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On the global C^∞ and Gevrey hypoellipticity on the torus of some classes of degenerate elliptic operators

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Abstract. In this paper we prove the global C^∞ and Gevrey hypoellipticity on the multidimensional torus for some classes of degenerate elliptic operators.

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Dedicated to the memory of V.B. Moscatelli

1 Introduction

In the last years many papers are concerned with the study of the global hypoellipticity and solvability of linear partial differential operators on compact manifolds, e.g., on the torus, in large scales of functional spaces (see e.g. [1–6,8–11,13–16,21–26,29] and the references listed therein). It is well-known that the theory of global properties of differential operators is not well-developped in comparison with the one of local properties. On the other hand, the local and global hypoellipticity/solvability are rather different in general.

In this paper we are interested in the problem of global C^∞ and Gevrey hypoellipticity for the following classes of linear partial differential operators on the multidimensional torus $\mathbb{T}^N = \mathbb{T}^{m+n}$:

$$P_1 = -\sum_{j=1}^{l}\left(\sum_{h=1}^{m} a_{jh}(y)\partial_{x_h} + \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}\right)^2, \tag{1}$$

$$P_2 = -\Delta_y - \sum_{j=1}^{m}\left(a_j(y)\partial_{x_j} + \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}\right)^2, \tag{2}$$

where the coefficients $a_{jh}(y)$, $a_j(y)$ and $b_{jk}(y)$ are real–valued functions defined on $\mathbb{T}^n$. Precisely, in Theorems 1 and 2 (in Theorems 4 and 5) we give sufficient conditions for the global C^∞ hypoellipticity (for the global Gevrey hypoellipticity) for the operator P_1 defined in (1).

We point out that in Theorem 2 we partially answer to the following conjecture of Petronilho [25]:

Let $X_1, \ldots, X_m$ be a family of real vector fields on $\mathbb{T}^N$. If there exist coordinates y on $\mathbb{T}^N$ in which the vector field X_1 admits the form $X_1 = \sum_{k=1}^N \lambda_k \partial_{y_k}$ with the numbers $\lambda_1, \ldots, \lambda_N$ satisfying the following condition: there exist $C > 0$, $K > 0$ such that

$$|\sum_{k=1}^N \lambda_k \eta_k| \geq \frac{C}{|\eta|^K}, \quad \eta \in \mathbb{Z}^N \setminus \{0\},$$

then the operator $P = -\sum_{j=1}^m X_j^2$ is globally hypoelliptic on $\mathbb{T}^N$.

Moreover, in case $l = 2$ and $m = n = 1$ in (1) we propose the Example 1 whose coefficients violate the conditions in Theorem 1 and despite of this fact the operator P_1 is still C^∞ globally hypoelliptic on the torus. Therefore, finding complete results for the global hypoellipticity for the operator P_1 remains an open difficult problem.

In Theorem 3 (Theorem 6) we give a necessary and sufficient condition for the global C^∞ hypoellipticity (for the global Gevrey hypoellipticity) for the operator P_2 defined in (2).

2 Statement of the results

Let $\mathbb{T}^N = \mathbb{R}^N / 2\pi\mathbb{Z}^N$ be the N–dimensional torus. A linear partial differential operator P defined on $\mathbb{T}^N$ with coefficients in $C^\infty(\mathbb{T}^N)$ is said to be *globally hypoelliptic* in $\mathbb{T}^N$ if the conditions $u \in \mathcal{E}'(\mathbb{T}^N)$ and $Pu \in C^\infty(\mathbb{T}^N)$ imply that $u \in C^\infty(\mathbb{T}^N)$ ($\mathcal{E}'(\mathbb{T}^N)$ denotes the topological dual of $C^\infty(\mathbb{T}^N)$). If P is defined on an open set Ω of $\mathbb{R}^N$, then P is said to be *locally hypoelliptic* in Ω if for any $U \subseteq \Omega$ open set the conditions $u \in \mathcal{D}'(U)$ and $Pu \in C^\infty(U)$ imply that $u \in C^\infty(U)$. We observe that local hypoellipticity implies global hypoellipticity. But, the converse is not true in general. For example, the operator $P = \partial_x + a\partial_y$, with $a \in \mathbb{R} \setminus \mathbb{Q}$ a non–Liouville number, is globally hypoelliptic in $\mathbb{T}^2$ but, it is not locally hypoelliptic in $\mathbb{R}^2$, see [12].

We also recall that a vector $a = (a_1, \ldots, a_N) \in \mathbb{R}^N \setminus \mathbb{Q}^N$ is said to be *non–Liouville* if there exist two positive constants C and L such that

$$|a \cdot \xi - \eta| \geq \frac{C}{|\xi|^L}, \quad \xi \in \mathbb{Z}^N \setminus \{0\}, \eta \in \mathbb{Z}.$$

If $N = 1$, then this is the definition of a *non–Liouville number.*

We can now state the main results of this paper.

Theorem 1. *Let P be the operator on $\mathbb{T}^{m+n}$ given by*

$$P = -\sum_{j=1}^l X_j^2, \tag{3}$$

where

$$X_j = \sum_{h=1}^m a_{jh}(y)\partial_{x_h} + \sum_{k=1}^n b_{jk}(y)\partial_{y_k}, \quad j = 1, \ldots, l,$$

with variables $(x, y) \in \mathbb{T}^m \times \mathbb{T}^n$, and suppose that the coefficients $\{a_{jh}\}_{h=1}^m$ and $\{b_{jk}\}_{k=1}^n$ are real valued functions in $C^\infty(\mathbb{T}^n)$ and that $\mathrm{div}_B X_j := \sum_{k=1}^n \partial_{y_k} b_{jk} \equiv 0$ on $\mathbb{T}^n$.

If the following conditions hold:

(i) the vector fields $\sum_{k=1}^n b_{jk}(y)\partial_{y_k}$, $j = 1, \ldots, l$, span $T_y(\mathbb{T}^n)$ for every $y \in \mathbb{T}^n$,

(ii) there exists $j_0 \in \{1, \ldots, l\}$ such that $a_{j_0,h} \equiv a_h$ for every $h = 1, \ldots, m$ and $b_{j_0 k} \equiv b \neq 0$ for every $k = 1, \ldots, n$, and the vector $(\frac{a_1}{b}, \frac{a_2}{b}, \ldots, \frac{a_m}{b})$ is non–Liouville,

then the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$.

