Note di Matematica **22,** n. 1, 2003, 49–58.

Legendre surfaces whose mean curvature vectors are eigenvectors of the Laplace operator

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Received: 13/9/2002; accepted: 17/12/2003.

Abstract. We study Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors of the Laplace operator.

Keywords: Sasakian manifold, Legendre surface, Laplace operator.

MSC 2000 classification: Primary 53C42; Secondary 53B25.

Introduction

Let $x : M^m \to N^n$ be an isometric immersion of an m-dimensional manifold M^m into an n-dimensional manifold N^n. Denote the Laplace operator acting on the sections of the induced bundle x^*TN^n (resp. normal bundle $T^{\perp}M^m$) by Δ (resp. Δ^D).

During the last two decades, the class of submanifolds satisfying the following condition in (pseudo-) Riemannian space forms has been investigated by many geometers:

$$\Delta H = \lambda H, \tag{1}$$

$$\Delta^D H = \lambda H, \tag{2}$$

where λ is a constant and H is the mean curvature vector field (see, for instance [2], [4]-[7], [12]).

However, in ambient spaces with non-constant curvature, very little is known for submanifolds satisfying such conditions. Recently, J. Inoguchi [13] has classified Legendre curves and Hopf cylinders which satisfy (1) or (2) in 3-dimensional Sasakian space forms. It is a generalization of the classification results about Hopf cylinders with harmonic mean curvature in the unit 3-sphere ([2]). Also, the author [14] studied surfaces satisfying (1) or (2) in a Sasakian space form of constant ϕ-sectional curvature -3.

If the mean curvature vector field satisfies $DH||\xi$ for the characteristic vector field ξ, then it is said to be *C-parallel*, where D is the normal connection. C. Baikoussis and D. E. Blair [1] classified Legendre surfaces in Sasakian space forms whose mean curvature vector fields are C-parallel. If the squared mean curvature of Legendre surfaces satisfying (1) or (2) in Sasakian space forms is constant, then H is C-parallel and therefore such surfaces are classified by the result due to C. Baikoussis and D. E. Blair.

In this article, we investigate Legendre surfaces with (1) or (2) in Sasakian space forms under the condition that the squared mean curvature is constant along a certain direction.

1 Preliminaries

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be an *almost contact manifold* if the structure group $\mathrm{GL}_{2n+1}\mathbf{R}$ of its linear frame bundle is reducible to $\mathrm{U}(n)\times\{1\}$. This is equivalent to the existence of a tensor field ϕ of type (1,1), a vector field ξ and one-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1. \tag{3}$$

It follows that

$$\eta\circ\phi = 0, \quad \phi\xi = 0. \tag{4}$$

Moreover, since $\mathrm{U}(n)\times\{1\}\subset \mathrm{O}(2n+1)$, there exists a Riemannian metric g which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\xi, X) = \eta(X), \tag{5}$$

for all $X, Y \in TM^{2n+1}$. The structure (ϕ, ξ, η, g) is called an *almost contact metric structure* and the manifold M^{2n+1} with an almost contact metric structure is said to be an *almost contact metric manifold*. If an almost contact metric manifold satisfies

$$d\eta(X, Y) = g(X, \phi Y), \tag{6}$$

for all $X, Y \in TM^{2n+1}$, then M is said to be a *contact metric manifold*. On a contact metric manifold, the vector field ξ is called the *characteristic vector field*.

A contact metric manifold is said to be a *Sasakian manifold* if it satisfies $[\phi, \phi] + 2d\eta\otimes\xi = 0$ on M^{2n+1}, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ.

The sectional curvature of a tangent plane which is invariant under ϕ is called *ϕ-sectional curvature*. If the sectional curvature is constant on all $p \in M^{2n+1}$ and all tangent planes in T_pM^{2n+1} which are invariant under ϕ, then

M^{2n+1} is said to be of *constant ϕ-sectional curvature*. Complete and connected Sasakian manifolds of constant ϕ-sectional curvature are called *Sasakian space forms*. We denote Sasakian space forms of constant ϕ-sectional curvature ϵ by $M^{2n+1}(\epsilon)$. The curvature tensor $\bar{R}$ of $M(\epsilon)$ is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{\epsilon+3}{4}\{g(Y,Z)X + g(Z,X)Y\} \\ &+ \frac{\epsilon-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\qquad + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\}.\end{aligned}$$

Let $x : N^m \to M^{2n+1}(\epsilon)$ be an isometric immersion. If η restricted to N^m vanishes, then N^m is an *integral submanifold*, in particular if $m = n$, it is called a *Legendre submanifold*.

