Note di Matematica **23,** n. 1, 2004, 7–13.

Null Bertrand curves in Minkowski 3-space and their characterizations

Handan Balgetir
Department of Mathematics, Firat University,
23119 Elazig, TURKEY
hbalgetir@firat.edu.tr

Mehmet Bektaş
Department of Mathematics, Firat University,
23119 Elazig, TURKEY
mbektas@firat.edu.tr

Jun-ichi Inoguchi
Department of Applied Mathematics, Fukuoka University,
Fukuoka 814-0180, JAPAN
inoguchi@bach.sm.fukuoka-u.ac.jp

Received: 5/3/2003; accepted: 30/4/2004.

Abstract. We introduce the notion of Bertrand curve for Cartan framed null curves in Minkowski 3-space and give a characterization.

Keywords: Bertrand curve, Cartan frame.

MSC 2000 classification: 53B30.

1 Introduction

In the mathematical study of relativity theory, a material particle in a space-time is understood as a future-pointing timelike curve of unit speed in a space-time, *i.e.*, a connected and time-oriented 4-dimensional Lorentz manifold. The unit-speed parameter is called the *proper time* of a material particle. Motivated by this fundamental observation, timelike curves in Lorentzian manifolds have been studied extensively by both physicists and differential geometers.

On the other hand, in relativity theory, a lightlike particle is a future-pointing *null geodesic* [7], [8].

More generally, from the differential geometric point of view, the study of null curves has its own geometric interest.

Many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied by a similar approach to that in positive definite Riemannian geometry.

However, null curves have many properties very different from spacelike or timelike curves. In other words, null curve theory has many results which have no Riemannian analogues.

The presence of null curves often causes important and interesting differences, as will be the case in the present study.

In this paper we shall give an example of such "different aspects" of null curves.

For other applications of null curve theory to general relativity, we refer to [2].

In the classical differential geometry of curves, J. Bertrand studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve. Such a curve is nowadays called a *Bertrand curve*. Bertrand curves have a characteristic property that curvature and torsion are in linear relation. For classical and basic treatments of Bertrand curves, we refer to [3].

In the recent work [1], the first and second named author and Ergüt studied spacelike and timelike Bertrand curves in Minkowski 3-space $\mathbb{R}^3_1$. (See also independently obtained results by N. Ekmekçi and K. İlarslan [4]).

In this paper, we study Cartan framed null curves with the Bertrand property in Minkowski 3-space $\mathbb{R}^3_1$.

We shall show that null Bertrand curves are null geodesics or Cartan framed null curves with constant second curvature.

2 Preliminaries

Let $\mathbb{R}^3_1 = (\mathbb{R}^3(t,x,y), g)$ be a *Minkowski* 3 *-space* with metric $g = -dt^2 + dx^2 + dy^2$. A tangent vector v of $\mathbb{R}^3_1$ is said to be

spacelike if $g(v,v) > 0$ or $v = 0$

timelike if $g(v,v) < 0$, or

lightlike or *null* if $g(v,v) = 0$ and $v \neq 0$.

A null frame of $\mathbb{R}^3_1$ is a *positively oriented* ordered triple (λ, N, W) of vectors satisfying

$$g(\lambda,\lambda) = g(N,N) = 0,\ g(\lambda,N) = 1,$$

$$g(\lambda,W) = g(N,W) = 0,\ g(W,W) = 1.$$

A parametrized curve $\alpha(p)$ in $\mathbb{R}^3_1$ is said to be *null* if its tangent vector field is null everywhere, *i.e.*, $g(\frac{d\alpha}{ds}, \frac{d\alpha}{ds}) = 0$ and $d\alpha/ds \neq 0$.

A null frame for a null curve $\alpha(s)$ is a frame field $F(s) = (\lambda(s), N(s), W(s))$ such that $d\alpha/ds$ is a positive scalar multiple of λ (p. 371, [5]). In such a case, α is said to be *framed by* $F(s)$. Frames for null curves are not unique. Moreover frames are changed under reparametrizations of a curve. Therefore, the curve and a frame must be given together.