In case there exists $j_0 \in \{1, \dots, l\}$ such that $a_{j_0,h} \equiv a_h$ for every $h = 1, \dots, m$ and $b_{j_0k} \equiv 0$ for every $k = 1, \dots, n$, the operator P given in (3) is globally hypoelliptic on $\mathbb{T}^{m+n}$ provided the condition (i) together a suitable Diophantine condition hold. Indeed, we have

Theorem 2. *Let P be the operator on $\mathbb{T}^{m+n}$ defined according to (3). If the following conditions hold:*

(i) the vector fields $\sum_{k=1}^n b_{jk}(y)\partial_{y_k}$, $j = 1, \dots, l$, span $T_y(\mathbb{T}^n)$ for every $y \in \mathbb{T}^n$,

(ii) there exists $j_0 \in \{1, \dots, l\}$ such that $a_{j_0,h} \equiv a_h$ for every $h = 1, \dots, m$ and $b_{j_0k} \equiv 0$ for every $k = 1, \dots, n$, and the numbers $a_1, a_2, \dots, a_m$ satisfy the following Diophantine condition: there exist $C > 0$, $L > 0$ such that

$$|\sum_{h=1}^m a_h \xi_h| \geq \frac{C}{|\xi|^L}, \quad \xi \in \mathbb{Z}^m \setminus \{0\}, \tag{4}$$

then the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$.

Finally, we prove

Theorem 3. *Let P be the operator on $\mathbb{T}^{m+n}$ given by*

$$P = -\Delta_y - \sum_{j=1}^m X_j^2, \tag{5}$$

where

$$X_j = a_j(y)\partial_{x_j} + \sum_{k=1}^n b_{jk}(y)\partial_{y_k}, \quad j = 1, \dots, m,$$

with variables $(x, y) \in \mathbb{T}^m \times \mathbb{T}^n$, and suppose that the coefficients a_j and $\{b_{jk}\}_{k=1}^n$ are real valued functions in $C^\infty(\mathbb{T}^n)$ and that $\mathrm{div}_B X_j := \sum_{k=1}^n \partial_{y_k} b_{jk} \equiv 0$ on $\mathbb{T}^n$.

Then the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$ if and only if $a_j \not\equiv 0$ for all $j = 1, \dots, m$.

3 Proof of Theorem 1

Proof. Without loss of generality, we may suppose that $j_0 = 1$, i.e., that $X_1 = \sum_{h=1}^m a_h \partial_{x_h} + b \sum_{k=1}^m \partial_{y_k}$ with $(\frac{a_1}{b}, \frac{a_2}{b}, \dots, \frac{a_m}{b})$ a non–Liouville vector.

Let $u \in \mathcal{E}'(\mathbb{T}^{m+m})$ which satisfies $Pu = h \in C^\infty(\mathbb{T}^{m+n})$. Taking partial transform with respect to $x \in \mathbb{T}^m$ we obtain

$$\left[-\sum_{j=1}^l \left(i \sum_{h=1}^m a_{jh}(y)\xi_h + \sum_{k=1}^n b_{jk}(y)\partial_{y_k} \right)^2 \right] \hat{u}(\xi, y) = \hat{h}(\xi, y), \quad \xi \in \mathbb{Z}^m,\, y \in \mathbb{T}^n. \tag{6}$$

Since the operator in (6) is elliptic with respect to y by assumption (i) and $\hat{h}(\xi, \cdot) \in C^\infty(\mathbb{T}^n)$ for every $\xi \in \mathbb{Z}^m$, we necessarily have that $\hat{u}(\xi, \cdot) \in C^\infty(\mathbb{T}^n)$ for every $\xi \in \mathbb{Z}^m$. We can then multiply (6) by $\bar{\hat{u}}(\xi, y)$ and integrate by parts with respect to $y \in \mathbb{T}^n$. So, by using the fact that $\mathrm{div}_B X_j \equiv 0$ for all $j = 1, \dots, l$, we obtain

$$\sum_{j=1}^l \|Y_j \hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} = \int_{T^n} \hat{h}(\xi, y)\bar{\hat{u}}(\xi, y)dy, \quad \xi \in \mathbb{Z}^m, \tag{7}$$

where $Y_j := i\sum_{h=1}^m a_{jh}(y)\xi_h + \sum_{k=1}^n b_{jk}(y)\partial_{y_k}$ for every $j = 1, \dots, l$.

In order to complete the proof we need the following lemma.

Lemma 1. *There exist two positive constants C and L such that*

$$\|\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)} \le C|\xi|^{2L}\|Y_1\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)}, \quad \xi \in \mathbb{Z}^m. \tag{8}$$

PROOF. For $\xi \in \mathbb{Z}^m$ fixed let $\varphi_\xi \in C^\infty(\mathbb{T}^n)$ and set

$$Y_1\varphi_\xi(y) = \left(i\sum_{h=1}^m a_h\xi_h + b\sum_{k=1}^n \partial_{y_k}\right)\varphi_\xi(y) =: \psi_\xi(y), \quad y \in \mathbb{T}^n. \tag{9}$$

Taking partial Fourier transform with respect to y, we obtain

$$i\left(\sum_{h=1}^m a_h\xi_h + b\sum_{k=1}^n \eta_k\right)\hat{\varphi}_\xi(\eta) = \hat{\psi}_\xi(\eta), \quad \eta \in \mathbb{Z}^n,$$

and hence,

$$|b|\left|\sum_{h=1}^m \frac{a_h}{b}\xi_h + \sum_{k=1}^n \eta_k\right||\hat{\varphi}_\xi(\eta)| = |\hat{\psi}_\xi(\eta)|, \quad \eta \in \mathbb{Z}^n. \tag{10}$$

Since $(\frac{a_1}{b}, \frac{a_2}{b}, \dots, \frac{a_m}{b})$ is non Liouville vector, there exist two positive constants C and L so that

$$\left|\sum_{h=1}^m \frac{a_h}{b}\xi_h + \eta\right| \ge \frac{C}{|\xi|^L}, \quad \xi \in \mathbb{Z}^m \setminus \{0\}, \eta \in \mathbb{Z}. \tag{11}$$

It follows from (10) and (11) that

$$|\hat{\varphi}_\xi(\eta)|^2 \le C'|\xi|^{2L}|\hat{\psi}_\xi(\eta)|^2, \quad \eta \in \mathbb{Z}^n. \tag{12}$$