Denote the Levi-Civita connection of $M^{2n+1}(\epsilon)$ (resp. N^m) by $\bar{\nabla}$ (resp. ∇). The formulas of Gauss and Weingarten are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{7}$$
$$\bar{\nabla}_X V = -A_V X + D_X V, \tag{8}$$

where $X,\ Y \in TM$, $V \in T^{\perp}M$. Here h, A and D are the second fundamental form, the shape operator and the normal connection, respectively. The mean curvature vector H is given by $H = \frac{1}{m}\mathrm{trace} h$.

If N^n is a Legendre submanifold, from [3] we have

$$A_{\phi Y}X = -\phi h(X,Y) = A_{\phi X}Y, \quad A_\xi = 0. \tag{9}$$

For more details see [3].

Denote by R the Riemann curvature tensor of N^m. Then the equations of Gauss, Codazzi and Ricci are given respectively by

$$\langle R(X,Y)Z,W\rangle = \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle + \langle \bar{R}(X,Y)Z,W\rangle, \tag{10}$$
$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{11}$$
$$\langle R^D(X,Y)V_1,V_2\rangle = \langle \bar{R}(X,Y)V_1,V_2\rangle + \langle [A_{V_1},A_{V_2}](X),Y\rangle, \tag{12}$$

where X,Y,Z,W (resp. V_1 and V_2) are vectors tangent (resp. normal) to N^m, $\langle,\rangle = g(,)$, $R^D(X,Y) = [D_X,D_Y] - D_{[X,Y]}$, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{13}$$

In case that N^n is a Legendre submanifold, the equation of Gauss, Codazzi, Ricci are equivalent to

$$\langle R(X,Y)Z,W\rangle = \langle [A_{\phi Z},A_{\phi W}]X,Y\rangle + \langle \bar{R}(X,Y)Z,W\rangle, \tag{14}$$
$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z). \tag{15}$$

The Laplace operator which acts on the sections of the induced bundle $x^*TM^{2n+1}(\epsilon)$ (resp. normal bundle $T^\perp N^m$) is defined by $\Delta = -\sum_{i=1}^m(\bar\nabla_{e_i}\bar\nabla_{e_i} - \bar\nabla_{\nabla_{e_i}e_i})$ (resp. $\Delta^D = -\sum_{i=1}^m(D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i})$), where $\{e_i\}$ is a local orthonormal frame of N^m.

One can obtain the following existence and uniqueness theorems by arguments similar to those given in [10, 11].

1 Theorem. *Let $(M^n, \langle\cdot,\cdot\rangle)$ be an n-dimensional simply connected Riemannian manifold. Let σ be a symmetric bilinear TM^n-valued form on M^n satisfying*

(1) *$\langle\sigma(X,Y),Z\rangle$ is totally symmetric,*

(2) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_XY,Z) - \alpha(Y,\nabla_XZ)$ is totally symmetric,*

(3) *$R(X,Y)Z = \epsilon(g(Y,Z)X - g(X,Z)Y) + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$.*

Then there exists a Legendre isometric immersion $x : (M^n, \langle\cdot,\cdot\rangle) \to N^{2n+1}(\epsilon)$ such that the second fundamental form h satisfies $h(X,Y) = \phi\sigma(X,Y)$.

2 Theorem. *Let $x^1, x^2 : M^n \to N^{2n+1}(\epsilon)$ be two Legendre isometric immersions of a connected Riemannian n-manifold into a Sasakian manifold $N^{2n+1}(\epsilon)$ with second fundamental forms h^1 and h^2. If*

$$\left\langle h^1(X,Y), \phi x^1_* Z\right\rangle = \left\langle h^2(X,Y), \phi x^2_* Z\right\rangle$$

for all vector fields X, Y, Z tangent to M^n, there exists an isometry A of $N^{2n+1}(\epsilon)$ such that $x^1 = A\circ x^2$.