Now suppose that α is framed by $F = (\lambda, N, W)$ with $\lambda = d\alpha/ds$. Then the vector fields N and W define line bundles $\mathrm{ntr}(\alpha)$ and $S(\alpha^{\perp})$ over α respectively. The line bundle $S(T\alpha^{\perp})$ is called the *screen vector bundle* and $\mathrm{ntr}(\alpha)$ the *null transversal vector bundle* of α with respect to $S(T\alpha^{\perp})$, respectively.

The Frenet formula of α with respect to the frame F is given by (p. 55, [2]):

$$\begin{aligned}\frac{d\lambda}{ds} &= h\lambda + \kappa_1 W,\\ \frac{dN}{ds} &= -hN + \kappa_2 W,\\ \frac{dW}{ds} &= -\kappa_2\lambda - \kappa_1 N.\end{aligned} \tag{1}$$

The functions h, κ_1 and κ_2 are called the *curvature functions* of α.

There always exists a parameter p of α such that $h = 0$ in (1). This parameter p is called a *distinguished parameter* of α [2]. The distinguished parameter is uniquely determined for prescribed screen vector bundle up to affine transformation.

In case that p is a distinguished parameter of a null curve α. Then we put

$$\ell(p) := \frac{d\alpha}{dp}(p),\ n(p) := -N(p),\ u(p) := W(p).$$

Then the Frenet formula of α with respect to $F = (\ell, n, u)$ become

$$\begin{aligned}\ell' &= \kappa_1 u,\\ n' &= -\kappa_2 u,\\ u' &= -\kappa_2\ell + \kappa_1 n.\end{aligned} \tag{2}$$

Here the prime "$'$" denotes differentiation with respect to the distinguished parameter p. The null frame F is called the *Cartan frame* of $\alpha(p)$. A parametrized null curve parametrized by the distinguished parameter p together with its Cartan frame is called a *Cartan framed null curve.*

Since we demanded that $\det(\lambda, N, W) > 0$, Cartan frames are *negatively oriented*, that is, $\det(\ell, n, u) < 0$.

For general theory of parametrized null curves, we refer to [2].

3 Bertrand curves in R^3_1

In [1], we have studied *non null* Bertrand curves in $\mathbb{R}^3_1$. In this section we introduce the notion of Bertrand curve for null curves in the following way:

1 Definition. Let $\alpha = (\alpha(p); \ell(p), n(p), u(p))$ and $\bar{\alpha} = (\bar{\alpha}(\bar{p}); \bar{\ell}(\bar{p}), \bar{n}(\bar{p}), \bar{u}(\bar{p}))$ be two Cartan framed null curves in $\mathbb{R}^3_1$. Then a pair of curves $(\alpha, \bar{\alpha})$ is said to be a (null) *Bertrand pair* if u and $\bar{u}$ are linearly dependent.

The curve $\bar{\alpha}$ is called a *Bertrand mate* of α and vice versa. A Cartan framed null curve is said to be a *Bertrand curve* if it admits a Bertrand mate.

By definition, for a null Bertrand pair $(\alpha, \bar{\alpha})$, there exists a functional relation $\bar{p} = \bar{p}(p)$ such that

$$\bar{u}(\bar{p}(p)) = \epsilon\, u(p), \ \epsilon = \pm 1.$$

The following is the main result of this paper.

2 Theorem. *Let α be a Cartan framed null curve. Then α is a Bertrand curve if and only if α is a null geodesic or a Cartan framed null curve with constant second curvature k_2.*

Proof. Let $(\alpha, \bar{\alpha})$ be a Bertrand pair. Then $\bar{\alpha}$ can be expressed as

$$\bar{\alpha}(p) := \alpha(p) + r(p)u(p) \tag{3}$$

for some function $r(p) \neq 0$ and some parametrization $\bar{p} = \bar{p}(p)$ with respect to the distinguished parameter p of α. Differentiating (3) with respect to p,

$$\bar{\ell}\,\frac{d\bar{p}}{dp} = \ell + r'\,u + ru'. \tag{4}$$

Here $\bar{p}$ is the distinguished parameter of $\bar{\alpha}$. By using the Frenet formula (2), we have

$$\bar{\ell}\,\frac{d\bar{p}}{dp} = (1 - r\kappa_2)\ell + r\kappa_1 n + r'\,u. \tag{5}$$

Since $\bar{\ell}$ is null,

$$(r')^2 = 2r\kappa_1(1 - r\kappa_2). \tag{6}$$

Next, since $\bar{\alpha}$ is a Bertrand mate of α, $\bar{u}$ is in the direction of u, thus $g(\bar{\ell}, u) = 0$, hence r is a constant.

