Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) no. 2, 107–111. doi:10.1285/i15900932v30n2p107

Groups with few non-((locally finite)-by-Baer) subgroups

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Received: 3.2.2010; accepted: 17.5.2010.

Abstract. In this note we study groups with few non-((locally finite)-by-Baer) subgroups and we prove that if G is a locally graded group satisfying the minimal condition on non-((locally finite)-by-Baer) subgroups or having finitely many conjugacy classes of non-((locally finite)-by-Baer) subgroups, then G is a (locally finite)-by-Baer group. We prove also that if G is a minimal non-((locally finite)-by-Baer) group then G is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group, where $Frat(G)$ stands for the Frattini subgroup of G.

Keywords: Locally graded groups, (locally finite)-by-Baer groups, conjugacy classes of subgroups, minimal condition on subgroups.

MSC 2000 classification: Primary 20F19; Secondary 20E07.

1 Introduction

The aim of this paper is to study groups that in some sense have few non-((locally finite)-by-Baer) subgroups, namely minimal non-((locally finite)-by-Baer) groups, or groups satisfying the minimal condition on non-((locally finite)-by-Baer) subgroups, or groups having finitely many conjugacy classes of non-((locally finite)-by-Baer) subgroups. Recall that a group G is said to be a Baer group if all its cyclic subgroups are subnormal in G.

Let $\mathfrak{X}$ be a class of groups. A group G is said to be a minimal non-$\mathfrak{X}$-group if it is not an $\mathfrak{X}$-group but all of whose proper subgroups are $\mathfrak{X}$-groups. Many results have been obtained on minimal non-$\mathfrak{X}$-groups, for various choices of $\mathfrak{X}$. In particular, in [5] (respectively, in [9]) it is proved that if G is a finitely generated minimal non-nilpotent (respectively, non-(finite-by-nilpotent) group, then G is a perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group, where $Frat(G)$ denotes the Frattini subgroup of G. Also in [7] it is proved that if G is a minimal non-(periodic-by-nilpotent) (respectively, non-((locally finite)-by-nilpotent)) group, then G is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group. We generalize this last result to minimal non-((locally finite)-by-Baer) groups. We will prove:

Theorem 1. *If G is a minimal non-((locally finite)-by-Baer) group, then G is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group.*

Groups satisfying the minimal condition on non-$\mathfrak{X}$-subgroups or groups having finitely many conjugacy classes of non-$\mathfrak{X}$-subgroups have been studied by several authors for many choices of $\mathfrak{X}$. In particular, in [8] it is proved that if G is a locally graded group satisfying the minimal condition on non-((locally finite)-by-nilpotent) subgroups or having finitely many conjugacy classes of non-((locally finite)-by-nilpotent) subgroups, then G is (locally finite)-by-nilpotent. We generalize this result to (locally finite)-by-Baer groups. We will prove:

Theorem 2. *Let G be a locally graded group satisfying the minimal condition on non-((locally finite)-by-Baer) subgroups. Then G is (locally finite)-by-Baer.*

Theorem 3. *Let G be a locally graded group having finitely many conjugacy classes of non-((locally finite)-by-Baer) subgroups. Then G is (locally finite)-by-Baer.*

2 Proof of Theorem 1

Lemma 1. *Let $G = \langle g, x\rangle$ be a torsion-free nilpotent group and let n, k be positive integers. If $[\langle g\rangle ,_k \langle x^n\rangle] = 1$ then $[\langle g\rangle ,_k \langle x\rangle] = 1$.*

Proof. Put $H_1 = K_1 = \langle g\rangle$, $H_2 = \langle x\rangle$ and $K_2 = \langle x^n\rangle$. Then H_i, K_i are subgroups of G such that K_i is a subgroup of H_i of finite index. We deduce by [4, Theorem 2.3.3] that $[K_1,_k K_2] = [\langle g\rangle ,_k \langle x^n\rangle]$ is of finite index in $[H_1,_k H_2] = [\langle g\rangle ,_k \langle x\rangle]$. Thus $[\langle g\rangle ,_k \langle x\rangle]$ is of finite order and hence it is trivial since G is torsion-free. QED

Lemma 2. *Let G be a torsion-free locally nilpotent group. If G has a Baer subgroup of finite index, then G is a Baer group.*

