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Dual deficiency one transitive partial parallelisms

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Abstract. It is shown that every transitive deficiency one finite partial parallelism in $PG(3,q)$ with standard group uniquely extends to a parallelism that is never isomorphic to its dual parallelism. The 'derived' parallelisms of standard parallelisms are also shown to be non-isomorphic to their duals.

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1 Introduction

Recently, there has been renewed interest in the study of parallelisms of $PG(3,q)$; coverings of the line set by a set of $1+q+q^2$ spreads, due both to new interconnections with spreads and flocks of quadric sets and their various constructions, and to group theoretic devices that enable the recognition of spreads of various types.

Biliotti, Jha and Johnson [1] have previously analyzed the so-called 'transitive deficiency one' partial parallelisms. A partial parallelism of deficiency one is a set of $q+q^2$ mutually line disjoint spreads. In this case, a transitive deficiency one partial parallelism is one that admits a group transitive on the spreads, with the additional assumption that the group is 'somewhat' linear'.

The main result of the analysis is as follows:

1 Theorem. *(Biliotti, Jha, Johnson [1]) Let $\mathcal{P}$ be a parallelism in $PG(3,q)$, $q=p^r$, p a prime, admitting an automorphism group G in $P\Gamma L(4,q)$, whose Sylow p-subgroups are in $PGL(4,q)$, or if $q=8$, that G itself is in $PGL(4,q)$. Assume G fixes one spread (the 'socle') and acts transitively on the remaining spreads.*

Then

(1) The socle is Desarguesian,

(2) the associated group G contains an elation group of order q^2 acting on the socle and

(3) the remaining spreads of the parallelism are isomorphic derived conical flock spreads.

With every finite spread, the associated dual spread obtained by taking a polarity of the associated projective space is also a spread. Similarly, given a finite parallelism, there is an associated 'dual parallelism' which is also a parallelism. The main fundamental question concerning parallelisms and their duals is how to distinguish between the two. That is, how does one find a class of parallelisms such that the dual parallelisms are always non-isomorphic to the original?

Recently, the author has considered this problem for transitive partial parallelisms of deficiency one.

2 Theorem. *(Johnson [3]) Let $\mathcal{P}$ be a parallelism in $PG(3,q)$. Assume that there exists a Desarguesian spread Σ in $\mathcal{P}$ and an collineation group G of $\mathcal{P}$ which fixes Σ and a component ℓ of Σ and acts transitively on the remaining spreads of $\mathcal{P}$.*

Assume that G contains an elation group E^+ of order q^2.

(1) If G contains a homology of odd order with axis ℓ which does not fix any spread of $\mathcal{P} - \{\Sigma\}$ then the dual parallelism $\mathcal{P}\delta$ is a parallelism that is not isomorphic to $\mathcal{P}$.

(2) If $q + 1 = 2^a$ for some integer a and G contains a homology of order $2^b \geq 8$ then the dual parallelism $\mathcal{P}\delta$ is a parallelism that is not isomorphic to $\mathcal{P}$.

In this article, we completely generalize the above result and show that deficiency one parallelisms also extend to parallelisms in $PG(3,q)$, whose dual parallelisms are never isomorphic to the original parallelism, provided the Sylow p-subgroups are linear, or more generally that there is an elation group E of order q^2 of the socle plane that acts on the parallelism.

It might be pointed out that if $\mathcal{P}$ is a parallelism admitting a group G that fixes one spread and acts transitively on the remaining spreads, then G need not be in $PGL(4,q)$, as there are many examples where the group is in $P\Gamma L(4,q)$ but not in $PGL(4,q)$. However, all of these examples have a corresponding Desarguesian plane Σ fixed and the group contains the full elation group E^+ of order q^2. We call the corresponding groups 'standard' in this situation.

On the other hand, generally speaking, it is an open question whether there can exist groups that can fix a spread and act transitively on the remaining spreads of a parallelism without the socle plane being Desarguesian and if the socle plane is Desarguesian, whether the group contains a elation group of order q^2 of the socle plane. That is, is every such transitive group a standard group?

We note that when $q = p^r$ and $(r,q) = 1$, certainly every transitive group is a standard group by the results of Biliotti, Jha and Johnson [1].

Thus, when the group is standard, there are always two non-isomorphic parallelisms for a parallelism arising from a transitive deficiency one partial parallelism.

Specifically, our main results are as follows:

3 Theorem. *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$, $q=p^r$, admitting a standard automorphism group G that fixes one spread Σ and acts transitively on the remaining spreads (for example, assume the Sylow p -subgroups are linear). Let $\mathcal{P}^{\perp}$ denote the associated dual spread.*

Then $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are never isomorphic.

4 Definition. Let $\mathcal{P}$ be a parallelism admitting a Desarguesian spread Σ and a group G fixing a component $x=0$. Let R be any regulus of Σ containing $x=0$. If the opposite regulus R^* is a subspread of one of the spreads of $\mathcal{P}$ then we shall say that the parallelism is 'standard'.

5 Theorem. *Let $\mathcal{P}$ be a standard parallelism in $PG(3,q)$, admitting a standard collineation group G that fixes one spread and is transitive on the remaining spreads. Let $\mathcal{P}_D$ be any 'derived' parallelism and let $\mathcal{P}_D^{\perp}$ be the corresponding dual spread.*

Then $\mathcal{P}_D$ is not isomorphic to $\mathcal{P}_D^{\perp}$.

2 The Group

In order to provide a proof to our main result, we begin by a further analysis of the group G in $\Gamma L(4,q)$ that fixes a spread Σ, the socle spread or plane, and acts transitively on the remaining spreads. When $q=8$, we assume that G is in $GL(4,q)$ and otherwise, we require that the Sylow p-subgroups of G, for $p^r=q$, are in $GL(4,q)$. In [1], it was shown that G also contains an affine elation group E of order q^2, say with axis $x=0$, acting on Σ, a Desarguesian spread. We begin with several fundamental results.

When $q=2$ then all spreads are Desarguesian and there are exactly two parallelisms in $PG(3,2)$. These parallelisms are non-isomorphic and dual to each other. Hence, we may assume that $q>2$.

6 Theorem. *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$ and let G be a collineation group of $\mathcal{P}$ that fixes a spread Σ, (the socle spread) and acts transitively on the set of spreads of $\mathcal{P}-\{\Sigma\}$. G is assumed to act in the translation complement of the associated translation plane π_{Σ}.*

(1) If G contains an elation subgroup E of order q^2 acting on Σ then Σ is Desarguesian and the group is standard.

(2) If G contains an elation group E of order q^2 then the elation axis $x=0$ is left invariant by the full collineation group of the parallelism.

Proof. Part (1) is proved in Biliotti, Jha and Johnson [1].

If the elation axis is not invariant then $SL(2,q^2)$ is generated by the elations. The order of $SL(2,q^2)$ is $q^2(q^4-1)$ and is a normal subgroup of G. Let k denote the length of the orbits of $SL(2,q^2)$ (the orbits of a normal subgroup are permuted by the full group G). Thus, k divides $q(q+1)$. Hence, there is a subgroup of order divisible by

$$q^2(q^4-1)/k = q(q-1)(q^2+1)(q(q+1)/k))$$

fixing a non-socle plane, which is a derived conical flock plane. If $q > 3$ then by Jha and Johnson [6], the group must fix the derived regulus net and hence be a subgroup of $\Gamma L(2,q)GL(2,q)$, where the product is central and the intersection is a central group of order $q-1$. But, q^2+1 does not divide the order of this group. Hence, the elation axis is invariant.

