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Soluble Products of Finite Groups

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Abstract. If G is the soluble product of its subgroups A and B, then a natural question is how the structure of G is influenced by the structure of A and B. In particular, can the derived length be bounded in terms of invariants of A and B. There are many results in this direction, but very little is known about best possible bounds. We survey some of these results.

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Suppose A and B are subgroups of a group G. We say that A permutes with B if the product of A and B, $AB = \{ab \ : \ a \in A, \ b \in B\}$, is a subgroup of G. If $AB = G$, we say G is the product of A and B and we call A and B factors of G. A natural question to ask is whether properties of $G = AB$ can be deduced from properties of A and B. There is an extensive literature on this question. Many properties have been considered- see for example the book of Amberg, Franciosi and de Giovanni [2]- and further restrictions on the products have also been considered. I want to concentrate on one particular property and will only consider finite groups, although some of the results do not need finiteness.

Suppose that G is soluble. Then A and B are certainly soluble but A and B soluble is not enough to ensure that G is soluble and so we may ask the following questions:

What further conditions on A and B will ensure that $G = AB$ is soluble?

If G is soluble, can we bound the derived length $d(G)$ of G in terms of invariants of A and B?

If $d(G)$ is bounded, can we find the best possible bound?

For the first question there is an extensive literature, both for restrictions on the factors and on the type of product. Producing bounds is harder and there is very little known about best possible bounds. I want to concentrate here on the second and third questions.

Here the story usually starts with the theorem of Ito:

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Theorem 1. (Ito [12]) *Let $G = AB$ with A and B abelian. Then G is metabelian.*

This actually does not need G finite and it is easy to construct groups which are nonabelian and so this bound is best possible.

The next step is due to Hall and Higman who proved

Theorem 2. ((Hall and Higman [9]) *Suppose that $G = AB$ is soluble and A, B are nilpotent of coprime order. Then $d(G) \leq c(A) + c(B)$, where $c(A)$ denotes the nilpotency class of A.*

(This result is often attributed to Pennington [20], who obtains it as a corollary to her main theorem. It first appears as a special case of Theorem 1.2.4 of [9].)

Much of the work since this result has concentrated on products of nilpotent groups.

Shortly after, Wielandt proved if $G = AB$ and A, B are nilpotent of coprime order then G is soluble. Then Kegel removed the restriction on coprimeness of A and B.

Theorem 3. (Wielandt [22]) *If $G = AB$ and A, B are nilpotent of coprime order then G is soluble.*

Theorem 4. (Kegel [16] *If $G = AB$ and A, B are nilpotent then G is soluble.*

There are many papers in the decade before Kegel's theorem that may be regarded as precursors of Kegel's result, proving solubility under further restrictions on the structure of the factors. Kegel's proof does not give a bound on the derived length of the product. For many years it was conjectured that the sum of the nilpotency classes would be an upper bound for the derived length and there are a number of partial results.

One of the first was

Theorem 5. (Pennington [20]) *If $G = AB$ and A, B are nilpotent then $d(G) \leq c(A) + c(B) + d(F(G))$.*

Perhaps more important from the viewpoint of finding a bound was the following result

Theorem 6. (Pennington [20], Amberg [1]) *If $G = AB$ with A, B nilpotent, then $F(G) = (F(G) \cap A)(F(G) \cap B)$.*

These two results tell us that it is enough to consider the case of G nilpotent (and hence a p-group) to find a bound (p will denote a prime throughout this paper).

When G is nilpotent, another approach was started by Kazarin in 1982, who showed that the derived length could be bounded in terms of the orders of the derived groups of A and B. Several authors have improved on his bounds in special cases:

Theorem 7. *Suppose that $G = AB$ is a p-group with $|A'| = p^m$ and $|B'| = p^n$ with $m \geq n$:*
i) (Kazarin [14]) $d(G) \leq 2(m+n)+1$;
ii) (Morigi [19]) $d(G) \leq m+2n+2$; and if B is abelian, $d(G) \leq m+2$;
iii) (McCann [18]) if A has class at most 2 and B is extraspecial, $d(G) \leq m+3$;
iv) (Mann [17]) if B is abelian, $d(G) \leq 2log_2(m+2)+3$.