The last inequality together the Parseval identity imply that

$$\begin{aligned}\int_{\mathbb{T}^n}|\varphi_\xi(y)|^2dy &= \sum_{\eta\in\mathbb{Z}^n}|\hat{\varphi}_\xi(\eta)|^2 \le C'|\xi|^{2L}\sum_{\eta\in\mathbb{Z}^n}|\hat{\psi}_\xi(\eta)|^2\\ &= C'|\xi|^{2L}\int_{\mathbb{T}^n}|\psi_\xi(y)|^2dy,\end{aligned}$$

i.e., that

$$\|\varphi_\xi\|^2_{L^2(\mathbb{T}^n)} \le C'|\xi|^{2L}\int_{\mathbb{T}^n}\left|\left(i\sum_{h=1}^m a_h\xi_h + b\sum_{k=1}^n \partial_{y_k}\right)\varphi_\xi(y)\right|^2 dy. \tag{13}$$

If we apply (13) with $\varphi_\xi(\cdot) = \hat{u}(\xi,\cdot)$ we obtain that

$$\begin{aligned}\|\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)} &\le C'|\xi|^{2L}\int_{\mathbb{T}^n}\left|\left(i\sum_{h=1}^m a_h\xi_h + b\sum_{k=1}^n \partial_{y_k}\right)\hat{u}(\xi,y)\right|^2 dy\\ &= C'|\xi|^{2L}\|Y_1\hat{u}(\xi,\cdot)\|^2_{L^2}.\end{aligned}$$

So, the proof is complete. *QED*

By applying Lemma 1 and identity (7) we deduce

$$\begin{aligned}\|\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)} &\le C'|\xi|^{2L}\|Y_1\hat{u}(\xi,\cdot)\|^2_{L^2} \le C'|\xi|^{2L}\sum_{j=1}^l\|Y_j\hat{u}(\xi,\cdot)\|^2_{L^2}\\ &= C'|\xi|^{2L}\int_{\mathbb{T}^n}\hat{h}(\xi,y)\hat{u}(\xi,y)dy, \quad \xi \in \mathbb{Z}^m.\end{aligned} \tag{14}$$

Thus, the Cauchy–Schwarz inequality togheter with (14) imply that

$$\|\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)} \leq C'|\xi|^{2L}\|\hat{h}(\xi,\cdot)\|_{L^2(\mathbb{T}^n)}\|\hat{u}(\xi,\cdot)\|_{L^2(\mathbb{T}^n)}, \quad \xi\in\mathbb{Z}^m,$$

and hence,

$$\|\hat{u}(\xi,\cdot)\|_{L^2(\mathbb{T}^n)} \leq C'|\xi|^{2L}\|\hat{h}(\xi,\cdot)\|_{L^2(\mathbb{T}^n)}, \quad \xi\in\mathbb{Z}^m. \tag{15}$$

Since $h\in C^\infty(\mathbb{T}^{m+n})$, for every $N\in\mathbb{N}$ there exists a positive constant C_N so that

$$\|\hat{h}(\xi,\cdot)\|_{L^2(\mathbb{T}^n)} \leq C_N|\xi|^{-(N+2L)}, \quad \xi\in\mathbb{Z}^m\setminus\{0\}. \tag{16}$$

Combining (15) and (16) we then obtain, that, for every $N\in\mathbb{N}$ there exists $C'_N>0$ such that

$$\|\hat{u}(\xi,\cdot)\|_{L^2(\mathbb{T}^n)} \leq C'_N|\xi|^{-N}, \quad \xi\in\mathbb{Z}^m\setminus\{0\}.$$

Since

$$\hat{u}(\xi,\eta)=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\hat{u}(\xi,y)e^{-iy\cdot\eta}dy, \quad \eta\in\mathbb{Z}^n,$$

the last inequality and the Cauchy–Schwarz inequality imply that

$$|\hat{u}(\xi,\eta)|\leq C'_N|\xi|^{-N}, \quad \xi\in\mathbb{Z}^m\setminus\{0\}. \tag{17}$$

Since the operator P is elliptic at $(x,y,0,\eta)$ for every $(x,y)\in\mathbb{T}^{m+n}$ and $\eta\in\mathbb{Z}^n\setminus\{0\}$ by assumption (i) (see [20, Theorem 8.3.1]), by using standard microlocal elliptic theory we obtain that for every $N\in\mathbb{N}$ there exists $C''_N>0$ such that

$$|\hat{u}(\xi,\eta)|\leq C''_N(|\xi|+|\eta|)^{-N}, \quad (\xi,\eta)\in\mathbb{Z}^m\times\mathbb{Z}^n\setminus\{0\}$$

which shows that $u\in C^\infty(\mathbb{T}^{m+n})$. So, the proof of Theorem 1 is complete. QED

Remark 1. The fact that the coefficients $\{a_{j_0h}\}_{h=1}^m$ and $\{b_{j_0k}\}_{k=1}^n$ satisfy the condition (ii) in Theorem 1 for some $j_0\in\{1,\dots,l\}$ is not necessary for the global hypoellipticity on $\mathbb{T}^{m+m}$ of the operator P given by (3) and satisfying condition (i) in Theorem 1 as the next example shows.

Example 1. Consider the operator P of the following type

$$P=-(a_1\partial_x+\partial_y)^2-(a_2(y)\partial_x+\partial_y)^2 \tag{18}$$

on $\mathbb{T}^2$ with variables (x,y), where $a_1\in\mathbb{Q}\setminus\{0\}$ and $a_2(y)\not\equiv 0$ on $\mathbb{T}$. If $a_2:=\frac{1}{2\pi}\int_{\mathbb{T}}a_2(y)dy$ is not a Liouville number, then the operator (18) is globally hypoelliptic on $\mathbb{T}^2$. In order to show this, we proceed as follows. Let $u\in\mathcal{E}'(\mathbb{T}^2)$ satisfy $Pu=h\in C^\infty(\mathbb{T}^2)$. Taking partial transform with respect to $x\in\mathbb{T}$ we obtain

$$-(ia_1\xi+\partial_y)^2\hat{u}(\xi,y)-(ia_2(y)\xi+\partial_y)^2\hat{u}(\xi,y)=\hat{h}(\xi,y), \quad \xi\in\mathbb{Z},\ y\in\mathbb{T}. \tag{19}$$