Proof. Let H be a normal Baer subgroup of G of finite index, say n. So $x^n \in H$ for all $x \in G$. Therefore $\langle x^n\rangle$ is subnormal in G and hence $[G,_k \langle x^n\rangle] = 1$ for some positive integer k (see [4, p.276]). Thus $[\langle g\rangle ,_k \langle x^n\rangle] = 1$ for all $g \in G$. By Lemma 1, we deduce that $[\langle g\rangle ,_k \langle x\rangle] = 1$ for all $g \in G$, hence $[G,_k \langle x\rangle] = 1$. It follows that $\langle x\rangle$ is subnormal in G (see [4, p.276]) which gives that G is a Baer group. QED

Proposition 1. *Let G be a torsion-free locally graded group. If every proper subgroup of G is a Baer group, then so is G.*

Proof. If G is finitely generated then it is nilpotent-by-finite as it is locally graded. So it satisfies the maximal condition on subgroups. Consequently, every proper subgroup of G is nilpotent. Hence G is nilpotent since by [5] a finitely generated minimal non-nilpotent group has no proper subgroup of finite index. Therefore G is a Baer group. Assume now that G is not finitely generated. Hence G is locally nilpotent. Let B denotes the Baer radical of G. If G is not a Baer group, then B is proper in G. Since B contains all subnormal Baer subgroups of G, G/B must be a simple group. Hence it is cyclic of prime order since it is locally nilpotent [6, Corollary 1 of Theorem 5.27] and this is a contradiction to Lemma 2. Therefore, G is a Baer group. QED

Proposition 2. *Let G be a group in which every proper subgroup is (locally finite)-by-Baer. Then G is (locally finite)-by-Baer if it satisfies one of the following two conditions:*
(i) G is finitely generated and has a proper subgroup of finite index, or
(ii) G is not finitely generated.

Proof. (i) Suppose that G is finitely generated and let H be a proper normal subgroup of finite index in G. So H is finitely generated and hence it is (locally finite)-by-nilpotent. It follows that $\gamma_{k+1}(H)$ is locally finite for some integer $k \geq 0$. Clearly, $G/\gamma_{k+1}(H)$ is a finitely generated nilpotent-by-finite group. So it satisfies the maximal condition on subgroups. Consequently, every proper subgroup of $G/\gamma_{k+1}(H)$ is finite-by-nilpotent. In [2, Lemma 4] it is proved that a finitely generated locally graded group in which every proper subgroup is finite-by-nilpotent is itself finite-by-nilpotent. We deduce that $G/\gamma_{k+1}(H)$ is finite-by-nilpotent which gives that G is (locally finite)-by-nilpotent, as claimed.

(ii) Suppose now that G is not finitely generated and let x, y be two elements of finite order in G. The subgroup $\langle x, y\rangle$, being proper in G, is (locally finite)-by-Baer hence it is finite. Thus xy^{-1} is of finite order, so G has a torsion subgroup T which is locally finite as G is not finitely generated. If G/T is not finitely generated, then it is a torsion-free locally nilpotent group in which every proper subgroup is a Baer group. By Proposition 1, we deduce that G/T is a Baer group hence G is (locally finite)-by-Baer, as desired. Now if G/T is finitely generated, then there exists a finitely generated subgroup X of G such that $G = XT$. We deduce that G/T is nilpotent since X is proper in G. Therefore, G is (locally finite)-by-nilpotent, as required. QED

The previous proposition admits the following immediate consequence.

Corollary 1. *If G is a locally graded group in which every proper subgroup is (locally finite)-by-Baer, then G is (locally finite)-by-Baer.*

Proof of Theorem 1. Let G be a minimal non-((locally finite)-by-Baer) group. By Proposition 2, G is a finitely generated group which has no proper subgroup of finite index. So $G/Frat(G)$ is infinite and G has no proper locally graded factor group. In particular, G is perfect. Suppose that $G/Frat(G)$ is not simple and let N be a normal subgroup of G such that $Frat(G) \lneq N \lneq G$. Therefore there is a maximal subgroup M of G such that $N \nleq M$ hence $G = MN$. Now $G/N \simeq M/M \cap N$ is (locally finite)-by-nilpotent hence G/N is locally graded which is a contradiction by the above remark. Therefore $G/Frat(G)$ is simple. QED

3 Proof of Theorem 2

Lemma 3. *Let G be a torsion-free locally graded group satisfying the minimal condition on non-Baer subgroups. Then G is a Baer group.*