Note that these results combined with the results of Pennington and Amberg above show that the derived length of the product is bounded (in terms of invariants of the factors). The bound that they give is generally very large. To find bounds in terms of the nilpotency classes of A and B seems more difficult and there are fewer results here. Of course the aim here was to prove Kegel's conjecture. Since soluble groups of derived length at most d form a formation, a minimal counterexample to Kegel's conjecture must have a unique minimal normal subgroup and so in particular the Fitting subgroup must be a p-group for some prime p. It seems natural to consider products in which the Fitting subgroup is a p-group and Stonehewer and I began to look at such groups in the late 90's.

Theorem 8. (Cossey and Stonehewer [5]) *Suppose $G = AB$, with A, B nilpotent, $F(G)$ a p-group. Suppose also $F(G)/\Phi(F(G))$ contains no central chief factors. Then $d(G) \leq c(A)+c(B)+1$.*

In analysing the structure of a minimal example of a group of derived length $c(A)+c(B)+1$ in the above theorem, Stonehewer and I produced an example of a group of derived length 4 which was the product of an abelian and a metabelian group. We were then able to find a few further examples . In particular we proved:

Theorem 9. (Cossey and Stonehewer [4]) *There are examples of groups $G = AB$ with A nilpotent of class m, B nilpotent of class n and G of derived length $d > m+n$ for the following triples (m, n, d): $(1, 2, 4)$, $(2, 2, 5)$ and $(2, 2, 6)$.*

Although the most surprising thing about these examples is the length of time it took to find them, it seems difficult to extend the construction to give more examples and it seems likely that a linear bound is the correct one.

Another invariant of nilpotent groups that would seem relevant to the derived length of G is the derived length of the factors and this was considered by Kazarin for factors of coprime order. When A and B have small class, the bound of Hall and Higman is best possible, but for larger classes it seems too large. Kazarin showed that it can be replaced by a bound involving the derived lengths of the factors and perhaps not surprisingly a better bound can be found if the product has odd order.

Theorem 10. (Kazarin [15]) *Suppose $G = AB$, A , B nilpotent of coprime order. Then*

$$d(G) \le 2d(A)d(B) + d(A) + d(B)$$

and if G has odd order then

$$d(G) \le d(A)d(B) + max\{d(A), d(B)\}.$$

If A is abelian, Kazarin's bound is $3d(B) + 1$ and $2d(B)$ if G has odd order. Are these best possible? If B is metabelian then Kazarin's bounds are 7 and 4 respectively. Wang and I showed that these bounds could be improved.

Theorem 11. (Cossey and Wang [6]) *Suppose $G = AB$ with A abelian and B nilpotent and metabelian. Suppose also that A and B have coprime orders. Then $d(G) \le 4$ and if G has odd order $d(G) \le 3$ and these bounds are best possible.*

We actually showed that examples of derived length 4 can be classified and the odd order result follows from the classification.

Recently Jabara has used a different invariant to bound the derived length of products of p-groups. Let $\mathcal{A}_1$ be the class of finite abelian p-groups. Then $G \in \mathcal{A}_n$ if and only if each chief series of G contains a nontrivial abelian term K such that $G/K \in \mathcal{A}_{n-1}$.

Theorem 12. (Jabara [13]) *Suppose $G = AB$ is a p-group, with A abelian, $B \in \mathcal{A}_n$. Then $d(G) \le 2n$.*

Another result which gives a bound for a product with restrictions on the type of product has been given by Dixon and Stonehewer.

Theorem 13. (Dixon and Stonehewer [7]) *Suppose $G = AB$, with A, B nilpotent quasinormal subgroups of G. Suppose also that $A \cap B = 1$. Then $d(G) \le max\{2, d(A), d(B)\}$.*

This bound is also clearly best possible. Dixon and Stonehewer ([8]) have also observed that the result holds with A and B soluble. Cossey and Ezquerro [3] have shown that the requirement that A and B be quasinormal can be replaced by G being the totally permutable product of A and B when G has odd order. (G is the totally permutable product of A and B if every subgroup of A permutes with every subgroup of B.)