Now assume that $q = 3$. We know that we have a Desarguesian socle Σ and an elation group of order 9 with axis $x = 0$. If $x = 0$ is not invariant then $SL(2,9)$ is a normal subgroup of G generated by the elations. Furthermore, there is a collineation group $SL(2,3)$ that fixes a regulus net and, since the group is generated by central collineations, fixes all Baer subplanes incident with the zero vector of this regulus. Hence, $SL(2,3)$ leaves invariant a spread Σ'. Since G is transitive on $3(3+1)$ spreads, it follows that the order of G is divisible by $3^2 \cdot 2^5 \cdot 5$ and the stabilizer of a spread also has order divisible by 5. Hence, the stabilizer $G_{\Sigma'}$ has order divisible by $3 \cdot 5 \cdot 2^3$. Since $SL(2,3)$ is normal in $G_{\Sigma'}$, it follows that an element g of order 5 must permute four elation axes, a contradiction unless g fixes each axis, but then g is a kernel homology and has order dividing 8. Hence, $x = 0$ is invariant. $\boxed{QED}$

7 Notation. We shall denote the translation plane associated with a spread S as π_S. The group G will normally be considered acting in the translation complement of π_Σ.

8 Corollary. *Let $\mathcal{P}$ be a parallelism admitting a standard group G in the translation complement of the socle Σ.*

(1) The stabilizer $G_{\pi_{\Sigma'}}$ of a non-socle plane $\pi_{\Sigma'}$ normalizes a regulus-inducing Baer group of order q, acting on $\pi_{\Sigma'}$.

(2) The order of the full group of the parallelism G is as follows:

$$q^2(q^2-1) \mid \quad |G| \quad \mid q^2(q^2-1)^2 2r,$$

where $q = p^r$.

Proof. We know that the group acting on the conical flock plane cannot move the elation axis by Theorem 6. The elation axis becomes a Baer axis when considered in the stabilizer of a non-socle plane. Furthermore, the stabilizer on a non-socle plane is still a subgroup of the stabilizer of a component in the socle plane; a subgroup of $\Gamma L(2,q^2)$ fixing a component—a group of order $q^2(q^2-1)^2 2r$. Since we know that we have an orbit of length $q(q+1)$, an elation group of order q^2 and a kernel group of order $q-1$, the group has order divisible by $q^2(q^2-1)$. $\boxed{QED}$

9 Theorem. *Under the previous assumptions, let π_Σ be coordinatized by a field F containing K. Let F^* denote the kernel homology group of π_Σ of order q^2-1. Let Σ' be a non-socle spread and let $G_{\pi_{\Sigma'}}$. Then*

$$\left|G_{\pi_{\Sigma'}} \cap F^*\right| \mid Ev(2,q-1)(q-1).$$

Proof. Assume there is an element g in the kernel homology group of Σ that fixes a non-socle plane Σ_2, whose order divides q^2-1. Initially assume that the order of g is a prime power u^a. If u is odd, then g must fix at least two components of Σ_2. Assume that g does not fix a component of Σ_2 then u divides $(q^2-1, q^2+1) = 2$. Hence, $u = 2$ or we are finished. But, even when $u = 2$, there is a subgroup of $\langle g \rangle$ of order 2^{a-1} that fixes two components. Hence, we have that g or g^2 fixes two components of Σ_2, which are Baer subplanes of Σ that are fixed by g^i, for $i = 1$ or 2, assuming the order of g is a prime power. But, g^i then induces a kernel homology group on the Baer subplanes, implying that g^i is in K^*. Hence, we are finished or all odd prime power elements are in K^*. But, then g is in K^* or g^2 is in K^* when q is odd.

So, what we have shown is that the stabilizer of a non-socle plane intersected with the kernel homology group has order dividing $2(q-1)$ if q is odd and dividing $(q-1)$ if q is even. This completes the proof of the lemma. $\boxed{QED}$

10 Remark. For the remainder of this section, we shall assume that G is a standard group acting on a Desarguesian spread Σ and transitive on the non-socle spreads of a parallelism $\mathcal{P}$. We note that G is then a subgroup of $\Gamma L(2,q^2)$ acting on π_Σ and acts in $\Gamma L(4,q)$ when considering acting on the parallelism. For notational purposes, Σ' shall always denote a non-socle spread.

11 Corollary. *G is a solvable group.*

Proof. G has an elementary Abelian normal p-group E of order q^2 that acts transitively on the components of $\Sigma - \{x = 0\}$. The stabilizer of a second component $y = 0$ of Σ in $GL(2,q^2)$ is a subgroup of a direct product of two cyclic groups of orders q^2-1. $\boxed{QED}$

12 Corollary.

$$\begin{aligned}\left|G \cap GL(2,q^2)\right| &\mid q^2(q^2-1)(q-1)(2,q-1) \text{ and} \\ \left|G_{\Sigma'} \cap GL(2,q^2)\right| &\mid q(q-1)^2(2,q-1).\end{aligned}$$

Thus,

$$|G| \mid q^2(q^2-1)(q-1)2r(2,q-1),$$

where $q = p^r$.

Proof. Assume that the order of G is $q^2(q^2-1)t$ where t divides $(q^2-1)2r$. First consider $G \cap GL(2,q^2)$. Let g be an element of this group. Since the Sylow p-subgroup of $G_{\Sigma'} \cap GL(2,q^2)$ of the stabilizer of a non-socle plane Σ' has order q and is normal in G, assume that g has order relatively prime to p. Then, the order of g divides $(q^2-1)^2$ and hence must fix two components of Σ $x = 0$ and say $y = 0$, by basis change, since g fixes one component. In $G_{\Sigma'} \cap GL(2,q^2)$, since this element normalizes a regulus-inducing elation group E_1 of order q, it follows that g fixes an E_1-invariant regulus of Σ and since $y = 0$ is fixed, the E_1-invariant regulus may be taken as the standard regulus. Then, this stabilizer group permutes $q-1$ components of a regulus of Σ and hence, the stabilizer of three components is a kernel homology group. But, this group must be a subgroup of the kernel subgroup of order $(2,q-1)(q-1)$. So, $G_{\Sigma'} \cap GL(2,q^2)$ has order dividing $q(q-1)^2(2,q-1)$. Hence, the order of G divides $q^2(q^2-1)(q-1)(2,q-1)(2r)$. $\boxed{QED}$

We shall require the following result proved in Biliotti, Jha and Johnson [1].

13 Theorem. *The elation group E may be partitioned into exactly $q+1$ elation subgroups E_i, $i = 1,2,\ldots,q+1$, of order q, each of which fixes exactly q spreads other than the socle spread. E acts transitively on the q spreads fixed by each subgroup E_i. G acts transitively on the set of $q+1$ elation subgroups.*

14 Corollary. *If Δ is the set of $q+1$ sets of q spreads each fixed by an elation subgroup then the stabilizer of a set J of Δ normalizes a regulus-inducing elation subgroup.*

15 Theorem. *$G\cap$(Kernel Homology group of Σ) has order dividing $(2,q-1)(q-1)$. Furthermore, this group leaves invariant each spread of the parallelism.*

Proof. Let g be a kernel homology of Σ. Then g normalizes each regulus-inducing elation subgroup (since it commutes with the linear subgroup). Each regulus-inducing group fixes exactly q non-socle spreads. Hence, g permutes $q+1$ sets Γ_i of q non-socle spreads each fixed by a regulus-inducing elation subgroup E_i, and E is covered by the E_i, $i = 1,2,\ldots,q+1$. Let g^j have prime power order so divides q^2-1. Then, g^j fixes one of the q non-socle spreads in each set Γ_i, and hence g^j must in $(2,q-1)K^*$. Hence, every subgroup of g of odd prime power order is in K^*, implying that g is in K^* or g^2 is in K^*, from a previous lemma.