Since the operator in (19) is elliptic with respect to y and $\hat{h}(\xi,\cdot)\in C^\infty(\mathbb{T})$ for all fixed $\xi\in\mathbb{Z}$, it follows that $\hat{u}(\xi,\cdot)\in C^\infty(\mathbb{T})$ for all fixed $\xi\in\mathbb{Z}$. We can then multiply (19) by $\overline{\hat{u}}(\xi,y)$ and integrate by parts with respect to $y\in\mathbb{T}$. So, setting $Y_1:=ia_1\xi+\partial_y$ and $Y_2:=ia_2(y)\xi+\partial_y$, we obtain

$$\|Y_1\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T})}+\|Y_2\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T})}=\int_{\mathbb{T}}\hat{h}(\xi,y)\hat{u}(\xi,y)dy, \quad \xi\in\mathbb{Z}. \tag{20}$$

For $\xi\in\mathbb{Z}$ fixed, let

$$Y_2\hat{u}(\xi,y)=(ia_2(y)\xi+\partial_y)\hat{u}(\xi,y)=:\psi_\xi(y), \quad y\in\mathbb{T}. \tag{21}$$

If we set $A_2(y) := \int_0^y a_2(z)dz - a_2 y$ and $v(\xi, y) := e^{iA(y)\xi}\hat{u}(\xi, y)$ for $y \in \mathbb{T}$, by (21) we obtain that the function $v(\xi, \cdot)$ satisfies the equation

$$\partial_y v(\xi, y) + ia_2\xi v(\xi, y) = e^{i\xi A(y)}\psi_\xi(y) =: g_\xi(y), \quad y \in \mathbb{T}. \tag{22}$$

For $\xi \neq 0$, from (22) it follows

$$v(\xi, y) = \frac{1}{e^{2\pi a_2\xi} - 1}\int_{\mathbb{T}} e^{ia_2\xi z} g_\xi(z + y)dz, \quad y \in \mathbb{T}, \tag{23}$$

where $e^{2\pi a_2\xi} - 1 \neq 0$ as a_2 is a not Liouville number and hence, $a_2 \notin \mathbb{Q}$.

For $\xi = 0$, from (22) it follows

$$v(0, y) = \int_0^y g_0(z)dz + c, \quad y \in \mathbb{T}. \tag{24}$$

Therefore, for $\xi \neq 0$, by (23) we obtain that

$$\hat{v}(\xi, \eta) = \frac{1}{2\pi}\frac{1}{e^{2\pi ia_2\xi} - 1}\frac{e^{2\pi i(a_2\xi+\eta)} - 1}{i(a_2\xi + \eta)}\hat{g}_\xi(\eta), \quad \eta \in \mathbb{Z}, \tag{25}$$

(see, e.g., [7, §20, (7), p.14]).

Since a_2 is a not Liouville number there exist two positive constants C and L such that

$$|a_2\xi + \eta| \geq \frac{C}{|\xi|^L}, \quad |e^{2\pi ia_2\xi} - 1| \geq \frac{C}{|\xi|^L}, \quad \xi \in \mathbb{Z} \setminus \{0\}, \eta \in \mathbb{Z},$$

(for a proof of the second inequality see, e.g., [18, Lemma 3.3]), by (25) we deduce that

$$|\hat{v}(\xi, \eta)| \leq C'|\xi|^{2L}|\hat{g}_\xi(\eta)|, \quad \xi \in \mathbb{Z} \setminus \{0\}, \eta \in \mathbb{Z}, \tag{26}$$

and hence, by Parseval identity, that

$$\|v(\xi, \cdot)\|^2_{L^2(\mathbb{T})} \leq C''|\xi|^{4L}\, \mathrm{l}g_\xi(\cdot)\|^2_{L^2(\mathbb{T})}, \quad \xi \in \mathbb{Z} \setminus \{0\}. \tag{27}$$

If we apply (27) with $\hat{u}(\xi, \cdot) = e^{-iA(\cdot)\xi}v(\xi, \cdot)$ we obtain, for every $\xi \in \mathbb{Z} \setminus \{0\}$, that

$$\begin{aligned}
\|\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} |e^{-iA(y)\xi}|^2|v(\xi, y)|^2 dy = \|v(\xi, \cdot)\|^2_{L^2(\mathbb{T})} \\
&\leq C''|\xi|^{4L}\, \mathrm{l}g_\xi(\cdot)\|^2_{L^2(\mathbb{T})} = C''|\xi|^{4L}\int_{\mathbb{T}} |e^{iA(y)\xi}|^2|\psi_\xi(y)|^2 dy \\
&= C''|\xi|^{4L}\int_{\mathbb{T}} |Y_2\hat{u}\xi, y)|^2 dy = C''|\xi|^{4L}\|Y_2\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

The last inequality together with (20) imply that

$$\|\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T})} \leq C''|\xi|^{4L}\int_{\mathbb{T}} \hat{h}(\xi, y)\overline{\hat{u}}(\xi, y)dy, \quad \xi \in \mathbb{Z} \setminus \{0\}. \tag{28}$$

So, (28) and Cauchy–Schwartz inequality yield that

$$\|\hat{u}(\xi, \cdot)\|_{L^2(\mathbb{T})} \leq C''|\xi|^{4L}\|\hat{h}(\xi, \cdot)\|_{L^2(\mathbb{T})}, \quad \xi \in \mathbb{Z} \setminus \{0\}. \tag{29}$$

Since $\hat{h}(\xi, \cdot) \in C^\infty(\mathbb{T})$, by (29) we can proceed as in the proof of Theorem 1 to show that $u \in C^\infty(\mathbb{T}^2)$. Thus, the operator P given in (18) is globally hypoelliptic on $\mathbb{T}^2$. QED

4 Proof of Theorem 2

PROOF. Without loss of generality, we may suppose that $j_0 = 1$, i.e., that $X_1 = \sum_{h=1}^m a_h \partial_{x_h}$ with the numbers $a_1, a_2, \ldots, a_m$ satisfying the condition (4).