Proof. If G is not a Baer group, then it has a subgroup which is a minimal non-Baer group and this is a contradiction to Proposition 1. Therefore G is a Baer group. QED

Proof of Theorem 2. (i) First suppose that G is finitely generated. If G is infinite, there is an infinite chain of subgroups, $G > G_1 > G_2 > ...$, such that $|G_i : G_{i+1}|$ is finite for all integers i. Thus there exists a positive integer k such that G_k is (locally finite)-by-Baer and $|G : G_k|$ is finite. So G_k is (locally finite)-by-nilpotent. Therefore there is an integer $c \geq 0$ such that $\gamma_{c+1}(G_k)$ is locally finite. Now $G/\gamma_{c+1}(G_k)$ is a finitely generated nilpotent-by-finite group. So it satisfies the maximal condition on subgroups. It follows that $G/\gamma_{c+1}(G_k)$ satisfies the minimal condition on non-(finite-by-nilpotent) subgroups. But Lemma 4 of [2] can easily be extended to that a finitely generated locally graded group which satisfies the minimal condition on non-(finite-by-nilpotent) subgroups is itself finite-by-nilpotent. Consequently, $G/\gamma_{c+1}(G_k)$ is finite-by-nilpotent hence G is (locally finite)-by-nilpotent, as required.

(ii) Now assume that G is not finitely generated. So G has a torsion subgroup T which is locally finite. By (i) every finitely generated subgroup of G is (locally finite)-by-nilpotent. So G has a torsion subgroup T which is locally finite and G/T is a torsion-free locally nilpotent group satisfying the minimal condition on non-Baer subgroups. It follows by Lemma 3 that G/T is a Baer group which gives that G is a (locally finite)-by-Baer group. QED

4 Proof of Theorem 3

Lemma 4. *Let G be a torsion-free locally graded group having finitely many conjugacy classes of non-Baer subgroups. Then G is a Baer group.*

Proof. In [3, Proposition 3.3] it is proved that if $\mathfrak{X}$ is a subgroup closed class of groups and K is a locally graded group having finitely many conjugacy classes of non-$\mathfrak{X}$-subgroups, then K is locally in the class $\mathfrak{X}\mathfrak{F}$, where $\mathfrak{F}$ denotes the class of finite groups. So every finitely generated subgroup of G is a Baer-by-finite group hence it is nilpotent-by-finite. So G satisfies locally the maximal condition on subgroups. Now Lemma 4.6.3 of [1] states that if K is a group locally satisfying the maximal condition on subgroups and if H is a subgroup of K such that $H^x \leq H$ for some element x of K, then $H^x = H$. We deduce that G satisfies the minimal condition on non-Baer subgroups. Now Lemma 3 gives that G is a Baer group. QED

Lemma 5. *Let $\mathfrak{X}$ be a quotient closed class of groups and let G be a group having finitely many conjugacy classes of non-$\mathfrak{X}$-subgroups. If N is a normal subgroup of G then G/N has finitely many conjugacy classes of non-$\mathfrak{X}$-subgroups.*

Proof. This follows from the fact that if $N \leq K \leq G$, then

$$\left\{(K/N)^{xN} : xN \in G/N\right\} \subseteq \{K^x/N : x \in G\}.$$

QED

Proof of Theorem 3. (i) First assume that G is finitely generated. So by [3, Proposition 3.3] G is ((locally finite)-by-nilpotent)-by-finite. Let N be a normal subgroup of G of finite index such that N is (locally finite)-by-nilpotent and let T its torsion subgroup. So T is locally finite and G/T is a finitely generated nilpotent-by-finite group having finitely many conjugacy classes of non-((locally finite)-by-Baer) subgroups by Lemma 5. Hence G/T has finitely many conjugacy classes of non-(finite-by-nilpotent) subgroups. But in [8, Proposition 1.1] it is proved that a finitely generated locally graded group which has finitely many conjugacy classes of non-(finite-by-nilpotent) subgroups is itself finite-by-nilpotent. Consequently, G/T is finite-by-nilpotent, which gives that G is (locally finite)-by-nilpotent, as claimed.

(ii) Now assume that G is not finitely generated. So by (i) every finitely generated subgroup of G is (locally finite)-by-nilpotent. Hence G has a torsion subgroup T such that T is locally finite and G/T is a locally nilpotent group having finitely many classes of non-Baer subgroups. We deduce by Lemma 4 that G/T is a Baer group which implies that G is (locally finite)-by-Baer, as claimed. QED

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