2 Main results

Let $x : M^2 \to N^5(\epsilon)$ be a non-minimal Legendre immersion and $\{e_i\}$ be an orthonormal frame along M^2 such that e_1, e_2 are tangent to M^2 and $e_3 = \phi e_1$ is parallel to H. Then the shape operators take the following forms,

$$A_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad A_2 = \begin{pmatrix} b & c \\ c & -b \end{pmatrix}, \quad A_\xi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{16}$$

where $A_i = A_{\phi e_i}$. The Codazzi equation becomes $(\nabla_{e_1}A_i)e_2 - A_{D_{e_1}\phi e_i}e_2 - (\nabla_{e_2}A_i)e_1 + A_{D_{e_2}\phi e_i}e_1 = 0, \quad i = 1, 2$, and hence we obtain

$$c_1 + 3b\omega_1^2(e_1) = b_2 + (a-2c)\omega_1^2(e_2), \tag{17}$$
$$-b_1 + 3c\omega_1^2(e_1) = c_2 + 3b\omega_1^2(e_2), \tag{18}$$
$$a_2 - 3b\omega_1^2(e_2) = b_1 + (a-2c)\omega_1^2(e_1). \tag{19}$$

where $\omega_i^j(e_k) = \langle\nabla_{e_k}e_i, e_j\rangle$ and $f_i = e_i f$ for a function f. Combining (18) and (19) yields

$$a_2 + c_2 = (a+c)\omega_1^2(e_1). \tag{20}$$

The allied mean curvature vector field $a(H)$ is defined by

$$\sum_{r=4}^{5}(\mathrm{trace}A_H A_{e_r})e_r.$$

By the Gauss and Weingarten formulas we have

$$\Delta H = \mathrm{tr}(\bar{\nabla} A_H) + \Delta^D H + (\mathrm{tr}A^2_{\phi e_1})H + a(H), \tag{21}$$

where $\mathrm{tr}(\bar{\nabla}A_H) = \sum_{i=1}^{2}(A_{D_{e_i}H}e_i + (\nabla_{e_i}A_H)e_i)$.

Assume that M^2 satisfies $\Delta H = \lambda H$. Since $\langle \Delta H - \lambda H, \xi\rangle = 0$, we get

$$a_1 + c_1 + (a+c)\omega_1^2(e_2) = 0. \tag{22}$$

3 Remark. If $a(H)$ vanishes identically on M^2, it is called a *Chen* (or $\mathcal{A}$) surface (see, [6], [7]). In this case, b in (16) vanishes. Using (20), (21) and (22), we find that in the class of Legendre surfaces with $\Delta H = \lambda H$, the condition "$||H||$ is constant" and "$\mathrm{tr}A^2_{\phi H} = (\lambda - 1)||H||^2$ and M^2 is a Chen surface" are equivalent.

By combining (17) and (22), we obtain

$$a_1 + b_2 = 3b\omega_1^2(e_1) - (2a - c)\omega_1^2(e_2). \tag{23}$$

Also since $\mathrm{tr}(\bar{\nabla}A_H) = 0$, we have

$$2a(a_1 + c_1) + 2b(a_2 + c_2) + (a+c)\{b_2 + a_1 + (a-b)\omega_1^2(e_1)\} = 0, \tag{24}$$

$$b(a_1 + c_1) + 2c(a_2 + c_2) + (a+c)\{c_2 + b_1 + (a+b)\omega_1^2(e_2)\} = 0. \tag{25}$$

By substituting (18), (20), (22) and (23) into (24) and (25), we get

$$4b\omega_1^2(e_1) - (3a - c)\omega_1^2(e_2) = 0, \tag{26}$$

$$(a + 5c)\omega_1^2(e_1) - 4b\omega_1^2(e_2) = 0. \tag{27}$$

Denote the unit vector field perpendicular to ϕH in TM^2 by $(\phi H)^{\perp}$. First we shall consider the case of $(\phi H)^{\perp}||H||^2 = 0$. Then $\omega_1^2(e_1) = 0$ by (20) and hence from (27) we obtain $b\omega_1^2(e_2) = 0$. We put $U = \{p \in M | \omega_1^2(e_2) \neq 0\}$. On U we have $b = 0$ and $3a = c$. It follows from (17) and (22) that $a = 0$ on U. Hence U is totally geodesic. It is a contradiction. Thus U is empty and $\omega_1^2 = 0$. Therefore H is C-parallel as in [1]. Moreover since $\langle \Delta H - \lambda H, \phi e_1\rangle = \langle \Delta H - \lambda H, \phi e_2\rangle = 0$, we have $\mathrm{tr}A^2_{\phi E_1} = \lambda - 1$ and $b = 0$.

