*Note di Matematica* **23,** n. 1, 2004, 39–46.

# The non-solvable triangle transitive planes

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Received: 26/4/2004; accepted: 26/4/2004.

**Abstract.** The class of finite translation planes admitting non-solvable triangle transitive groups is completely determined as the class of irregular nearfield planes admitting non-solvable groups.

**Keywords:** triangle transitive, non-solvable group, autotopism, anti-autotopism

**MSC 2000 classification:** 51E23 (primary), 51A40.

## 1 Introduction.

A 'triangle transitive plane' is a translation plane that induces a collineation group $G$ in the translation complement that fixes or interchanges two components (the autotopisms and anti-autotopisms, respectively), $x = 0, y = 0$ such that $G_{x=0}$, $G_{y=0}$ and $G$ are transitive on the non-fixed points of $x = 0, y = 0, \ell_\infty$ (the line at infinity), respectively. A triangle transitive plane is 'solvable' or 'non-solvable' if and only if $G$ is solvable or non-solvable, respectively. The projective triangle with sides $x = 0, y = 0$ and $\ell_\infty$ and vertex points $(0, 0), (0), (\infty)$ is called the 'autotopism triangle'.

There are a variety of triangle transitive planes. For example, various generalized twisted field planes, whose right, middle, and left nuclei are equal (see, e.g. [4]) the Suetake planes [30], certain generalized André planes. Furthermore, there are a great variety of triangle transitive planes of order $p^u$ in Williams [32] that admit a cyclic autotopism group of order $p^u - 1$. These planes are discussed further in Kantor [25], who explicates the connections with these and the construction of Blokhuis, Coulter, Henderson and O'Keefe [5]. In particular, the number of non-isomorphic planes is not bounded by any polynomial in $q = p^u$. All of these planes are solvable triangle transitive planes.

Of course, certain of the irregular nearfield planes, one of order $11^2$ and the two of order $29^2$ and $59^2$ are non-solvable triangle transitive planes since the direct product of the two homology groups is an autotopism group transitive on the non-vertex points on each side of the autotopism triangle.

In this article, we show that these three planes are exactly the non-solvable triangle transitive planes.

**1 Theorem.** *Let $\pi$ be a finite non-solvable triangle transitive plane then $\pi$ is an irregular nearfield plane of order* $11^2, 29^2$ *or* $59^2$.

In the following sections, we give the proof to Theorem 1. We assume the above hypothesis for the remainder of the article.

## 2 The Initial Reduction.

We begin with a reduction of orders.

**2 Theorem.** *Let $\pi$ be a triangle transitive plane of order $q^n$. If the group induced on any leg of the autotopism triangle is non-solvable then*

$$q^n \in \{11^2, 19^2, 29^2, 59^2, 3^4, 3^6\}.$$

Proof. If the group acting on a finite leg of the autotopism triangle is non-solvable then we are finished by the theorem of Ganley, Jha and Johnson [11]. Hence, assume the group acts transitively and non-solvably on the infinite leg of the autotopism triangle.

Hence, we have a non-solvable group which induces solvable groups on both $x = 0$ and $y = 0$. First assume that $G$ is an autotopism group (fixes both $x = 0, y = 0$). Hence, there is a normal subgroup $N_x$ that fixes $x = 0$ pointwise and if the group $G$ induces $G/N_x$ is solvable on $x = 0$ then the group $N_x$ acts faithfully on $y = 0$ and hence must itself be solvable. Then, $N_x$ and $G/N_x$ are both solvable, contrary to our initial assumptions. Now assume that $G$ is not an autotopism group, implying that $G$ is transitive on $\{(\infty), (0)\}$. Hence, $G_{x=0}$ is a normal subgroup of index 2 and is thus non-solvable. The above argument shows that the group induced on $x = 0$ or $y = 0$ by $G$ is non-solvable. This means that we may apply the theorem of Ganley, Jha and Johnson [11], implying that the plane is Hall, Dempwolff of order 16, or Walker of order 25. However, for the Hall planes of order $q^n \neq 9$, the full collineation group leaves invariant a regulus net of degree $\sqrt{q^n} + 1$, so the plane cannot be triangle transitive. Furthermore, although the Hall plane of order 9 admits a non-solvable group, the global stabilizer of an autotopism triangle is solvable. In the Walker planes, $SL(2, 5)$ is reducible but not completely reducible. The Dempwolff planes of order 16 admitting $SL(2, 4)$ are not triangle transitive. $\boxed{QED}$