From (6), we get the following equation:

$$\kappa_1(1 - r\kappa_2) = 0.$$

Thus we conclude that $\kappa_1 = 0$ or $\kappa_2 = 1/r =$ constant.

We investigate these curves in more detail.

Case 1 $\kappa_1 = 0$: In this case α is a null geodesic. Thus α is represented as $\alpha(p) = \alpha_0 + p\,\ell$. Here α_0 is a constant vector and ℓ is a constant null vector. Moreover α is framed by a constant frame (ℓ, n, u). Hence $k_2 = 0$. Thus $\bar{\alpha}$ differs from α only by translation. Hence $\bar{\alpha}$ is congruent to α.

Note that two Cartan frames are related by

$$\bar{\ell} = \mu_1 \ell, \bar{n} = \mu_1^{-1} n,\ \ \bar{u} = u$$

for some constant μ_1.
Case 2 $\kappa_2 = 1/r$:
In this case, by (5), we notice that

$$\bar{\ell}(\bar{p}(p)) = \mu\, n(p), \quad \mu(p) = r\, \kappa_1(p)\, \frac{dp}{d\bar{p}}(p) \neq 0. \tag{7}$$

This equation implies that

$$\bar{n}(\bar{p}(p)) = \mu p^{-1} \ell(p),\ \ \bar{u}(\bar{p}(p)) = -u(p).$$

Differentiating (7) with respect to p,

$$\frac{d\bar{p}}{dp}\, \frac{d\bar{\ell}}{d\bar{p}} = \frac{d\mu}{dp} n + \mu\, \frac{dn}{dp}.$$

Using the Frenet formulae for α and $\bar{\alpha}$, we obtain

$$\frac{d\bar{p}}{dp}\, \bar{\kappa}_1\, \bar{u} = \frac{d\mu}{dp}\, n - \frac{\mu}{r}\, u.$$

This formula implies μ is constant and

$$\bar{\kappa}_1(\bar{p}(p)) = \frac{\mu}{r} / \frac{d\bar{p}}{dp}(p), \quad \bar{\kappa}_2(\bar{p}(p) = \frac{\kappa_1(p)}{\mu} / \frac{d\bar{p}}{dp}(p).$$

Inserting $\mu = r\kappa_1/(d\bar{p}/dp)$ into these equations, we get

$$\kappa_1(p)\bar{\kappa}_1(\bar{p}(p)) = \left(\frac{\mu}{r}\right)^2, \quad \bar{\kappa}_2(\bar{p}(p)) = \kappa_2(p) \equiv \frac{1}{r}.$$

Conversely let α be a Cartan framed null curve and r a nonzero constant. The case, "α is a null geodesic" is trivial. We only have to investigate Cartan framed null curves with $\kappa_1 \neq 0$ and $\kappa_2 = 1/r$.

For a non-zero constant μ define a function $\bar{p}$ by

$$\bar{p} := \frac{r}{\mu} \int \kappa_1(p) dp.$$

Next put

$$\bar{\alpha}(\bar{p}) := \alpha(p) + r u(p).$$

Then $\bar{\alpha}$ is a Cartan framed null curve with distinguished parameter $\bar{p}$ and framed by

$$\bar{\ell}(\bar{p}(p)) = \mu n(p),\ \bar{n}(\bar{p}(p) = \mu^{-1}\ell(p),\ \bar{u}(\bar{p}(p) = -u(p).$$

Thus $(\alpha, \bar{\alpha})$ is a Bertrand pair. The curvature functions of $\bar{\alpha}$ are computed as

$$\bar{\kappa}_1(\bar{p}(p)\kappa_1(p) = \left(\frac{\mu}{r}\right)^2,\ \bar{\kappa}_2(\bar{p}(p) = \kappa_2(p) = \frac{1}{r}.$$

This completes the proof. $\boxed{QED}$

3 Corollary. *Let $(\alpha, \bar{\alpha})$ be a Bertrand pair of Cartan framed null curves which are not geodesics. Then their curvature functions satisfy the following relations:*

$$\bar{\kappa}_1 \cdot \kappa_1 = \text{constant} > 0, \quad \bar{\kappa}_2 = \kappa_2 = \text{constant} \neq 0.$$

Theorem 2 implies that every Cartan framed proper null helix admits a Bertrand mate. Moreover, by Corollary 3 the Bertrand mate is also a proper null helix.