Next we shall consider the case of $\phi H||H||^2 = 0$. Then $\omega_1^2(e_2) = 0$ by (22) and hence from (26) we obtain $b\omega_1^2(e_1) = 0$. Similarly we see that H is C-parallel and $\mathrm{tr}A^2_{\phi e_1} = \lambda - 1$ and $b = 0$.

Conversely if H is C-parallel, then $||H||^2$ is constant. Consequently we can state the following.

4 Proposition. *Let M^2 be a non-minimal Legendre surface of $N^5(\epsilon)$ satisfying $\Delta H = \lambda H$ for some constant λ. Then H satisfies $\phi H||H||^2 = 0$ or $(\phi H)^{\perp}||H||^2 = 0$ if and only if M^2 is a Chen surface and H is C-parallel, and moreover* $\mathrm{tr}A^2_{\phi H} = (\lambda - 1)||H||^2$.

5 Remark. By remark 3 and proposition 4, we obtain that in the class of Legendre surfaces with $\Delta H = \lambda H$, the condition "$\phi H||H||^2 = 0$ or $(\phi H)^{\perp}||H||^2 = 0$" and "$||H||$ is constant" are equivalent.

In the rest of this section we study Legendre surfaces satisfying $\Delta^D H = \lambda H$.

6 Proposition. *Let M^2 be a non-minimal Legendre surface satisfying $\Delta^D H = \lambda H$ in $N^5(\epsilon)$. Then $||H||^2$ is a constant if and only if H is C-parallel and $\lambda = 1$*

Proof. Since $\langle \Delta^D H - \lambda H, \phi e_1 \rangle = \langle \Delta^D H - \lambda H, \phi e_2 \rangle = 0$, we have

$$\Delta\alpha + \alpha\{1 - \lambda + (\omega_1^2(e_1))^2 + (\omega_1^2(e_2))^2\} = 0, \tag{28}$$

$$2\alpha_1\omega_1^2(e_1) + 2\alpha_2\omega_1^2(e_2) + \alpha\{(\omega_1^2(e_1))_1 + (\omega_1^2(e_2))_2\} = 0, \tag{29}$$

where $\alpha = a + c$. The condition $\langle \Delta^D H - \lambda H, \xi \rangle = 0$ is equivalent to (22). If $||H||^2$ is constant, we have $\omega_1^2 = 0$ from (20) and (22). Hence H is C-parallel by the same way as [1]. Moreover by (28) we obtain $\lambda = 1$. The converse is trivial. $\boxed{QED}$

In view of this proposition, it is interesting to investigate whether there exists a Legendre surface satisfying $\Delta^D H = \lambda H$ with non-constant squared mean curvature.

By using (17), (18), (19) and (22), we can prove the following lemma.

7 Lemma. *If $a = 2c$, then $||H||^2$ is constant.*

Suppose that M^2 is a Legendre Chen surface in $N^5(\epsilon)$ satisfying $\Delta^D H = \lambda H$ with non-constant squared mean curvature. Then $a \neq 2c$ from lemma 7.

First we shall consider the case of $\alpha_2 = 0$. Then $\omega_1^2(e_1) = 0$ by (20). Hence there exists a local coordinate system $\{x, \tilde{y}\}$ such that the metric tensor takes the form

$$g = dx^2 + G^2 d\tilde{y}^2 \tag{30}$$

and $e_1 = \frac{\partial}{\partial x}$, $e_2 = G^{-1}\frac{\partial}{\partial \tilde{y}}$. Since $a_2 = c_2 = 0$ by (18) and (19), we have $a = a(x)$ and $c = c(x)$. We obtain $G = f(\tilde{y})\exp\int^x (\frac{c'}{(a-2c)})dx$ for some function $f(\tilde{y})$ because $\omega_1^2(e_2) = \frac{G_x}{G}$ holds, where $c' = c_1$. By using the coordinate change:

$$y = \int^{\tilde{y}} f(\tilde{y})d\tilde{y},$$

G takes the form $G = \exp \int^x (\frac{c'}{(a-2c)})dx$. The relation (28) implies

$$-\alpha\alpha^{''} + 2(\alpha^{'})^2 = (\lambda - 1)\alpha^2. \tag{31}$$

By (22), (31) and the equation of Gauss, we obtain

$$\begin{aligned} ac - c^2 + \frac{\epsilon+3}{4} &= -(\omega_1^2(e_2))^{'} - (\omega_1^2(e_2))^2 \\ &= 1 - \lambda. \end{aligned} \tag{32}$$

