $A^{-1}M^\sigma B$ we have that $M_{\sigma,k}$ coincides with the determinant pattern of S_2.

Let Π be a semifield plane of order 5^4 and kernel containing GF(5^2) with matrix spread set S and determinant pattern M. Then there exist $\sigma \in$ Aut(GF(5^2)) and $k \in$GF(5^2) such that $M_{\sigma,k}$ coincides with one of the determinant patterns given in Table 3.

When we calculated the determinants for a matrix spread set for Primi 1 we obtained the following determinant pattern:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	6	6	6	6	6	36	36	36	36	6	36	36	36	36	6	36	36	36	36	6	36	36	36	36

Denote this pattern by M. Next we computed the patterns $M_{\sigma,k}$ for all $\sigma \in$ Aut(GF(5^2)) and $k \in$GF(5^2). We found that for all the choices of σ and k for which $M_{\sigma,k}$ was a pattern in Table 3, then the pattern was that of Plane (b). Therefore the plane coordinatized by (b) is isomorphic to the one coordinatized by Primi 7.

Similarly after comparing the determinant patterns of all the p-primitive semifield planes with the patterns of the planes in Table 1 we found the following isomorphisms:

Plane in Table 1	Isomorphic to
(b)	Primi 7
(c)	Primi 1
(d)	Primi 9
(e)	Primi 6
(f)	Primi 8
(h)	Primi 3
(i)	Primi 4
(j)	Primi 2
(k)	Primi 5

In order to complete the classification of semifield planes of order 5^4 we now consider the generalized twisted field planes .

Let Π be a generalized twisted field plane of order p^r. The product in Π is given by $x \circ y = xy - cx^\alpha y^\beta$ where $\alpha, \beta \in$ Aut(GF(p^r)) and $c \neq x^{\alpha-1}y^{\beta-1}$

for all $x, y \in$GF(p^r). If $\alpha \neq I$, $\beta \neq I$, (I = identity) and $\alpha \neq \beta$,then the right nucleus is the subfield of GF(p^r) fixed by β; that is, $\{x \in GF(p^r) : x^\beta = x\}$. The left nucleus (or kernel) is the subfield fixed by α, Albert [1], Theorem 1. Moreover, Π is Desarguesian if and only if $\alpha = \beta$, Albert [1]. Therefore, if Π is a non-Desarguesian semifield plane of order p^4 with kernel GF(p^2) we must have $\alpha : x \to x^{p^2}$ and the product is one of the following:

(i) $x \circ y = xy - cx^{p^2}y^p$

(ii) $x \circ y = xy - cx^{p^2}y^{p^2}$

(iii) $x \circ y = xy - cx^{p^2}y^{p^3}$

The product in (iii) gives a pre-semifield isotopic to the one given by (i) by Albert [2], Lemma 6. The product in (ii) yields a pre-semifield isotopic to GF(p^4). For the case (i), let $\mathcal{D}(c)$ be the pre-semifield with product given in (i). Since x and y are powers of a primitive element $\theta \in$ GF(p^4), the condition $c \neq x^{1-p^2}y^{1-p}$ is equivalent to $c \neq \theta^{i(1-p^2)}\theta^{j(1-p)} = \theta^{(p-1)(-j-i(p+1)}$. Therefore c cannot be a power of θ^{p-1}. Hence $\theta, \theta^2, \ldots, \theta^{p-2}$ are possible values for c. Now $\mathcal{D}(c)$ is isotopic to $\mathcal{D}(d)$ if and only if $d = ch^{p^2-1}k^{p-1}$ for some nonzero $h, k \in$ GF(p^4), Albert [2]. Thus $d = c\theta^{(p-1)m}$ for some integer m. Therefore there are exactly $p - 2$ nonisotopic generalized twisted fields of order p^4 with product given in (i), namely $\mathcal{D}(\theta), \ldots, \mathcal{D}(\theta^{p-2})$. In particular, when p=5 there are 3 nonisotopic generalized twisted fields. Therefore the remaining three planes on the list, namely planes (g), (L) and (m) must be the three generalized twisted field planes of order 5^4.

We have the following result.

4 Theorem. *There are exactly 13 nonisomorphic semifield planes of order 5^4 and kernel containing $GF(5^2)$; one is the Desarguesian plane, three are Generalized Twisted Field planes and nine are p-primitive planes.*

In the case when $p = 3$, Boerner [3] showed that there are six nonisomorphic semifield planes of order 3^4 and kernel containing $GF(3^2)$. Of these, one is the Desarguesian plane, one is a Generalized Twisted Field plane and four are $p-$primitive planes. We have the following conjecture:

5 Conjecture. *There are exactly $1 + (p - 2) + \left(\frac{p+1}{2}\right)^2$ non isomorphic semifield planes of order p^4 and kernel containing $GF(p^2)$ for each prime number $p > 2$. Of these, one is the Desarguesian plane, $p - 2$ are Generalized Twisted Field planes and $\left(\frac{p+1}{2}\right)^2$ are p-primitive planes.*

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