Let $u \in \mathcal{E}'(\mathbb{T}^{m+n})$ which satisfies $Pu = h \in C^\infty(\mathbb{T}^{m+n})$. Argumenting as in the proof of Theorem 1, one shows that the partial Fourier transfom $\hat{u}(\xi, \cdot)$ with respect to x of u belongs to $C^\infty(\mathbb{T}^n)$ for every $\xi \in \mathbb{Z}^m$ and that $\hat{u}(\xi, \cdot)$ satisfies the following identity

$$\sum_{j=1}^{l} \|Y_j \hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} = \int_{T^n} \hat{h}(\xi, y)\bar{\hat{u}}(\xi, y) dy, \quad \xi \in \mathbb{Z}^m, \tag{30}$$

where $Y_j := i \sum_{h=1}^m a_{jh}(y)\xi_h + \sum_{k=1}^n b_{jk}(y)\partial_{y_k}$ for every $j = 1, \ldots, l$.

In order to complete the proof we need the following lemma.

Lemma 2. *There exist two positive constants C and L such that*

$$\|\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} \leq C|\xi|^{2L} \|Y_1 \hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)}, \quad \xi \in \mathbb{Z}^m. \tag{31}$$

PROOF. For $\xi \in \mathbb{Z}^m$ fixed let $\varphi_\xi \in C^\infty(\mathbb{T}^n)$ and set

$$Y_1 \varphi_\xi(y) = \left(i \sum_{h=1}^m a_h \xi_h \right) \varphi_\xi(y) =: \psi_\xi(y), \quad y \in \mathbb{T}^n. \tag{32}$$

Taking partial Fourier transform with respect to y, we obtain

$$i \left(\sum_{h=1}^m a_h \xi_h \right) \hat{\varphi}_\xi(\eta) = \hat{\psi}_\xi(\eta), \quad \eta \in \mathbb{Z}^n,$$

and hence,

$$\left| \sum_{h=1}^m a_h \xi_h \right| |\hat{\varphi}_\xi(\eta)| = |\hat{\psi}_\xi(\eta)|, \quad \eta \in \mathbb{Z}^n. \tag{33}$$

Since the vector $(a_1, a_2, \ldots, a_m)$ satisfies the condition (4), there exist two positive constants C and L so that

$$\left| \sum_{h=1}^m a_h \xi_h \right| \geq \frac{C}{|\xi|^L}, \quad \xi \in \mathbb{Z}^m \setminus \{0\}. \tag{34}$$

It follows from (33) and (34) that

$$|\hat{\varphi}_\xi(\eta)|^2 \leq C'|\xi|^{2L} |\hat{\psi}_\xi(\eta)|^2, \quad \eta \in \mathbb{Z}^n. \tag{35}$$

The last inequality together the Parseval identity imply that

$$\int_{\mathbb{T}^n} |\varphi_\xi(y)|^2 dy = \sum_{\eta \in \mathbb{Z}^n} |\hat{\varphi}_\xi(\eta)|^2 \leq C'|\xi|^{2L} \sum_{\eta \in \mathbb{Z}^n} |\hat{\psi}_\xi(\eta)|^2 = C'|\xi|^{2L} \int_{\mathbb{T}^n} |\psi_\xi(y)|^2 dy,$$

i.e., that

$$\|\varphi_\xi\|^2_{L^2(\mathbb{T}^n)} \leq C'|\xi|^{2L} \int_{\mathbb{T}^n} \left| \left(i \sum_{h=1}^m a_h \xi_h \right) \varphi_\xi(y) \right|^2 dy. \tag{36}$$

If we apply (36) with $\varphi_\xi(\cdot) = \hat{u}(\xi, \cdot)$ we obtain that

$$\|\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} \leq C'|\xi|^{2L} \int_{\mathbb{T}^n} \left| \left(i \sum_{h=1}^m a_h \xi_h \right) \hat{u}(\xi, y) \right|^2 dy = C'|\xi|^{2L} \|Y_1 \hat{u}(\xi, \cdot)\|^2_{L^2}.$$

So, the proof is complete. QED

Using Lemma 2 and argumenting as in the end of the proof of Theorem 1, one shows that the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$ and so, the proof of Theorem 2 is complete. QED

Remark 2. We recall the following conjecture of Petronilho [25]:

Let $X_1, \ldots, X_m$ be a family of real vector fields on $\mathbb{T}^N$. If there exist coordinates y on $\mathbb{T}^N$ in which the vector field X_1 admits the form $X_1 = \sum_{k=1}^N \lambda_k \partial_{y_k}$ with the numbers $\lambda_1, \ldots, \lambda_N$ satisfying the following condition: there exist $C > 0$, $K > 0$ such that

$$|\sum_{k=1}^N \lambda_k \eta_k| \geq \frac{C}{|\eta|^K}, \quad \eta \in \mathbb{Z}^N \setminus \{0\},$$

then the operator $P = -\sum_{j=1}^m X_j^2$ is globally hypoelliptic on $\mathbb{T}^N$.

So, our Theorem 2 gives a partial positive answer to the conjecture above and it is hoped that it also provides some insight into this problem.

5 Proof of Theorem 3

Proof. The condition is clearly necessary. Indeed, if $a_j \equiv 0$ for some $j_0 \in \{1, \ldots, m\}$, then every $u(x, y) = u(x_{j_0}) \in \mathcal{E}'(\mathbb{T}^{m+n}) \setminus C^\infty(\mathbb{T}^{m+n})$ satisfies $Pu = 0$.

Suppose that $a_j \not\equiv 0$ for all $j = 1, \ldots, m$. In order to show that P is globally hypoelliptic on $\mathbb{T}^{m+n}$ we proceed as follows. Let $u \in \mathcal{E}'(\mathbb{T}^{m+n})$ such that $Pu = h \in C^\infty(\mathbb{T}^{m+n})$. Taking partial Fourier transform with respect to x, we obtain

$$-\Delta_y \hat{u}(\xi, y) - \sum_{j=1}^m \left(i a_j(y)\xi_j + \sum_{k=1}^n b_{jk}(y)\partial_{y_k} \right)^2 \hat{u}(\xi, y) = \hat{h}(\xi, y), \quad \xi \in \mathbb{Z}^m. \tag{37}$$

For every fixed $\xi \in \mathbb{Z}^m$ the operator in (37) is elliptic with respect to $y \in \mathbb{T}^n$ and $\hat{h}(\xi, \cdot) \in C^\infty(\mathbb{T}^n)$. It necessarily follows that $\hat{u}(\xi, \cdot) \in C^\infty(\mathbb{T}^n)$. We can then multiply (37) by $\overline{\hat{u}}(\xi, y)$ and integrate by parts with respect to $y \in \mathbb{T}^n$. So, by setting $Y_j := i a_j(y)\xi_j + \sum_{k=1}^n b_{jk}(y)\partial_{y_k}$ and using the fact that $\mathrm{div}_B X_j \equiv 0$ for $j = 1, \ldots, l$, we obtain

$$\sum_{k=1}^n \|\partial_{y_k}\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} + \sum_{j=1}^n \|Y_j \hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} = \int_{\mathbb{T}^n} \hat{h}(\xi, y)\overline{\hat{u}}(\xi, y)dy, \quad \xi \in \mathbb{Z}^m. \tag{38}$$

For each $j = 1, \ldots, m$ and $\varphi \in C^\infty(\mathbb{T}^n)$ we set

$$\|\varphi\|_j^2 := \int_{\mathbb{T}^n} |i a_j(y)\varphi(y)|^2 dy + \sum_{k=1}^n \|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)}.$$

In order to complete the proof we need the following results.