We put $W = \{p \in M^2 | c \neq 0\}$. On $M - W$ we see that α is constant by using (17) and (22). It is a contradiction. Hence $M^2 = W$. From (32) we have

$$\alpha = \frac{1 - 4\lambda - \epsilon + 8c^2}{4c}. \tag{33}$$

Combining (17) and (22) yields

$$\frac{c^{'}}{\alpha - 3c} = -\frac{\alpha^{'}}{\alpha}. \tag{34}$$

By substituting (33) into (34) we obtain

$$(1 - 4\lambda - \epsilon)(1 - 4\lambda - \epsilon - 16c^2) = 0. \tag{35}$$

If $\lambda \neq \frac{1-\epsilon}{4}$, then c is constant and hence α is constant. It is a contradiction. Hence $\lambda = \frac{1-\epsilon}{4}$. Thus $\alpha = 2a = 2c$ from (32). Then by solving (31), we find that $\alpha(x)$ is one of the following function:

$$\frac{1}{s\cos\frac{\sqrt{\epsilon+3}}{2}x + t\sin\frac{\sqrt{\epsilon+3}}{2}x} \quad (\epsilon > -3), \tag{36}$$

$$\frac{1}{sx + t} \quad (\epsilon = -3), \tag{37}$$

$$\frac{1}{s\exp(\frac{\sqrt{-\epsilon-3}}{2}x) + t\exp(-\frac{\sqrt{-\epsilon-3}}{2}x)} \quad (\epsilon < -3), \tag{38}$$

where s, t are integration constants.

Conversely, suppose that s, t, $\epsilon = 1 - 4\lambda$ are constants and $\alpha(x)$ is a function satisfying one of (36)-(38) defined on an open interval I. Let g be the metric tensor on a simply-connected domain $V \subset I \times \mathbf{R}$ defined by (30), where $G = \frac{1}{\alpha(x)}$. We define a symmetric bilinear form σ on (V, g) by $\sigma(\partial_x, \partial_x) = \frac{\alpha}{2}\partial_x$, $\sigma(\partial_y, \partial_y) = \frac{1}{2\alpha}\partial_x$, $\sigma(\partial_x, \partial_y) = \frac{\alpha}{2}\partial_y$. Here $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$. By a straightforward computation, we can see that $((V, g), \sigma)$ satisfies $(1) - (3)$ of theorem 1.

By applying theorem 1 and 2, we conclude that up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendre Chen immersion of (V, g) into $N^5(\epsilon)$. Moreover we can show that such an immersion satisfies $\Delta^D H = \lambda H$ and $(\phi H)^{\perp}||H||^2 = 0$.

Next we shall consider the case of $\alpha_1 = 0$. Then $\omega_1^2(e_2) = 0$ by (22). Hence there exists a local coordinate system $\{\tilde{x}, y\}$ such that the metric tensor takes the form

$$g = E^2 d\tilde{x}^2 + dy^2 \tag{39}$$

and $e_1 = E^{-1}\frac{\partial}{\partial \tilde{x}}$, $e_2 = \frac{\partial}{\partial y}$. Since $a_1 = c_1 = 0$ by (17) and (22), we have $a = a(y)$ and $c = c(y)$. We obtain $E = g(\tilde{x})\exp\int^y(\frac{-a'}{(a-2c)})dy$ for some function $g(\tilde{x})$ because $\omega_1^2(e_1) = -\frac{E_y}{E}$ holds. By using the coordinate change:

$$x = \int^{\tilde{x}} g(\tilde{x})d\tilde{x},$$

E takes the form $E = \exp\int^y(\frac{-a'}{(a-2c)})dy$.

By the same way as the case $\alpha_2 = 0$, we obtain (31) and (32). On W we have

$$1 - \lambda - \frac{\epsilon + 3}{4} + 2c^2 = 0. \tag{40}$$

It follows that $\alpha = 0$. It is a contradiction. Thus W is empty and hence $c = 0$ and $\epsilon = 1 - 4\lambda$ on M^2.