4 Example. Let α be a parametrized null curve defined by

$$\alpha(p) = (\frac{1}{2}\sinh(2p), \frac{1}{2}\cosh(2p), p).$$

Then α is framed by a Cartan frame $F = (\ell, n, u)$:

$$\begin{aligned} \ell(p) &= \alpha'(p) = (\cosh(2p), \sinh(2p), 1), \\ n(p) &= (\frac{1}{2}\cosh(2p), \frac{1}{2}\sinh(2p), -\frac{1}{2}), \\ u(p) &= (\sinh(2p), \cosh(2p), 0). \end{aligned}$$

Note that $\det(\ell, n, u) < 0$. The curvature functions of α with respect to F are $\kappa_1 = 2,\ \kappa_2 = -1$. Define $\bar{p} = (-2/\mu)p$. Then the curve

$$\bar{\alpha}(\bar{p}) = \alpha(p) - u(p) = -\frac{1}{2}(\sinh(-\mu\bar{p}), \cosh(-\mu\bar{p}), \mu\,\bar{p})$$

gives a one-parameter family of Bertrand mates of α framed by

$$\bar{\ell}(\bar{p}) = \mu\, n(p),\ \bar{n}(\bar{p}) = \mu^{-1}\,\ell(p),\ \bar{u}(\bar{p}) = -u(p).$$

The curvature functions of $\bar{\alpha}$ are

$$\bar{\kappa}_1 = \frac{\mu^2}{2},\ \bar{\kappa}_2 = -1.$$

5 Remark. Let $(\alpha, \bar{\alpha})$ be a pair of Cartan framed null curves such that n and $\bar{n}$ are linearly dependent. Then both the curves are null geodesics or Cartan framed null curves with same constant second curvatures [6].

6 Remark. In Euclidean 3-space, Bertrand curves are characterized as follows (See [3], p. 41):

> Let α be a curve in Euclidean 3-space parametrized by arclength. Then α is a Bertrand curve if and only if α is a plane curve or a curve whose curvature κ and torsion τ are in linear relation:
>
> $$\mu\kappa + \nu\tau = 1$$
>
> for some constant μ and ν.

The product of torsions of a Bertrand pair is constant.

Our Corollary 3 says null Bertrand curves provide us an example of peculiarity of Lorentz geometry.

Acknowledgements. The first and second named authors would like to thank professor Mahmut Ergüt for his careful reading of the manuscript.

References

[1] H. Balgetir, M. Bektaş, M. Erg üt: *Bertrand curves for nonnull curves in 3-dimensional Lorentzian space*, Hadronic Journal **27** (2004), 229–236.

[2] K. L. Duggal, A. Bejancu: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, Dordrecht, 1996.

[3] L. P. Eisenhart: A Treatise on the Differential Geometry of Curves and Surfaces, Ginn and Company, Boston, 1909, reprinted by Dover, 1960.

[4] N. Ekmekçi, K. İlarslan: *On Bertrand curves and their characterization*, Differential Geometry–Dynamical Systems **3** (2001), 17–24.

[5] L. K. Graves: *Codimension one isometric immersions between Lorentz spaces*, Trans. Amer. Math. Soc. **252** (1979), 367–392.

[6] K. Honda, J. Inoguchi: *On a characterisation of generalised null cubics*, Differential Geometry-Dynamical Systems **5** (2003), to appear.

[7] B. O'Neill: Semi-Riemannian Geometry with Application to Relativity, Academic Press, New York, 1983.

[8] R. K. Sachs, H. Wu: General Relativity for Mathematicians, Graduate Texts in Math. **48**, Springer Verlag, New York-Heidelberg, 1977.