Lemma 3. *There exists a positive constant C such that*

$$\|\varphi\|^2_{L^2(\mathbb{T}^n)} \leq C\|\varphi\|_j^2, \quad \varphi \in C^\infty(\mathbb{T}^n), \tag{39}$$

for all $j = 1, \ldots, m$.

Proof. Let $j \in \{1, \ldots, m\}$ be fixed. Since $a_j \not\equiv 0$ on $\mathbb{T}^n$ there exists $I_j \subseteq T^n$ with $\mathring{I}_j \neq \emptyset$ such that

$$|a_j(y)| \geq \alpha_j > 0, \quad y \in I_j. \tag{40}$$

So, for $y \in \mathbb{T}^n$ and $z \in I_j$ by the fundamental theorem of calculus we have, for every $\varphi \in C^\infty(\mathbb{T}^n)$, that

$$\varphi(y) = \varphi(z) + \sum_{k=1}^{n} \int_{z_k}^{y_k} \varphi(z_1, \dots, z_{k-1}, \tau_k, y_{k+1}, \dots, y_n)\, d\tau_k.$$

Applying the Cauchy–Schwarz inequality we obtain

$$|\varphi(y)|^2 \leq C_1 \left(|\varphi(z)|^2 + \sum_{k=1}^{n} \int_{\mathbb{T}} |\varphi(z_1, \dots, z_{k-1}, \tau_k, y_{k+1}, \dots, y_n)|^2\, d\tau_k \right)$$

for some suitable constant $C_1 > 0$ independent on φ. If we integrate the last inequality first with respect to $y \in \mathbb{T}^n$ and then with respect to $z \in I_j$ and using the fact that $|I_j| > 0$, then we obtain

$$\|\varphi\|^2_{L^2(\mathbb{T}^n)} \leq C_1' \left(\int_{I_j} |\varphi(z)|^2 dz + \sum_{k=1}^{n} \|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)} \right). \tag{41}$$

Since by (40) we have

$$\int_{I_j} |\varphi(z)|^2 dz \leq \frac{1}{\alpha_j^2} \int_{I_j} |ia_j(z)\varphi(z)|^2 dz \leq \frac{1}{\alpha_j^2} \int_{\mathbb{T}^n} |ia_j(z)\varphi(z)|^2 dz,$$

inequality (41) implies that there exists a constant $C_j > 0$ such that

$$\|\varphi\|^2_{L^2(\mathbb{T}^n)} \leq C_j \|\varphi\|_j^2.$$

If we set $C := \max_{j=1}^m C_j$, then inequality (39) holds for every $j = 1, \dots, m$ and hence, the proof of Lemma 3 is complete. QED

Then next result follows from Lemma 3 and permits us to conclude that $u \in C^\infty(\mathbb{T}^{m+n})$.

Proposition 1. *The exists a positive constant D such that*

$$\|\hat{u}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)} \leq D \|\hat{h}(\xi, \cdot)\|^2_{L^2(\mathbb{T}^n)}, \quad \xi \in \mathbb{Z}^m \setminus \{0\}, \tag{42}$$

where $\hat{u}(\xi, \cdot)$ and $\hat{h}(\xi, \cdot)$ are as in (38).

Proof. Let $\xi \in \mathbb{Z}^m \setminus \{0\}$. Then $\xi_j \neq 0$ for some $j \in \{1, \dots, m\}$ and hence,

$$|ia_j(y)| \leq |ia_j(y)\xi_j|, \quad y \in \mathbb{T}^n.$$

Using the last inequality and the Cauchy–Schwarz inequality we obtain, for any $\varphi \in C^\infty(\mathbb{T}^n)$, that

$$\begin{aligned} \|\varphi\|_j^2 &= \int_{\mathbb{T}^n} |ia_j(y)\varphi(y)|^2 dy + \sum_{k=1}^{n} \|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)} \\ &\leq \int_{\mathbb{T}^n} |ia_j(y)\xi_j\varphi(y)|^2 dy + \sum_{k=1}^{n} \|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)} \\ &= \int_{\mathbb{T}^n} \left| ia_j(y)\xi_j\varphi(y) + \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}\varphi(y) - \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}\varphi(y) \right|^2 dy + \\ &\quad + \sum_{k=1}^{n} \|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)} \end{aligned} \tag{43}$$

$$\begin{aligned} &\leq C\int_{\mathbb{T}^n}\left|ia_j(y)\xi_j\varphi(y)+\sum_{k=1}^n b_{jk}(y)\partial_{y_k}\varphi(y)\right|^2 dy+\\ &\quad +C\sum_{k=1}^n\int_{\mathbb{T}^n}|b_{jk}(y)\partial_{y_k}\varphi(y)|^2dy+\sum_{k=1}^n\|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)}\\ &\leq C'\left(\|Y_j\varphi\|^2_{L^2(\mathbb{T}^n)}+\sum_{k=1}^n\|\partial_{y_k}\varphi\|^2_{L^2(\mathbb{T}^n)}\right),\end{aligned}$$

with $C' := 1 + C\max\{1, \max_{k=1}^n \|b_{jk}\|_\infty\}$. If we apply (43) with $\varphi(y) = \hat{u}(\xi, y)$, then it follows that

$$\|\hat{u}(\xi,\cdot)\|^2_j \leq C'\left(\|Y_j\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)}+\sum_{k=1}^n\|\partial_{y_k}\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)}\right). \quad (44)$$

Combining inequalities (38), (39) and (44) we deduce

$$\begin{aligned}\|\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)} &\leq C\|\hat{u}(\xi,\cdot)\|^2_j\\ &\leq CC'\left(\|Y_j\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)}+\sum_{k=1}^n\|\partial_{y_k}\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)}\right)\\ &\leq CC'\int_{\mathbb{T}^n}\hat{h}(\xi,y)\overline{\hat{u}}(\xi,y)dy,\end{aligned}$$

and hence, by the Cauchy–Schwarz inequality that

$$\|\hat{u}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)} \leq CC'\|\hat{h}(\xi,\cdot)\|^2_{L^2(\mathbb{T}^n)}.$$

This completes the proof. QED

Using Proposition 1 and argumenting as in the end of the proof of Theorem 1, one shows that the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$ and so, the proof of Theorem 3 is complete. QED

6 Final Remarks

We point out that Theorems 1, 2 and 3 continue to hold in the setting of analytic and Gevrey classes. First to give the proper versions of such results, we recall the necessary definitions.