Conversely, suppose that s, t, $\epsilon = 1-4\lambda$ are constants and $\alpha(y)$ is a function satisfying one of (36)-(38) (replace x with y) defined on an open interval I. Let g be the metric tensor on a simply-connected domain $V \subset I \times \mathbf{R}$ defined by (39), where $E = \frac{1}{a(y)}$. We define a symmetric bilinear form σ on (V, g) by $\sigma(\partial_x, \partial_x) = \partial_x$, $\sigma(\partial_y, \partial_y) = \sigma(\partial_x, \partial_y) = 0$. By a straight-forward computation, we can see that $((V, g), \sigma)$ satisfies $(1) - (3)$ of theorem 1. By applying theorem 1 and 2, we conclude that up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendre Chen immersion of (V, g) into $N^5(\epsilon)$. Moreover we can show that such an immersion satisfies $\Delta^D H = \lambda H$ and $\phi H||H||^2 = 0$.

Consequently we can state the following.

8 Theorem. *Let M^2 be a Legendre Chen surface of $N^5(\epsilon)$ satisfying $\Delta^D H = \lambda H$ with non-constant squared mean curvature. If $(\phi H)^{\perp}||H||^2 = 0$ (resp. $\phi H||H||^2 = 0$), then $1 - 4\lambda - \epsilon = 0$ and there exists a coordinate system $\{x, y\}$ defined in a neighborhood $V \subset I \times \mathbf{R}$ of $p \in M^2$ and a function $\alpha : I \to \mathbf{R} : x \to \alpha(x)$ satisfying one of $(36) - (38)$ for some constants s, t. Moreover the metric tensor of M^2 is given by*

$$g = dx^2 + \frac{1}{\alpha(x)^2}dy^2 \tag{41}$$

and the second fundamental form is given by

$$h(\partial_x,\partial_x)=\frac{\alpha}{2}\phi\partial_x,\quad h(\partial_y,\partial_y)=\frac{1}{2\alpha}\phi\partial_x,\quad h(\partial_x,\partial_y)=\frac{\alpha}{2}\phi\partial_y,\tag{42}$$

$$(resp.\quad h(\partial_x,\partial_x)=h(\partial_x,\partial_y)=0,\quad h(\partial_y,\partial_y)=\phi\partial_y)\,,\tag{43}$$

Conversely, suppose that s, t, $\epsilon=1-4\lambda$ are constants and $\alpha(x)$ is a function defined on an open interval I satisfying one of (36)$-$(38)*. Let g be the metric tensor on a simply-connected domain $V\subset I\times\mathbf{R}$ defined by* (41)*. Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendre Chen immersion of (V,g) into $N^5(\epsilon)$ whose second fundamental form is given by* (42) *(resp.* (43))*. Moreover such a surface satisfies $\Delta^D H=\lambda H$ and $(\phi H)^{\perp}||H||^2=0$ (resp. $\phi H||H||^2=0$).*

By the same method as [8] and [9] and using the suitable coordinate change in theorem 8, if $\epsilon=1$, i.e., the ambient space is the unit 5-sphere $S^5(1)$, we obtain the explicit representation of the position vector of surfaces in theorem 8.

9 Corollary. *Let $\psi: M^2\to S^5(1)\subset\mathbf{C}^3$ be a Legendre Chen surface which satisfies $\Delta^D H=\lambda H$ for a constant λ. Suppose that the mean curvature is nonconstant. Then H satisfies $(\phi H)^{\perp}||H||^2=0$ if and only if up to rigid motions of $S^5(1)$, the immersion ψ is locally given by*

$$\psi(x,y)=\frac{1}{\sqrt{2}}(i+\mathrm{sin}x,(\mathrm{sec}x+\mathrm{tan}x)^i\mathrm{cos}x\mathrm{cos}y,(\mathrm{sec}x+\mathrm{tan}x)^i\mathrm{cos}x\mathrm{sin}y).\tag{44}$$

Also, H satisfies $\phi H||H||^2=0$ if and only if up to rigid motions of $S^5(1)$, the immersion ψ is locally given by

$$\psi(x,y)=\left(\frac{1}{\sqrt{2}}\mathrm{exp}(\frac{1+\sqrt{5}}{2}iy)\mathrm{cos}x,\frac{1}{\sqrt{2}}\mathrm{exp}(\frac{1-\sqrt{5}}{2}iy)\mathrm{cos}x,\mathrm{sin}x\right).\tag{45}$$

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