If $G = AB$ is soluble, even when bounds can be found for $d(G)$, best possible bounds have only been found for special cases, either for small values of some invariant or strong restrictions on the type of product. Perhaps the simplest case where the answer is not known is $G = AB$, where A and B are normal in G.

Of course, if A and B are soluble then G is soluble - this is an easy undergraduate problem, as is the bound $d(G) \leq d(A) + d(B)$. So we may ask if this bound is best possible. Perhaps surprisingly this seems to be a difficult question.

If A, B are abelian, Ito's Theorem gives a bound of 2 and it is easy to find examples where this bound is reached. We can also find examples where A is abelian, $d(B)$ is arbitrary and G has derived length $1 + d(B)$. An easy set of examples is given by the following.

Let $T_n(p)$ be the group of upper unitriangular $n \times n$ matrices over $GF(p)$. Weir [21] showed in 1955- see also Huppert [10] III.16.6- that $T_n(p)$ can be written as the product of $n-1$ abelian normal subgroups $A_1, ..., A_{n-1}$ where A_j consists of the matrices of the form

$$\begin{pmatrix} I_j & S \\ 0 & I_{n-j} \end{pmatrix}.$$

Now given an integer d we can find an n such that $T_n(p)$ has derived length at least $d+1$. For some $i < n$ we will have $N = \langle A_1, ..., A_i \rangle$ of derived length d and $\langle A_1, ..., A_{i+1} \rangle = A_{i+1}N$ of derived length $d+1$.

When A and B are both nonabelian, I know of no examples where the bound is attained. It appears to be a difficult problem, even when A and B are both metabelian. Note that all these examples are groups of prime power order. We might ask if this is an essential feature of such a product in the following sense. If $G = AB$ is the product of normal subgroups and P is a Sylow p-subgroup of G, p a prime, then P is the normal product of $P \cap A$ and $P \cap B$. If $d(G) = d(A) + d(B)$ is it true that for some prime p, $d(G) = d(P) = d(P \cap A) + d(P \cap B)$. This also seems a difficult question in general. It is true if both A and B are abelian but for A metabelian and B abelian it is not true and an example is given below.

Let p be an odd prime, C be a group of order p and $X = \langle x, y \rangle$ be the nonabelian group of order p^3 and exponent p. Put $[x, y] = z$. Put $H = C\mathrm{wr}\mathrm{X}$ and denote the base group of H by Y. Then Y is an elementary abelian group. We let α be the automorphism of H which fixes X and inverts every element of Y and put $K = H\langle \alpha \rangle$. As a $\langle y, z \rangle$ module, $Y = Y_1 \times ... \times Y_p$ where each Y_i is isomorphic to the regular module and so has a unique minimal submodule

M_i. We may assume that $Y_i = Y_1^{x^{i-1}}$. By the dual of [11] Theorem VII.15.5 Y_i/M_i has a submodule N_i/M_i of order p^2 on which $\langle y, z\rangle$ acts trivially and $[N_i, z] \neq 1$. Let M be the product of the M_i's and N the product of the N_i's. Then M and N are normal subgroups of K. We let $G = NX\langle\alpha\rangle$, $A = N\langle x, z, \alpha\rangle$ and $B = M\langle y, z\rangle$. Modulo M, B is normalised by x and centralised by N, z and α and so B is normal in G. Also modulo M, A centralises N and α and normalises $\langle x, z\rangle$ and so B is normal in G. We then have $G' = \langle z\rangle N$. Since z does not centralise N, $G'' \neq 1$ and so G has derived length 3. The Sylow p-subgroup of G is just NX and is the normal product of $N\langle x, z\rangle$ (which is not abelian) and B. Since $(NX)' = [NX, NX] = [N, X]X' = [N, X]\langle z\rangle$ and $[N, X] \leq M \leq B$ we have $(NX)'' = 1$.

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