Now it also follows that E commutes with each regulus-inducing group E_i and commutes with g, so E permutes the q spreads fixed by E_i. It follows that E is transitive on these spreads since the maximum Baer group fixing a spread has order q. Also, E permutes the spreads fixed by g^j, so g^j fixes each spread fixed by every subgroup E_i, so that g^j fixes all $q(q+1)$ spreads other than Σ

and, of course, fixes Σ. As all elements of prime power order fix all spreads of the parallelism, it then follows that g fixes all spreads of the parallelism. $\boxed{QED}$

3 The Dual Parallelisms.

In this and the next sections, we ask if the parallelism obtained by taking a duality of the associated projective space, the dual parallelism, is isomorphic to the original. In this first section, we consider matrix representations of the dual parallelisms.

16 Lemma. *Represent $PG(3,q)$ as the lattice of subspaces of the 4-dimensional $GF(q)$-space V_4, with vectors as (x_1,x_2,y_1,y_2) and 3 -dimensional subspaces written as*

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

and a vector or 'point' incident with a 3-space or 'plane' exactly when

$$ax_1+bx_2+cy_1+dy_2=0.$$

Dualize by

$$(x_1,x_2,y_1,y_2) \leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

(1) We note that $x=0$ as $x_1=x_2=0$ and $y=0$ as $y_1=y_2=0$ are interchanged by the duality. A line that may be written in the form $y=xM$, where M is a 2×2 $GF(q)$-matrix as a set of vectors becomes a line that is the intersection of a set of 3-dimensional $GF(q)$-subspaces (intersection of a set of planes). We shall use the notation $\perp$ to denote the images under the associated mapping.

We have that

$$(y=xM)^{\perp}=(y=x(-M^{-t}),$$

where M^t denotes the transpose of M.

(2)

(a) If

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

is a group element of a parallelism $\mathcal{P}$ then

$$\begin{bmatrix} I & 0 \\ -A^t & I \end{bmatrix}$$

is a group element of the dual parallelism $\mathcal{P}^{\perp}$.

(b) If

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

is an element of the group of $\mathcal{P}$ then

$$\begin{bmatrix} B^{-t} & 0 \\ 0 & C^{-t} \end{bmatrix}$$

is an element of the group of $\mathcal{P}^{\perp}$.

PROOF. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

The vector (x, xM) maps under the duality to $\begin{bmatrix} x^t \\ M^t x^t \end{bmatrix}$.

Since

$$xx^t + x(-M^{-t})M^t x^t = 0,$$

it follows that

$$(y = xM)^{\perp} = (y = x(-M^{-t})),$$

which proves part (1).

In part (2)(a), (x, y) maps to $(x, xA + y)$, which dualizes to $\begin{bmatrix} x^t \\ A^t x^t + y^t \end{bmatrix}$.

Thus,

$$\begin{bmatrix} x^t \\ y^t \end{bmatrix} \longmapsto \begin{bmatrix} x^t \\ A^t x^t + y^t \end{bmatrix}.$$

To obtain the associated mapping on the vectors, we note that

$$(a, b) \text{ is on } \begin{bmatrix} x^t \\ y^t \end{bmatrix} \text{ if and only if } (a - bA^t, b) \text{ is on } \begin{bmatrix} x^t \\ A^t x^t + y^t \end{bmatrix}.$$

This proves part (a).

Similarly, if $(x, y) \longmapsto (xB, yC)$, then

$$\begin{bmatrix} x^t \\ y^t \end{bmatrix} \longmapsto \begin{bmatrix} B^t x^t \\ C^t y^t \end{bmatrix}.$$

Since it follows that

$$(a, b) \text{ is on } \begin{bmatrix} x^t \\ y^t \end{bmatrix} \text{ if and only if } (aB^{-t}, yC^{-t}) \text{ is on } \begin{bmatrix} B^t x^t \\ C^t y^t \end{bmatrix},$$

part (2)(b) is also proved. $\boxed{QED}$

17 Notation. If $\mathcal{P}$ is a parallelism then we denote the associated dual parallelism as $\mathcal{P}^{\perp}$. Hence, we may also use the notation $\mathcal{P}^{-t}$.

18 Lemma. *If $f : (x, y) \longmapsto (y, -x)$, then $\mathcal{P}^{-t}$ maps to $\mathcal{P}^{t}$ and*

(a) If

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

is a group element of a parallelism $\mathcal{P}$ then

$$\begin{bmatrix} I & A^t \\ 0 & I \end{bmatrix}$$

is a group element of the dual parallelism $\mathcal{P}^t$.

(b) If

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

is an element of the group of $\mathcal{P}$ then

$$\begin{bmatrix} C^{-t} & 0 \\ 0 & B^{-t} \end{bmatrix}$$

is an element of the group of $\mathcal{P}^t$.

(c) If $y = xM$ is a line of $\mathcal{P}$ then $y = xM^t$ is a line of $\mathcal{P}^t$.

19 Lemma. *Simply apply the indicated mapping.*

20 Notation. We may assume that for any parallelism $\mathcal{P}$, then $\mathcal{P}^t$, is isomorphic to the dual parallelism $\mathcal{P}^{\perp} = \mathcal{P}^{-t}$. Hence, we now refer to $\mathcal{P}^t$ as the dual parallelism of $\mathcal{P}$.

21 Lemma. *We may represent the spread of Σ by the following matrix spread set:*

$$x = 0, y = x \begin{bmatrix} u + \rho t & t\gamma \\ t & u \end{bmatrix}; u, t \in GF(q), \rho, \gamma \text{ constants in } GF(q),$$

where the indicated matrix set is a field of order q^2. Furthermore, the elation axis of E is denoted by $x = 0$ and we may take $\rho = 0$ if q is odd.

Proof. Simply note when q is odd that we may take γ as a non-square to represent a field. Since all finite Desarguesian spreads are isomorphic, we may begin with a representation of the given form. $\boxed{QED}$

22 Lemma. *The dual parallelism consists of the dual spreads of all of the spreads of the original parallelism.*

(1) If a given spread is represented as a matrix spread set $x = 0, y = xM$ then the dual spread may be represented in the form $x = 0, y = xM^t$, where M^t is the transpose of the matrix M.

(2) The elation group E has the general form:

$$\left\langle \begin{bmatrix} I & U \\ 0 & I \end{bmatrix}; U = \begin{bmatrix} u+\rho t & t\gamma \\ t & u \end{bmatrix} \right\rangle,$$

considered as in the previous lemmas.

Then, an elation group in $\Sigma^{\perp}$, the dual spread of Σ, is given as

$$\left\langle \begin{bmatrix} I & U^t \\ 0 & I \end{bmatrix}; U = \begin{bmatrix} u+\rho t & t\gamma \\ t & u \end{bmatrix}^t \right\rangle.$$

(3) If a group of Σ has the form:

$$\left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\rangle,$$

where A and B are non-zero elements of the field of order q^2, then in $\Sigma^{\perp}$, there is a group of the form

$$\left\langle \begin{bmatrix} B^t & 0 \\ 0 & A^t \end{bmatrix} \right\rangle.$$

Proof. This follows exactly as above. $\boxed{QED}$

23 Lemma. *Assume that a parallelism $\mathcal{P}$ is isomorphic to its dual parallelism $\mathcal{P}^t$, and σ is any element of $\Gamma L(4,q)$ then $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\perp}\sigma$.*

Proof. Simply note that $\mathcal{P}^{\perp}\sigma$ is isomorphic to $\mathcal{P}^{\perp}$ and use transitivity of isomorphism.