## 3 Order $9^2$.

If the order of $q^n$ is 81, then the only possible group acting involves $SL(2, 5)$. Furthermore, since 60 does not divide 80, it follows that no subgroup of $SL(2, 5)$

can act as a homology group with affine axis (since $SL(2,5)$ must fix two components, also showing that there must be a collineation group isomorphic to $SL(2,5)$). Since the 3-elements fix two components, the 3-elements are planar. This means, since the plane is triangle transitive, and we have a group of order 3 stabilizing a third component, that the order of the full collineation group is divisible by $80 \cdot 3 = 2^4 \cdot 3 \cdot 5$. Moreover, by a Lemma 25 of Biliotti, Jha, Johnson [3], each element of order 3 is either Baer, an elation or fixes at least 27 points on a unique component. Hence, we must have that each 3-element is Baer. Suppose that two of the 3-elements that generate $SL(2,5)$ on a fixed component share all of their components. Then, it follows easily that the Baer axes corresponding to $SL(2,5)$ define a derivable net. However, it now follows from Johnson and Prince [24], that the derivable net is invariant, so that plane cannot be triangle transitive. Thus, no two Baer subplanes that generate a copy of $SL(2,5)$ can share all of their components. Now since each 3-element is Baer and $SL(2,5)$ fixes two components $L$ and $M$, it follows that the 3-elements are elations on both $L$ and $M$. It then follows that $SL(2,5)$ acts transitively on 10 mutually disjoint subspaces, implying that there is a spread on $L$, which must be Desarguesian since the Hall spread does not admit elations. So L and M are Desarguesian spreads admitting $SL(2,5)$. Hence, there are 10 Baer subplanes that are disjoint on $L$ and $M$. Suppose two Baer subplanes intersect non-trivially. Then there is a subplane fixed pointwise by some element of $SL(2,5)$ which has $x = 0$ and $y = 0$ as components. Since this cannot happen, it follows that the 10 Baer subplanes are mutually disjoint and share two components $x = 0$ and $y = 0$. It follows that this set of 10 Baer subplanes form a derivable partial spread (although, it not yet clear that this is a derivable partial spread of the plane). Each Baer subplane has 8 components other than $x = 0, y = 0$. It is possible that these Baer subplanes lay completely across the plane. Since it seems clear that $SL(2,5)$ is normal so that group permutes the subplanes, the stabilizer of a Baer subplane admits a group of order 8 acting transitively on the components not equal to $x = 0, y = 0$. The Baer subplane is either Hall or Desarguesian, since it has order 9. Anyway, $L$ and $M$ are Desarguesian spreads, $SL(2,5)$ is normal on $L$, implying the spread on $L$ is preserved by the collineation group $G$. Hence, the group induced on $L$ is a subgroup of $\Gamma L(2,9)$. Since we have the kernel is at least $GF(3)$, we have a linear subgroup of order at least $2^5 \cdot 3 \cdot 5/2 = 2^4 \cdot 3 \cdot 5$ in $GL(2,9)$ and we know that this group times $GF(9)^*$ mod $GF(9)$ is $A_5$. Also, the stabilizer of the Baer subplane is transitive on the points on $x = 0$ and transitive on the 8 infinite points. Hence, the stabilizer of the Baer subplane has order on $x = 0$ divisible by $8 \cdot 8$. But, this group is in $\Gamma L(2,9)$ on $x = 0$, implying that we have a homology group of order divisible by 4. The same argument shows we have a homology group of order divisible by 4) on

$y = 0$. But, on the subplane we still have transitivity on the 8 components and on the 8 points on $x = 0$ and we have a homology group of order 4, this pumps up the group to order $8 \cdot 8 \cdot 4$. However, now modulo $GF(9)$ and in $GL(2, 9)$, this means we have a homology group of order 16 —which is too big. Hence, this case does not occur.

## 4 Orders $11^2, 19^2, 29^2, 59^2$

Now consider order $p^2$, for $p = 11, 19, 29, 59$.

Since the group $G_{x=0}$ is non-solvable and reducible and transitive, by Hiramine [15] (2.2), it follows that we have either $SL(2,p)$, $SL(2,5)$ or $SL(2,5) \times SL(2,5) = H$. When $SL(2,5)$ occurs, as in the latter two cases, Hiramine notes that $SL(2,5)$ is a homology group or $SL(2,5) \times SL(2,5)$ is a product of homology groups. Then, we consider the orbit structure on the line at infinity. If there is an orbit of length 120 under $H$, then Hiramine determines all of the planes. There are some new planes constructed as well as some of the exceptional planes of Lüneburg [26] (see section 18, particularly (18.16) and (18.17)). However, none of these planes other than the irregular nearfield planes are triangle transitive (see also Hiramine Theorem (4.14) and the following remark (4.15).