Let $s \geq 1$. A function $f \in C^\infty(\mathbb{T}^N)$ is said to belong to the Gevrey class $G^s(\mathbb{T}^N)$ of order s if there exists a constant $C > 0$ such that $|\partial^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$ for every $\alpha \in \mathbb{Z}^N_+$ and $x \in \mathbb{T}^N$ (see, e.g., [28]). In particular, $G^1(\mathbb{T}^N)$ is the space of all analytic functions on $\mathbb{R}^N$ which are 2π–periodic with respect to each variable, usually denoted by $C^\omega(\mathbb{T}^N)$ or by $\mathcal{A}(\mathbb{T}^N)$. Therefore, $G^s(\mathbb{T}^N)$ can be described as $G^s(\mathbb{T}^N) = \operatorname{indlim}_{\eta\to 0} G^s(\mathbb{T}^N, \eta)$, where

$$G^s(\mathbb{T}^N,\eta)=\left\{\varphi\in G^s(\mathbb{T}^N);\ |\varphi;s,\eta|=\sum_{\alpha\in\mathbb{N}_0^n}\left|\left|\partial^\alpha\varphi\right|\right|_{L^2(\mathbb{T}^N)}\frac{\eta^{|\alpha|}}{(\alpha!)^s}<\infty\right\}.$$

Since the inclusions maps $G^s(\mathbb{T}^N, \eta) \hookrightarrow G^s(\mathbb{T}^N, \eta')$ for all $\eta > \eta' > 0$, are compact, $G^s(\mathbb{T}^N)$ is a dual Fréchet–Schwartz space. We denote by $\mathcal{E}'_s(\mathbb{T}^N) = (G^s(\mathbb{T}^N))'$ its topological dual.

Using Fourier expansion, one proves that $u \in \mathcal{E}'_s(\mathbb{T}^N)$ belongs to the Gevrey class $G^s(\mathbb{T}^N)$ if and only if there exist two positive constants ε and C such that

$$|\hat{u}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/s}}, \quad \xi \in \mathbb{Z}^N \setminus \{0\}.$$

A linear partial differential operator P defined on $\mathbb{T}^N$ with coefficients in $G^s(\mathbb{T}^N)$, $s \geq 1$, is said to be *s–globally hypoelliptic* (*globally analytic hypoelliptic* if $s = 1$) in T^N if the conditions $u \in \mathcal{E}'_s(T^N)$ and $Pu \in G^s(T^N)$ imply that $u \in G^s(T^N)$.

We also recall that a vector $a = (a_1, \dots, a_N) \in \mathbb{R}^N \setminus \mathbb{Q}^N$ is said to be *an exponentially non–Liouville vector with exponent s* if for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|a \cdot \xi - \eta| \geq C_\varepsilon e^{-\varepsilon|\xi|^{1/s}}, \quad \xi \in \mathbb{Z}^N \setminus \{0\}, \eta \in \mathbb{Z}.$$

If $N = 1$, then this is the definition of a *not exponentially Liouville number with exponent s.*

Now, we can state the Gevrey and analytic proper analogues to Theorems 1, 3 and 2.

Theorem 4. *Let P be the operator on $\mathbb{T}^{m+n}$ given by*

$$P = -\sum_{j=1}^{l} X_j^2, \tag{45}$$

where

$$X_j = \sum_{h=1}^{m} a_{jh}(y)\partial_{x_h} + \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}, \quad j = 1, \dots, l,$$

with variables $(x, y) \in \mathbb{T}^m \times \mathbb{T}^n$, and suppose that the coefficients $\{a_{jh}\}_{h=1}^m$ and $\{b_{jk}\}_{k=1}^n$ are real valued functions in $G^s(\mathbb{T}^n)$ with $s \geq 1$ and that $\mathrm{div}_B X_j := \sum_{k=1}^n \partial_{y_k} b_{jk} \equiv 0$ *on $\mathbb{T}^n$.*

If the following conditions hold:

(*i*) *the vector fields $\sum_{k=1}^n b_{jk}(y)\partial_{y_k}$, $j = 1, \dots, l$, span $T_y(\mathbb{T}^n)$ for every $y \in \mathbb{T}^n$,*

(*ii*) *there exists $j_0 \in \{1, \dots, l\}$ such that $a_{j_0,h} \equiv a_h$ for every $h = 1, \dots, m$ and $b_{j_0 k} \equiv b \neq 0$ for every $k = 1, \dots, n$, and the vector $(\frac{a_1}{b}, \frac{a_2}{b}, \dots, \frac{a_m}{b})$ is an exponentially non–Liouville vector with exponent s,*

then the operator P is s–globally hypoelliptic on $\mathbb{T}^{m+n}$.

Theorem 5. *Let P be the operator on $\mathbb{T}^{m+n}$ defined according to (45). If the following conditions hold:*

(*i*) *the vector fields $\sum_{k=1}^n b_{jk}(y)\partial_{y_k}$, $j = 1, \dots, l$, span $T_y(\mathbb{T}^n)$ for every $y \in \mathbb{T}^n$,*

(*ii*) *there exists $j_0 \in \{1, \dots, l\}$ such that $a_{j_0,h} \equiv a_h$ for every $h = 1, \dots, m$ and $b_{j_0 k} \equiv 0$ for every $k = 1, \dots, n$, and the numbers $a_1, a_2, \dots, a_m$ satisfy the following Diophantine condition: for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|\sum_{h=1}^{m} a_h \xi_h| \geq C_\varepsilon e^{-\varepsilon|\xi|^{1/s}}, \quad \xi \in \mathbb{Z}^m \text{ with } |\xi| \geq C_\varepsilon, \tag{46}$$

then the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$.