24 Lemma. *Let*

$$\sigma : (x,y) \longmapsto (x,y) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

(1) $\mathcal{P}^{\perp}\sigma$ contains Σ in the form given originally.

(2) $\mathcal{P}^{\perp}\sigma$ contains E, in the original form and if

$$\left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\rangle$$

is in G acting on $\mathcal{P}$ then

(a) if q is odd

$$\left\langle \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} \right\rangle,$$

acts on $\mathcal{P}^{\perp}\sigma$ and

(b) if q is even

$$\left\langle \begin{bmatrix} B/\det B & 0 \\ 0 & A/\det A \end{bmatrix} \right\rangle,$$

acts on $\mathcal{P}^{\perp}\sigma$.

$\boxed{QED}$

Proof. Consider the action of σ on the representation (the transposed spread; the dual spread of Σ)

$$x = 0, y = x \begin{bmatrix} u + \rho t & t \\ t\gamma & u \end{bmatrix}; u, t \in GF(q).$$

We note that σ fixes $x = 0$ and $y = 0$ and

$$y = x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u + \rho t & t \\ t\gamma & u \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} u & t\gamma \\ t & u + \rho t \end{bmatrix}.$$

Hence, for q odd, $\rho = 0$ and we have verified that Σ is within $\mathcal{P}^{\perp}\sigma$. Assume that q is even and let $v = u + \rho t$ so that $u = v + \rho t$. That is,

$$\begin{bmatrix} u & t\gamma \\ t & u + \rho t \end{bmatrix} = \begin{bmatrix} v + \rho t & t\gamma \\ t & v \end{bmatrix}.$$

If q is even, ρ is non-zero and $\begin{bmatrix} u & t\gamma \\ t & u + \rho t \end{bmatrix} = \begin{bmatrix} u + \rho t & t\gamma \\ t & u \end{bmatrix}^{-1} \Delta_{u,t}$, where $\Delta_{u,t}$ denotes the determinant of $\begin{bmatrix} u + \rho t & t\gamma \\ t & u \end{bmatrix}$. This proves (1). To prove (2), we merely take the conjugate of the groups by σ. Hence,

$$\begin{aligned}
&\left\langle \sigma^{-1}\begin{bmatrix} I & U^t \\ 0 & I \end{bmatrix}\sigma; U = \begin{bmatrix} u+\rho t & t\gamma \\ t & u \end{bmatrix}\right\rangle \\
= &\left\langle \begin{bmatrix} I & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 & I \end{bmatrix}; U = \begin{bmatrix} u+\rho t & t\gamma \\ t & u \end{bmatrix}\right\rangle \\
= &\left\langle \begin{bmatrix} I & U \\ 0 & I \end{bmatrix}; U = \begin{bmatrix} u+\rho t & t\gamma \\ t & u \end{bmatrix}\right\rangle, \text{ when } q \text{ is odd and } \rho = 0 \\
= &\left\langle \begin{bmatrix} I & U^* \\ 0 & I \end{bmatrix}; U^* = \begin{bmatrix} u & t\gamma \\ t & u+\rho t \end{bmatrix}\right\rangle, \text{ where } q \text{ is even.}
\end{aligned}$$

However, since E has order q^2, we obtain the same general form for either q odd or even.

$$\begin{aligned}
&\left\langle \sigma^{-1}\begin{bmatrix} B^t & 0 \\ 0 & A^t \end{bmatrix}\sigma\right\rangle \\
= &\left\langle \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}\right\rangle \\
= &\left\langle \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\right\rangle, \text{ or } \left\langle \begin{bmatrix} B/\det B & 0 \\ 0 & A/\det A \end{bmatrix}\right\rangle,
\end{aligned}$$

respectively as q is odd or even, noting that B, A are in

$$\left\{ \begin{bmatrix} u+\rho t & \gamma t \\ t & u \end{bmatrix}; u, t \in GF(q) \right\}.$$

$\boxed{QED}$

25 Lemma. *We assume that E_1 is*

$$E_1 = \left\langle \tau_u = \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u \in GF(q) \right\rangle,$$

the standard regulus-inducing group.

Then, there is a unique partition of E into regulus-inducing subgroups E_c, where

$$E_c = \left\langle \begin{bmatrix} I & uc \\ 0 & I \end{bmatrix}; u \in GF(q) \right\rangle,$$

and the set of $q+1$ regulus-inducing elation groups corresponds to the cosets of $cGF(q)^*$ of $GF(q^2)^*/GF(q)^*$.

PROOF. Since the regulus-inducing groups are in an orbit under a collineation group of Σ, it follows that a regulus-inducing subgroup is exactly an image of E_1 under a collineation of Σ of the form $(x,y) \longmapsto (xa, yb)$, where the action is conjugation. Hence, E_1 maps onto

$$E_{a^{-1}b} = \left\langle \begin{bmatrix} I & ua^{-1}b \\ 0 & I \end{bmatrix}; u \in GF(q) \right\rangle.$$

Hence, the $(q+1)$ regulus-inducing elation subgroups then have the form

$$E_c = \left\langle \begin{bmatrix} I & uc \\ 0 & I \end{bmatrix}; u \in GF(q) \right\rangle,$$

for $c \in GF(q^2)$. Moreover, two such elation groups E_c and E_d are identical if and only if $cd^{-1} \in GF(q)$. Hence, the $q+1$ regulus-inducing elation groups correspond to the $q+1$ cosets of $GF(q^2)^*/GF(q)^*$. $\boxed{QED}$

4 The Parallelism and the Dual are not Isomorphic, $q \neq 3, 7$.

Assume that $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\perp}$. Then $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\perp}\sigma$ by an automorphism ρ in $\Gamma L(4,q)$ mapping $\mathcal{P}$ onto $\mathcal{P}^{\perp}\sigma$. Since Σ is in both parallelisms and is the unique Desarguesian spread in each, it follows that Σ must be left invariant by ρ. That is, ρ is a collineation group of Σ. Since both parallelisms admit the collineation group E and the axis cannot be moved by an automorphism group of either parallelism, it follows that ρ must leave invariant the axis $x = 0$ of E. Since E has order q^2, it is transitive on the remaining components and we may assume that ρ fixes both $x = 0$ and $y = 0$. Represent ρ by $(x,y) \longmapsto (x^{\tau}c, y^{\tau}d)$, for c, d in $GF(q^2)$ associated with Σ and in the form defined in the previous lemmas, and where τ is an automorphism of $GF(q^2)$.

We recall that

$$\begin{array}{rcl} \left|G \cap GL(2,q^2)\right| & \mid & q^2(q^2-1)(q-1)(2,q-1), \\ \left|G_{\Sigma'} \cap GL(2,q^2)\right| & \mid & q(q-1)^2(2,q-1), \\ |G| & \mid & q^2(q^2-1)(q-1)2r(2,q-1). \end{array}$$

Furthermore, we know that

$$G \cap GL(2,q^2) = E(G \cap GL(2,q^2)_{(y=0)}.$$

In the following lemmas, we assume the above conditions.

26 Lemma. *If $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\perp}$ then every element of*

$$(G \cap GL(2, q^2))_{(y=0)}$$

may be represented in the following form:

$$g_{a,\alpha} : (x, y) \longmapsto (xa, ya^{-1}\alpha),$$

for some elements a and α of $F \simeq GF(q^2)$, where a is non-zero and the order of α divides $(2, q-1)(q-1)$.