We now just check the numbers and show that the second SL(2,5) must permute the orbits of the first SL(2,5) and the numbers are such that the second SL(2,5) must fix one of the orbits of the first SL(2,5): $p = 11$, $p^2 - 1 = 120$, done. $p = 19, p^2 - 1 = 20 \cdot 18 = 120 \cdot 3$, so $SL(2,5)$ must fix all three orbits, since all 5-elements must fix all three. $p = 29, p^2 - 1 = 30 \cdot 28 = 120 \cdot 7$. There are six groups of order 5 and each must fix two of the seven orbits. If no two groups of order 5 fix the same orbit then there must be at least 12 orbits. Hence, $SL(2,5)$ fixes an orbit. $p = 59, p^2 - 1 = 60 \cdot 58 = 120 \cdot 29$. Hence, each 5-group fixes at least 4 orbits, say $4 + 5a$. If $SL(2,5)$ does not fix an orbit then $a = 0$ and $SL(2,5)$ permutes a set of 24 orbits and 5 orbits. The set of 24 orbits are permuted in six sets of four and $SL(2,5)$ acts transitively on this set of degree six. Consider the stabilizer of a set of four, of order 20. This is the normalizer of a Sylow 5-subgroup $N$. The number of orbits of $N$ of the set of four is $1, 2$ or $4$. Hence, we have orbits of lengths $\{6^4, 5\}$, or $\{12^2, 5\}$ or $\{24, 5\}$. Hence, we have orbits of lengths $\{(120 \cdot 6)^4, 120 \cdot 5\}$, $\{(120 \cdot 12)^2, 120 \cdot 5\}$ or $\{120 \cdot 24, 120 \cdot 5\}$, orbits under $SL(2,5) \times SL(2,5)$ and this group is clearly normal in $G$. Hence, the orbit of length $120 \cdot 5$ must be fixed by $G$, a contradiction as $G$ is triangle transitive.

So, we do not have $SL(2,5) \times SL(2,5)$ unless the plane has is an irregular nearfield.

And, $SL(2,p)$ gives the Hall planes –again, not triangle transitive. So, we are left with $SL(2,5)$ on a component. And since $SL(2,5)$ is a homology group on $L$ and as we have dealt with the $SL(2,5) \times SL(2,5)$ case, it follows that $G$ cannot interchange $x = 0, y = 0$ and hence is an autotopism group.

Hence, we have that $SL(2,5)$ is a homology group with axis $L$ and coaxis $M$. Hence, the group induced on $L$ is solvable and the group induced on $M$ is non-solvable. Since $G$ is transitive on $L$, this means that $G$ has order divisible by $120(p^2 - 1)$.

Then, the group has order on the coaxis is $120(p^2 - 1)/j$, where $j$ is the order of the group fixing $M$ pointwise. Since we know that $SL(2,5)K^*/K^*$ on $M$ induces $A_5$ in $PGL(2,p^2)$, $K^*$ the kernel homology group of order $p - 1$, it follows that $SL(2,5) \mid M$ contains $-1$, and the $(p^2 - 1)/j$ part can contribute at most $(p-1)/2$ to the kernel $K^*$, acting on $M$. Inducing on the coaxis means that we have a homology group of order at least 2(p+1)), forcing a cyclic group of order p+1. By recent results on cyclic homology groups of order $q+1$ acting on spreads in $PG(3,q)$, there is an associated regular hyperbolic fibration with constant back half (see the main result of Johnson [21]), we get an associated flock of a quadratic cone that admits a collineation group fixing one regulus and transitive on the remaining $q - 1$ reguli; a so-called 'transitive deficiency one' type. Moreover, this cyclic group $C_{p+1}$ of order $p+1$ is a normal subgroup of $G$ (see also Johnson [22]). Then, it turns out that $G/C_{p+1}$ is a normal subgroup of the collineation of the associated flock of a quadratic cone (see e.g. Baker, Ebert, Penttila [2]). Since $SL(2,5)$ is a normal subgroup of $G$ and normalizes But, we now have SL(2,5) as a normal subgroup and hence normalizes the regulus-inducing elation group $E$, of order $q = p$, so that SL(2,5) commutes with E, since $E$ must normalize $SL(2,5)$. Note that $SL(2,5)C_{p+1}/C_{p+1}$ is isomorphic to $SL(2,5)$. Look at the group acting on the axis of E and observe that we end up getting a cyclic homology group of order p-1. Moreover, $G/C_{p+1}$ acting on the flock spread has order divisible by $120(p-1)^2/(p+1)$ We have a group acting transitively on the $p-1$ orbits of $C_{p+1}$ and these are the hyperbolic quadrics in the associated regular hyperbolic fibration. We have a group of order divisible by p(p-1) transitive on the components not in the fixed regulus, in the flock plane. When $p = 11$ all flocks are obtained by computer, and only the Desarguesian admit non-solvable groups and of course this would imply that $SL(2,5)$ is a kernel homology group –a contradiction.