Theorem 6. *Let P be the operator on $\mathbb{T}^{m+n}$ given by*

$$P = -\Delta_y - \sum_{j=1}^{m} X_j^2, \tag{47}$$

where

$$X_j = a_j(y)\partial_{x_j} + \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}, \quad j = 1, \dots, m,$$

with $(x, y) \in \mathbb{T}^m \times \mathbb{T}^n$, and suppose that the coefficients a_j and $\{b_{jk}\}_{k=1}^n$ are real valued functions in $G^s(\mathbb{T}^n)$ with $s \geq 1$ and that $\mathrm{div}_B X_j := \sum_{k=1}^n \partial_{y_k} b_{jk} \equiv 0$ *on $\mathbb{T}^n$.*

Then the operator P is globally hypoelliptic on $\mathbb{T}^{m+n}$ if and only if $a_j \not\equiv 0$ for all $j = 1, \dots, m$.

To prove the above results it suffices to follow the same lines of the proofs of the corresponding theorems in the C^∞–setting and hence, we leave the details to the interested reader.

References

[1] A. A. Albanese, P. Popivanov: *On the global solvability in Gevrey classes on the n–dimensional torus*, J. Math. Anal. Appl., **297** (2004), 659–672.

[2] A. A. Albanese, P. Popivanov: *Global analytic and Gevrey solvability of sublaplacians under Diophantine conditions*, Ann. Mat. Pura e Appl., **185** (2006), 395–409.

[3] A. A. Albanese, P. Popivanov: *Gevrey hypoellipticity and solvability on the multidimensional torus of some classes of linear partial differential operators*, Ann. Univ. Ferrara, **52** (2006), 65–81.

[4] A. A. Albanese, L. Zanghirati: *Global hypoellipticity and global solvability in Gevrey classes on the n–dimensional torus*, J. Diff. Eq., **199** (2004), 256–268.

[5] A. P. Bergamasco: *Remarks about global analytic hypoellipticity*, Trans. Amer. Math. Soc., **351** (1999), 4113–4126.

[6] A. P.Bergamasco, P. D. Cordaro, G. Petronilho: *Global solvability for certain classes of underdetermined system of vector fields*, Math. Z., **223** (1996), 261–274.

[7] H. Bohr: Almost periodic functions, Chelsea Publishing Company, New York, 1947.

[8] D. Dickinson, T. Gramchev, M. Yoshino: *Perturbations of vector fields on tori: resonant normal forms and Diophantine phenomena*, Proc. Edinb. Math. Soc., **45** (2002), 731–759.

[9] W. Chen, M. Y. Chi: *Hypoelliptic vector fields and almost periodic motions on the torus* $\mathbb{T}^n$, Comm. Part. Diff. Eq., **25** (2000), 337-354.

[10] T. Gramchev, P. Popivanov, M. Yoshino: *Some note on Gevrey hypoellipticity and solvability on torus*, J. Math. Soc. Japan, **43** (1991), 501-514.

[11] T. Gramchev, P. Popivanov, M. Yoshino: *Global properties in spaces of generalized functions on the torus for second order differential operators with variable coefficients*, Rend. Sem. Univ. Pol. Torino, **51** (1993), 145–172.

[12] S. Greenfield, N. Wallach: *Global hypoellipticity and Liouville numbers*, Proc. Amer. Math. Soc., **31** (1972), 112–114.

[13] A. A. Himonas: *Global analytic and Gevrey hypoellipticity of sublaplacians under diophantine conditions*, Proc. Amer. Math. Soc., **129** (2000), 2001–2007.

[14] A. A. Himonas, G. Petronilho: *On global hypoellipticity of degenerate elliptic operators*, Math. Z., **230** (1999), 241–257.

[15] A. A. Himonas, G. Petronilho: *Global hypoellipticity and simultaneous approximability*, J. Func. Anal., **170** (2000), 356–365.

[16] A. A. Himonas, G. Petronilho: *Propagation of regularity and global hypoellipticity*, Michigan Math. J., **50** (2002), 471–481.

[17] A. A. Himonas, G. Petronilho: *On Gevrey regularity of globally C^∞ hypoelliptic operators*, J. Diff. Eq., **207** (2004), 267–284.

[18] A. A. Himonas, G. Petronilho, L. A. C. dos Santos: *Regularity of a class of sub-Laplacians on the 3–dimensional torus*, J. Funct. Anal., **240** (2006), 568–591.

[19] L. Hörmander: *Hypoelliptic second order differential equations*, Acta Mat., **119** (1967), 147–171.

[20] L. Hörmander: The Analysis of Linear Partial Differential Operators, Springer–Verlag, Vol. I, Berlin Heidelbergh Nwe York, 2nd Edition, 1990.

[21] H. Omori, T. Kobayashi: *Global hypoellipticity of subelliptic operators on closed manifolds*, Hokkaido Math. J., **28** (1999), 613–633.

[22] G. Petronilho: *Global solvability on the torus for certain classes of formally self–adjoint operators*, Manuscripta Math., **103** (2000), 9–18.

[23] G. Petronilho: *Global solvability on the torus for certain classes of operators in the form of a sum of squares of vector fields*, J. Diff. Eq., **145** (1998), 101–118.

[24] G. Petronilho: *Global solvability and simultaneously approximable vectors*, J. Diff. Eq., **184** (2002), 48–61.

[25] G. Petronilho: *Simultaneous reduction of a family of commuting real vector fields and global hypoellipticity*, Israel J. Math., **155** (2006), 81–92.

[26] G. Petronilho: *On Gevrey solvability and regularity*, Math. Nachr., **282** (2009), 470–481.

[27] P. Popivanov: *On the solvability of the linear PDE with costant coefficients in the class of generalized periodic functions*, Annuaire de l'Université de Sofia, **76**, Livre 1, (1982), 41–49.

[28] L. Rodino: Linear Partial Differential Operators in Gevrey Classes, World Scientific, Singapore New Jersey London Hong Kong, 1993.

[29] D. Tartakoff: *Global (and local) analyticity for second order operators constructed from rigid vector fields on product of tori*, Trans. Amer. Math. Soc., **348** (1996), 2577–2583.