PROOF. Assume that $g \in (G \cap GL(2, q^2))_{(y=0)}$ and represent $g = g_{a,b}$: $(x, y) \longmapsto (xa, yb)$, for $a, b \in F^* = F - \{0\}$. Our previous section shows that $g_{b/\delta_b, a/\delta_a}$ acts on $\mathcal{P}^{\perp\sigma}$. Furthermore, we note that $E^\rho = E$, and $\langle g_{a,b}\rangle^\rho = \langle g_{a^\tau, b^\tau}\rangle$. Hence, we have in $\mathcal{P}^{\perp}\sigma$ the automorphism groups $\left\langle g_{b/\delta_b, a/\delta_a}\right\rangle$ and $\langle g_{a^\tau, b^\tau}\rangle$, where for q odd, we have $\delta_b = \delta_a = 1$ and for q even, δ_c is the determinant of c, for $c = a$ or b. We note that $\langle g_{a,b}\rangle = \langle g_{a,b}\rangle^\tau = \langle g_{a^\tau, b^\tau}\rangle$, since $(p^c, q^2 - 1) = 1$. Hence, we must have $\left\langle g_{b/\delta_b, a/\delta_a}\right\rangle$ and $\langle g_{a,b}\rangle$ are collineation groups of the parallelism $\mathcal{P}^{\perp}\sigma$.

First assume that q is odd. Then

$$g_{a,b}g_{b,a} : (x, y) \mapsto (x(ab), y(ab)),$$

is a collineation of the parallelism $\mathcal{P}^{\perp}\sigma$. However, since Σ has exactly the same form in both parallelisms, it follows that the above element is in the kernel homology group of Σ. But this means that $g_{a,b}^{\perp} = g_{a,b}g_{b,a} \in 2K^*$. This implies that the order of ab divides $2(q-1)$. Let $2K^* = \langle z\rangle$, z of order $2(q-1)$ if q is odd Thus, we obtain a collineation

$$h_{a,\alpha} : (x, y) \longmapsto (xa, ya^{-1}\alpha), a \in GF(q^2) - \{0\} ,$$

where $\alpha \in 2K^*$.

Now assume that q is even and consider

$$g_{a,b}^{q-1} g_{b/\delta_b, a/\delta_a}^{q-1} : (x, y) \longmapsto (x(ab)^{q-1}, y(ab)^{q-1}).$$

Since this is a kernel homology of Σ and q is even, then $(ab)^{q-1}$ is in K^*. This implies that $(ab)^{(q-1)^2} = 1$. Since $((q-1)^2, q^2-1) = (q-1)(q-1, q+1) = (q-1)$, it follows that the order of ab divides $q-1$, so that $ab \in K^*$. Letting $ab = \beta$ in K^* , we have a collineation:

$$k_{a,\beta} : (x, y) \longmapsto (xa, ya^{-1}\beta), a \in GF(q^2) - \{0\}.$$

So, in general, we may assume that the collineation

$$h_{a,\alpha} : (x, y) \longmapsto (xa, ya^{-1}\alpha), a \in GF(q^2) - \{0\} ,$$

exists, where $\alpha \in (2, q-1)K^*$. This proves the lemma. $\boxed{QED}$

27 Lemma. *If $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\perp}$ then there is a non-socle spread Σ' such that*

(1) $G_{\Sigma'} \cap ((G \cap GL(2,q^2))_{(y=0)})$ is a subgroup of index dividing $(2,q-1)(q-1)$.

(2) $(G \cap GL(2,q^2))_{(y=0)}$ has order dividing $(q-1)^3(2,q-1)^2$ and

(2) $\left|(G \cap GL(2,q^2)_{(y=0)})^{(2,q-1)(q-1)} = G_{\Sigma',y=0} \cap GL(2,q^2)\right| \mid (q-1)^2(2,q-1)$.

Proof.

$$h_{a,\alpha}^{(2,q-1)(q-1)} : (x,y) \longmapsto (xa^{(2,q-1)(q-1)}, ya^{-(2,q-1)(q-1)}), a \in GF(q^2) - \{0\}.$$

We note that

$$h_{a,\alpha}^{(2,q-1)(q-1)} : (c,c^q) \longmapsto (ca^{(2,q-1)(q-1)}, c^q a^{-(2,q-1)(q-1)})$$

and $(ca^{(2,q-1)q-1}, c^q a^{-(2,q-1)(q-1)})$ is a point of $y = x^q$ since

$$(ca^{(2,q-1)(q-1)})^q = c^q a^{(2,q-1)(1-q)} = c^q a^{-(2,q-1)(q-1)}.$$

Hence, $h_{a,\alpha}^{(2,q-1)(q-1)}$ fixes a Baer subplane of Σ, implying that $h_{\alpha,\alpha}^{(2,q-1)(q-1)}$ also fixes a spread Σ' containing $y = x^q$ as a component.

We know that $\left|G_{\Sigma'} \cap GL(2,q^2)\right|$ divides $q(q-1)^2(2,q-1)$. Hence, $h_{a,\alpha}^{(2,q-1)(q-1)}$ has order dividing $(q-1)^2(2,q-1)$, so that $h_{a,\alpha}$ has order dividing $(q-1)^3(2,q-1)^2$. So

$$(G \cap GL(2,q^2)_{(y=0)})^{(2,q-1)(q-1)} = G_{\Sigma',y=0} \cap GL(2,q^2)$$

is a subgroup of order dividing $(q-1)^2(2,q-1)$. $\boxed{QED}$

28 Lemma. *If q^2-1 has a p-primitive divisor u then $\mathcal{P}$ cannot be isomorphic to $\mathcal{P}^{\perp}$.*

Proof. Assume that q^2-1 has a p-primitive prime divisor u. Then u divides $q+1$ and there is an element g of order u, which is necessarily in $GL(4,q)$. But, g is in $\Gamma L(2,q^2)$ and $GL(4,q) \cap GL(2,q^2)$ has index 2 in $GL(4,q) \cap \Gamma L(2,q^2)$. Since u odd is odd, g must in $GL(2,q^2)$ (note that u divides $p^{u-1}-1$). However, if the parallelism and its dual are isomorphic then u must divide $(q-1)^3(2,q-1)^2$, a contradiction. $\boxed{QED}$

29 Lemma. *If q^2-1 does not have a p-primitive divisor and q is not $3,7$ or 8 then $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are not isomorphic.*

Proof. Hence, $q = p$ and $q+1 = p+1 = 2^a$. All elements of $(G \cap GL(2,q^2))_{(y=0)}$ have orders dividing $(q-1)^3(2,q-1)^2$. In this situation, $(q-1)_2^3(2,q-1)^2 = 2^5$. Now we have a subgroup $G_{y=0}$ of order divisible by $(q+1)(q-1)_2 = 2^{a+1}$, the linear part (the intersection with $GL(2,p^2)$) of which is divisible by 2^a.But, the linear part divides 2^5. Hence, $a = 1,2,3,4$ or 5, so that

$p = 3, 7$, or 31, as $2^4 - 1 = 15$. Part of the order involves $2K^*$, which fixes each spread. Thus, in the case where the part referred to as $(2, q-1)$ actually exists is when each spread is stabilizer by a group of order $2(p-1)_2 = 4$. This means that we must have a 2-group or order at least $2(q+1)(q-1)_2/2$ in the linear part. This implies that the case $p = 31$ does not actually occur. $\boxed{QED}$

30 Lemma. *If $q = 8$ then $\mathcal{P}$ is not isomorphic to $\mathcal{P}^\perp$.*

PROOF. For $q = 8$, we have a subgroup of $GL(2, q^2)$ of order divisible by 63. Take an element g of order 3. Then, $g^{(q-1)} = g^7$ has order 3 and must divide $(q-1)^3 = 7^3$, a contradiction. $\boxed{QED}$

Hence, the only special cases are $q = p = 3$ or 7.

5 $q = 3$ or 7.