Assume that $p = 19, 29, 59$. We have a homology group of order $2(p+1)$ with $(p-1)/2$ component orbits of length $2(p+1)$, permutes by $SL(2,5)$. When $p = 19$, $(p-1)/2 = 9$, so that each 5-element must fix four such orbits but since there are six 5-subgroups, it follows that $SL(2,5)$ must fix an orbit of length $2(p+1) = 40$, a contradiction since $SL(2,5)$ acts semi-regularly. If $p = 29$,

$(p-1)/2 = 14$ and again the 5-elements fix four orbits, implying that $SL(2,5)$ must an orbit so that 120 divides $2(p+1) = 60$, a contradiction.

Hence, let $p = 59$. Since $G$ is transitive on the line at infinity, it follows that $120 \cdot 29$ must divide the order of the group $G$ and we also have the kernel homology that fixes each component so that group has order divisible by $120 \cdot 29 \cdot 58$. The group on the coaxis of $SL(2,5)$ group then must have a homology group of order 29. Thus, we have a homology group of order $2(p+1)(p-1)/2 = p^2 - 1$, implying that the plane is a nearfield plane and hence an irregular nearfield plane.

This shows that for 19,29,59, we only get the irregular nearfield planes, when $p = 29$ or 59 are triangle transitive planes, see Lüneburg [26] (18.17) for the groups of these planes.

**Conclusion.** The only non-solvable triangle transitive planes of order $p^2$, for $p$ a prime, are the three irregular nearfield planes of orders $11^2, 29^2$ and $59^2$.

## 5 Order $3^6$.

Assume that the order of the plane is $3^6$.

By Hering [13], we know that there is a normal subgroup $H$ of $G$ that acts transitively on a component $L$, isomorphic to $SL(2,13)$. Since $SL(2,13)$ has order $13 \cdot 7 \cdot 3 \cdot 8$, it follows that $SL(2,13)$ is generated by planar 3-elements. Moreover, there are exactly $13 \cdot 14/2 = 13 \cdot 7$ groups of order 3. Hence, if $SL(2,13)$ is to act transitively on $L$, it follows that the planar 3-elements fix subplanes of order $3^2$ pointwise, as if the planar 3-elements fix 3 points, the set of fixed points fall into a set of $13 \cdot 7 \cdot 2 < 3^6 - 1$. Also, the planar 3-elements cannot be Baer since then these would then fix $3^3$ points each on $L$ and since they must fix mutually disjoint subspaces on $L$, it follows that there are at least $(3^3 + 1)13 \cdot 7 + 1$ points on $L$, a contradiction.

By Hering [14], we know that we have a transitive group $Z$ on $L$ isomorphic to $SL(2,13)$ such that the 3-elements fix 1-dimensional $GF(3)$-subspaces pointwise. However, this group cannot be the group that we are considering. So, if there is a representation $H$ of $SL(2,13)$ on $L$ with such that the 3-elements fix 2-dimensional $GF(3)$-subspaces pointwise, then $H$ and $Z$ are disjoint, except possibly for the unique involution in $Z$ or $H$. However, since both groups are normal by Hering [13], it follows that there is a subgroup of order $|SL(2,13)|^2/2$ of $GL(6,3)$. However, the order of $GL(6,3)$

$$\begin{aligned} & 3^3(3^6-1)(3^5-1)(3^4-1)(3^3-1)(3^2-1)(3-1) \\ = \ & 2^{13}3^3 5 \times 7 \times 11^2 13^2. \end{aligned}$$

However, this is a contradiction as $7^2$ cannot divide the order of $GL(6,3)$. Thus, it cannot be that the plane has order $3^6$.

The previous three sections completes the proof of the main result.

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