Now assume that $q = 3$ or 7 and assume that the parallelism $\mathcal{P}$ is isomorphic to the dual parallelism $\mathcal{P}^\perp$. Since -1 is a non-square in $GF(q)$, we may represent Σ as follows:

$$\Sigma : x = 0, y = x \begin{bmatrix} u & -t \\ t & u \end{bmatrix}.$$

Note that Σ appears in $\mathcal{P}$ and $\mathcal{P}^\perp$ is exactly the same form (even before the σ-map) since the spread itself is self-transpose.

31 Lemma. *If $(x, y) \longmapsto (x^q b, y^q c)$ of $\Gamma L(2, q^2)$ acts on $\mathcal{P}$ then $(x, y) \longmapsto (x^q c, y^q b)$ acts on $\mathcal{P}^\perp$.*

PROOF. With appropriate choice of basis coordinates,

$$x^q = x \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $x^q \begin{bmatrix} u & -t \\ t & u \end{bmatrix} = x \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & -t \\ t & u \end{bmatrix} = x \begin{bmatrix} -u & t \\ t & u \end{bmatrix}$. Hence, it follows easily that $(x, y) \longmapsto (xc^t \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, yb^t \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix})$ acts on $\mathcal{P}^\perp$.

But, if $x^q b = x \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & -t \\ t & u \end{bmatrix}$ then

$$\begin{aligned} xb^t \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} &= x \begin{bmatrix} u & t \\ -t & u \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= x \begin{bmatrix} -u & t \\ t & u \end{bmatrix}. \end{aligned}$$

Therefore,

$$x^q b = x b^{tq}.$$

$\boxed{QED}$

32 Lemma. *$G \cap GL(2, q^2)$ cannot act transitively on $\mathcal{P} - \{\Sigma\}$, if $\mathcal{P}$ is isomorphic to $\mathcal{P}^{\perp}$.*

Proof. From our previous analysis, we know that $G \cap GL(2, q^2)$ has elements of the following form:

$$g_a : (x, y) \longmapsto (xa, ya^{-1}k),$$

where $|k|$ divides $2(q-1)$, for $q = 3$ or 7. If $G \cap GL(2, q^2)$ is transitive on $\mathcal{P} - \{\pm\}$, we also know that the set of these elements is transitive on the set of $q+1$ regulus-inducing elation groups E_c.

Note that an element $\begin{bmatrix} I & uI \\ 0 & I \end{bmatrix}$ of E_1 maps under $g_a^{(q-1)}$ to

$$\begin{bmatrix} I & ua^{-2(p-1)}k^{(p-1)} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & ua^{-2(p-1)} \\ 0 & I \end{bmatrix}.$$

If $q = 3$ then $a^{-4} = \pm 1$, implying that g_a^2 normalizes E_1. Since $G \cap GL(2, q^2)$ is Abelian, it follows that, when $q = 3$, the stabilizer of E_1 has index dividing 2. Hence, this group cannot act transitively on four subgroups. Let $q = 7$ and consider $g_a^{2(q-1)}$ with corresponding element $a^{-4(6)} = \pm 1$. Hence, g_a^{12} has index dividing 2, implying that the orbit lengths are less than 4, a contradiction if the group is to be transitive.

This completes the proof of the lemma. $\boxed{QED}$

Hence, we must have a non-linear element

$$\theta : (x, y) \longmapsto (x^q b, y^q c),$$

on $\mathcal{P}$ implying, as in our previous analysis, that we have an element

$$\theta^{\perp} : (x, y) \longmapsto (x^q c, y^q b)$$

that acts on $\mathcal{P}^{\perp}$.

Let $\rho : (x, y) \longmapsto (x^{\tau} e, y^{\tau} f)$ be the isomorphism from $\mathcal{P}$ onto $\mathcal{P}^{\perp}$. Since E exists as a group of the parallelism, it follows that we may assume that ρ fixes both $x = 0$ and $y = 0$ and is a collineation of Σ, as Σ is common to both $\mathcal{P}$ and $\mathcal{P}^{\perp}$. Furthermore, since $G_{y=0}$ is transitive on the parallelism-inducing subgroups, we may assume that ρ leaves invariant the standard regulus-inducing group. Hence, it follows that $e^{-1} f \in GF(q)$. The conjugate

$$\theta^{\rho} : (x, y) \longmapsto (x^q e^{-q+1} b^{\tau}, y^q f^{-q+1} c^{\tau}).$$

then acts on $\mathcal{P}^{\perp}$, and note that $e^{-q+1} = f^{-q+1}$.

Since $q = 3$ or 7, it follows that $\tau = 1$ or q.

33 Lemma. *There must be a non-linear element $\theta : (x, y) \longmapsto (x^q b, y^q c)$ such that $b^{-1}c$ has order either 8 if $p = 3$, or 16 or 48 if $p = 7$.*

PROOF. $\begin{bmatrix} I & uI \\ 0 & I \end{bmatrix}$ maps to $\begin{bmatrix} I & ua^{-2}k \\ 0 & I \end{bmatrix}$, under an element of the form g_a.

Furthermore, $\begin{bmatrix} I & uI \\ 0 & I \end{bmatrix}$ is mapped to $\begin{bmatrix} I & u^p b^{-1}c \\ 0 & I \end{bmatrix}$ under a non-linear element θ. Since k has order dividing $2(q-1)$, it follows that $b^{-1}c$ cannot be a square, since $G_{y=0}$ is transitive on the regulus-inducing elation groups. QED

34 Lemma. $\tau = 1$.

PROOF. Consider

$$\theta^{\rho}\theta : (x, y) \longmapsto (xc^q e^{1-q} b^{\tau}, yb^q e^{1-q} c^{\tau}).$$

Assume that $\tau = q$, then $\theta^{\rho}\theta$ is a kernel homology group of Σ acting on a parallelism containing Σ, so that $c^q e^{1-q} b^q$ has order dividing $2(q-1)$. If $q = 3$ then $(bc)^q e^{1-q}$ has order dividing 4, implies that the order of bc divides 4 and the order of $b^{-1}c$ and the order of bc are equal since $(bc) = b^2(b^{-1}c)$ and the order of b^2 divides 4. If $q = 7$ then $(bc)^q e^{1-q}$ has order dividing $2(7-1) = 12$. Since the order of e^6 divides 8, then $(bc)^{28}e^{24}$ has order dividing 3, implying that $(bc)^4$ has order dividing 3, so that the order of bc divides 12. But, $b^{-1}c$ has order either 16 or 48 and $b^2(b^{-1}c) = bc$. Thus, $(b^2(b^{-1}c))^{12} = 1 = b^{24}(b^{-1}c)^{12}$ and $b^{24} = \pm 1$. Hence, $(b^{-1}c)^{12} = \pm 1$, implying that the order of $b^{-1}c$ divides 24, a contradiction. QED

35 Lemma. *If $q = 3$ then $\mathcal{P} \ncong \mathcal{P}^{\perp}$.*

PROOF. Recall that

$$\theta^2 : (x, y) \longmapsto (xb^4, yc^4).$$

If $c^4 = b^4$ then the order of $b^{-1}c$ is 4, a contradiction. Hence, we always obtain an element

$$z : (x, y) \longmapsto (x, -y)$$

by multiplication by a kernel homology of order 2, if necessary.

$$\eta = z\theta^{\rho}\theta^{-1} : (x, y) \longmapsto x(c^{-1}b), -y(c^{-1}b)^{-1}.$$

Note that $(c^{-1}b)^4 = -1$. Since $(c^{-1}b)^3 = -(c^{-1}b)^{-1}$, it follows immediately that η fixes $y = x^q$, which is a 2-dimensional $GF(q)$-space so that η fixes a non-socle spread Σ'. There is a unique regulus-inducing group $E_{\Sigma'}$ that fixes Σ', implying that η must normalize $E_{\Sigma'}$. But for $d = c^{-1}b$, we have that $\begin{bmatrix} I & uw \\ 0 & I \end{bmatrix}$ maps under η to $\begin{bmatrix} I & uw(-d^{-2}) \\ 0 & I \end{bmatrix}$, which means that for some $w \in GF(q^2) - \{0\}$, $d^{-2} \in GF(q)$, a contradiction. $\boxed{QED}$

36 Lemma. *If* $q = 7$ *then* $\mathcal{P} \not\cong \mathcal{P}^{\perp}$.

Proof. So, we have

$$\theta^2 : (x, y) \longmapsto (xb^8, yc^8).$$

First assume that the order of $b^{-1}c$ is 16. Then $(b^{-1}c)^8 = -1$, so that $b^8 = -c^8$. Since we may multiply by the $GF(7)$-kernel homologies, it follows that we obtain an element

$$z : (x, y) \longmapsto (x, -y).$$

Just as before, we obtain an element μ where

$$\mu : (x, y) \longmapsto (x(c^{-1}b), -(c^{-1}b)^{-1}.$$

Since $(c^{-1}b)^8 = -1$, then $(c^{-1}b)^7 = -(c^{-1}b)^{-1}$, so that μ fixes $y = x^q$ and hence fixes a non-socle spread Σ'. But, μ then must normalize a regulus-inducing group, so that $\begin{bmatrix} I & uw \\ 0 & I \end{bmatrix}$ maps under μ to $\begin{bmatrix} I & uw(-d^{-2}) \\ 0 & I \end{bmatrix}$, where $c^{-1}b = d$. This implies that $d^{-2} \in GF(7) - \{0\}$ so that the order of d^{-2} divides 6, and the order of d divides 12, a contradiction since the order of d is 16. Now let $d = c^{-1}b$ have order 48. Then

$$\theta^2 : (x, y) \longmapsto (xb^8, yc^8),$$

and note that $c^{-8}b^8$ has order 6 and thus, we may assume by multiplication of a $GF(q)$-kernel homology that we obtain a collineation

$$v_\alpha : (x, y) \longmapsto (x, y\alpha),$$

where α generates $GF(7) - \{0\}$. Thus, we obtain a collineation

$$\omega : (x, y) \longmapsto (xd, yd^{-1}\alpha).$$

Since d has order 48, $d^8 = \alpha$, without loss of generality. Since $d^7 = \alpha d^{-1}$, it follows that ω fixes $y = x^q$ and hence fixes a non-socle spread Σ' and thus must normalize a regulus-inducing group. Just as before, it now follows that $\alpha d^{-2} \in GF(7) - \{0\}$, implying that d^{-2} has order dividing 6, a contradiction. $\boxed{QED}$

Hence, we have shown that if $q = 3$ or 7, the parallelism $\mathcal{P}$ and the parallelism $\mathcal{P}^{\perp}$ cannot be isomorphic.

6 The Main Results.

Thus, we have proved the following theorem.

37 Theorem. *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$, $q = p^r$, admitting a standard automorphism group G that fixes one spread Σ and acts transitively on the remaining spreads (for example, assume the Sylow p -subgroups are linear). Let $\mathcal{P}^{\perp}$ denote the associated dual spread.*

Then $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are never isomorphic.

6.1 The 'Derived' Parallelisms

Let $\mathcal{P}$ be a parallelism of $PG(3,q)$ containing a spread Σ and admitting a standard collineation group G in $\Gamma L(4,q)$ acting transitively on the spreads of $\mathcal{P}$ not equal to Σ. Choose any spread Σ' and realize that Σ' is a derived conical flock spread admitting a Baer collineation group of order q and the fixed point space of this Baer group defines a regulus net R'. Derive R' to R'^* producing the conical flock spread Σ'^*. If R'^* is a regulus of Σ we shall say that the parallelism is of 'standard type'. All of the known examples of parallelisms admitting standard groups are of standard type. If the parallelism is of standard type, denote the Hall spread by Σ^* by the derivation of R'^*.

38 Theorem. *For any standard parallelism $\mathcal{P}$, The set of spreads*

$$\Sigma^* \cup \Sigma'^* \cup (\mathcal{P} - \{\Sigma, \Sigma'\}$$

is a parallelism, called a 'derived parallelism'.

Proof. Choose any line L of $PG(3,q)$. Assume that L is in $\Sigma \cup \Sigma'$. If L is in R'^* then L is in Σ^*. If L is in R' then L is in Σ^*. If L is not in $\Sigma \cup \Sigma'$ then L is in a unique spread Σ'' of $\mathcal{P} - \{\Sigma, \Sigma'\}$. $\boxed{QED}$

39 Theorem. *Let $\mathcal{P}$ be a standard parallelism in $PG(3,q), q > 3$, admitting a standard collineation group G that fixes one spread and is transitive on the remaining spreads. Let $\mathcal{P}_D$ be any 'derived' parallelism and let $\mathcal{P}_D^{\perp}$ be the corresponding dual spread.*

Then $\mathcal{P}_D$ is not isomorphic to $\mathcal{P}_D^{\perp}$.

Proof. In $\mathcal{P}_D$, there is a unique conical flock spread Σ'^*, the remaining spreads are either Hall, i.e. Σ^*, or derived conical flock spreads $\Sigma'' \in \mathcal{P} - \{\Sigma, \Sigma'\}$. Suppose that τ is an isomorphism from $\mathcal{P}_D$ onto $\mathcal{P}_D^{\perp}$. Since we may assume that Σ is in both parallelism $\mathcal{P}$ and $\mathcal{P}^{\perp}$, in exactly the same form, we may assume that Σ^* is in both parallelisms, in exactly the same form. It follows that τ maps Σ^* onto Σ^*. Since $q > 3$, the full collineation group of Σ^* is the inherited group; i.e. leaves invariant the relevant derivable net. Hence, τ induces a mapping from

Σ onto Σ. Furthermore, there is a unique conical flock spread Σ'^{*} in $\mathcal{P}_D$ which must map to the unique conical flock spread $\Sigma'^{*\perp}$ in $\mathcal{P}_D^{\perp}$. Since it is also true that for conical flock spreads, the inherited group is the full group, it follows that Σ' in $\mathcal{P}$ maps to $\Sigma'^{*\perp *}$ in $\mathcal{P}^{\perp}$. We note from Johnson [2] that the processes of derivation and transpose are commutative, in that the spread obtained from derivation then transpose is identical to the spread obtained from transpose and then derivation. Hence, $\Sigma'^{*\perp *} = \Sigma'^{**\perp} = \Sigma'^{\perp}$.

Hence, we have an induced isomorphism from $\mathcal{P}$ to $\mathcal{P}^{\perp}$, a contradiction. $\boxed{QED}$

7 Non-Standard Groups and Non-Standard Parallelisms.

We have analyzed parallelisms of $PG(3,q)$ admitting a group G that fixes one spread and is transitive on the remaining spreads. If G is a standard group then we have shown that the parallelism and its dual can never be isomorphic. However, it is an open question is there can be such transitive deficiency one partial parallelisms that do not admit standard groups. All of the known examples admit standard groups and for example, if $(p,r)=1$, all transitive groups are also standard.

40 Problem. Study deficiency one transitive partial parallelisms. Show that the spread extending the deficiency one partial parallelism to a parallelism is Desarguesian. Furthermore, show that the remaining spreads of the parallelism are derived conical flock spreads.

Note that all of these would be true if it could be shown that any such transitive group is standard.

We also mentioned 'standard' parallelisms. In this setting, we require that the socle plane have $q(q+1)$ reguli sharing $x=0$, and this will force the socle plane Σ to be Desarguesian. Furthermore, if one of the reguli R of Σ has its opposite regulus R^* as a subspread of a spread Σ' of the parallelism then Σ' cannot contain another opposite regulus R_1^* to a regulus R_1 of Σ. To see this note that $x=0$ is a Baer subplane of Σ' uniquely defining a regulus R', which we are requiring is an opposite regulus of a regulus of Σ. Hence, there are $q(q+1)$ such reguli one each in the non-socle spreads of the parallelism.

Although it might appear that if there is a parallelism admitting a Desarguesian socle plane Σ, where the remaining spreads are derived conical flock spreads, it is not clear that the corresponding conical flock spread share reguli of Σ. If they do not then the 'derived structures' of the previous section would not actually be parallelisms. That is, if the parallelism is not standard, this is a very wild situation.

7.1 Non-Standard Parallelisms

When $q = 2$, there are exactly two parallelisms in $PG(3, 2)$, and these parallelisms admit $PSL(2, 7)$ as a collineation group acting two-transitively on the spreads. Hence, the stabilizer G of a spread Σ has order 24. So, there is a Sylow 2-subgroup of Σ that necessarily fixes a component. Since the Baer 2-groups in $\Gamma L(2, 4)$ have orders dividing 2, it follows that there is an elation group E of order 4. Hence, the group is standard. However, there are exactly $q(q-1)/2$ Desarguesian spreads in $PG(3, q)$ containing a given regulus R. Hence, when $q = 2$, there is a unique such Desarguesian spread. Hence, the parallelisms, considered as extensions to transitive deficiency one partial parallelisms are non-standard but admit standard groups.

41 Problem. Show that any deficiency one transitive partial parallelism in $PG(3, q)$, for $q > 2$ lifts to a standard parallelism, or find a class of non-standard parallelisms.

42 Theorem. *Let $\mathcal{P}$ be a parallelism in $PG(2, q)$ with a standard group G, for $q > 2$.*

(1) If q is odd and $\mathcal{P}$ admits a group $2K^*$ of order $2(q-1)$ that is in the kernel homology group of the socle spread Σ then $\mathcal{P}$ is a standard parallelism.

(2) Let Σ' be a non-socle spread of $\mathcal{P}$. If Σ' is left invariant by an affine homology of Σ then $\mathcal{P}$ is a standard parallelism.

More generally, if Σ is left invariant by a collineation of Σ that fixes exactly two components $x = 0, y = 0$ of Σ but fixes no 2 -dimensional subspaces disjoint from $x = 0$ or $y = 0$ then $\mathcal{P}$ is a standard parallelism.

Proof. Let E^- fix a spread non-socle spread Σ' and act as a Baer group of order q on Σ'. The net R' defined by the fixed point space of E^- is a regulus net. Since $2K^*$ acts on the parallelism and fixes each spread, and $FixE^-$ (i.e. $x = 0$) is left invariant under the full group of the parallelism, it follows that the q Baer subplanes incident with the zero vector of R' are permuted by $2K^*$. Since K^* fixes each such Baer subplane, it follows that there is a fixed subplane under $2K^*$ as 2 divides q. But, a fixed subplane that is fixed by $2K^*$ is a component of Σ. Since E^- acts regularly on these q subplanes, it follows that all subplanes of R' are components of Σ, implying that Σ' contains an opposite regulus to a regulus of Σ. This proves (1).

If Σ' is left invariant by an affine homology g then g fixes exactly one 2-dimensional $GF(q)$-space that is disjoint from the axis of g, namely the co-axis, a component of Σ. An affine homology g becomes a Baer p'-group, for $q = p^r$, which must fix a second Baer subplane of the net defined by $Fixg$. Hence, there is a second Baer subplane of R' that is a line of Σ and E^- is transitive on the set of Baer subplanes different from $x = 0$. Thus, the parallelism is standard, completing the proof of (2). $\boxed{QED}$

43 Theorem. *Assume that $\mathcal{P}$ is a non-standard parallelism in $PG(3,q)$ admitting a standard group G.*

(1) If $q = 2^r$ is even then the order of G/K^* divides $q^2(q+1)2r$.

Hence, $G \cap GL(2,q^2)/K^*$(the $GF(q)$-kernel) has order $q^2(q+1)$.

(2) If $q = p^r$ is odd then the order of G/K^* divides $q^2(q+1)4r$.

So, $G \cap GL(2,q^2)/K^*$ has order either $q^2(q+1)$ or $2q^2(q+1)$.

Proof. Consider a non-socle spread Σ' and let g be a non-kernel collineation in $GL(2,q^2)$ and $G_{y=0}$. We may assume that g normalizes some regulus-inducing group, and so we may take the group to be the standard group E_1. Since g will have order dividing $q-1$ and permutes q spreads (the q spreads fixed by E_1), it follows that g will fix a spread Σ', also fixed by E_1. Since we may assume that $q > 2$, it follows that the axis of the elation group, $x = 0$, is G-invariant. There are q remaining 2-dimensional $GF(q)$-subspaces that lie as Baer subplanes in the regulus net R' containing $x = 0$ as a Baer subplane. Let g have prime order. Since the order of g is prime and divides $q-1$, it follows that g fixes one of these Baer subplanes. What we are trying to show is that this Baer subplane is actually $y = 0$, implying that the parallelism is standard. Since the parallelism is assumed to be non-standard, the fixed subplane must be a Baer subplane of Σ that is disjoint from $x = 0$ and hence has the form $y = x^q m + xn$, where $m, n \in GF(q^2)$ and $m \neq 0$.

Since g fixes $x = 0, y = 0$ and normalizes E_1, it follows that the form for g is $(x,y) \longmapsto (xb, xb\alpha)$, where $\alpha \in GF(q) - \{0\}$ and $b \in GF(q^2) - \{0\}$. If $\alpha = 1$, then g is a kernel homology and if g is not in K^* then the parallelism is standard from a previous result. Hence, $\alpha \neq 1$. Since g fixes $y = x^q m + xn$, we have the following two conditions:

$$\begin{aligned} b^q m &= mb\alpha, \\ bn &= nb\alpha. \end{aligned}$$

If $n \neq 0$ then $\alpha = 1$. Hence, $n = 0$. Thus, $b^{q-1} = \alpha$, implying that $b^{(q-1)^2} = 1$. Since $(q^2-1,(q-1)^2) = (2,q-1)(q-1)$, it follows that $\alpha = -1$ and q is odd, or we are finished.

But, then $g^2 : (x,y) \longmapsto (xb^2, yb^2)$ which is in K^*. This completes the proof. $\boxed{QED}$

References

[1] M. Biliotti, V. Jha, N. L. Johnson: *Classification of transitive deficiency one parallelisms*, Bulletin Belgian Math. Soc. (to appear).

[2] N. L. Johnson: *A note on net replacement in transposed spreads*, Bull. Canad. J. Math., **28** (4) (1985), 469–471.

[3] N. L. Johnson: Dual Parallelisms, Note di Mat. **21** (1) (2002), 137–150.

[4] M. Biliotti, V. Jha, N.L. Johnson: Foundations of Translation Planes. Pure and Applied Mathematics, Vol. 243, Marcel Dekker, New York, Basel, 2001, pp. 1–552.

[5] V. Jha: *On tangentially transitive translation planes and related systems*, Geometriae Dedicata **4** (1975), no. 2/3/4, 457–483.

[6] V. Jha, N. L. Johnson: *The classification of spreads in $PG(3, q)$ admitting linear groups of order $q(q + 1)$, I. Odd Order*, to appear.