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# An absolutely continuous function whose inverse function is not absolutely continuous

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**Abstract.** We construct a strictly increasing function $f\colon [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$, $f$ is absolutely continuous, and $f^{-1}$ is not absolutely continuous. Functions of this type are very scarce in the literature.

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## 1 Introduction

Throughout this paper, $\lambda$ denotes the Lebesgue measure on the real line $\mathbb{R}$, and "a.e." means "$\lambda$-almost everywhere". We recall that a function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$ whenever $]a_1, b_1[, \ldots ]a_n, b_n[$ are pairwise disjoint subintervals of $[a, b]$ for which $\sum_{i=1}^{n}(b_i - a_i) < \delta$. It turns out that any absolutely continuous function $f$ on $[a, b]$ is continuous, and has a finite derivative $f'$ on $[a, b]$. Moreover,

$$f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t, \qquad x \in [a, b]. \tag{1}$$

The purpose of this note is to construct a strictly increasing function $f : [0, 1] \longrightarrow [0, 1]$ such that $f(0) = 0$, $f(1) = 1$, $f$ is absolutely continuous, and the inverse function $f^{-1}$ is not absolutely continuous. To do this, we need the following standard result due to M. A. Zareckii (see, e.g., [1]).

**Theorem.** *Let $f : [a, b] \longrightarrow [c, d]$ be a strictly increasing function that maps $[a, b]$ onto $[c, d]$. Then the following hold:*

(i) *$f$ is absolutely continuous if and only if $\lambda(f(\{x : f'(x) = \infty\})) = 0$;*

(ii) *$f^{-1}$ is absolutely continuous if and only if $\lambda(\{x : f'(x) = 0\}) = 0$.*

## 2 The construction

First, we construct a Cantor-like set $B \subset [0,1]$ as follows. Remove an open interval $I_{11}$ of length $\alpha < 1/3$ from the center of $[0,1]$. This leaves 2 disjoint closed intervals $J_{11}$ and $J_{12}$ each having length $< 1/2$. This completes the first stage of the construction. If the $n$-th step of the construction has been completed, leaving $2^n$ disjoint close intervals $J_{n1}, \ldots, J_{n2^n}$ (numbered from left to right), each of length $< 1/2^n$, we perform the $(n+1)$-st step by removing an open interval $I_{n+1,k}$ of length $\alpha^{n+1}$ from the center of $J_{nk}$, $1 \leq k \leq 2^n$. This leaves $2^{n+1}$ closed intervals $J_{n+1,1}, \ldots, J_{n+1,2^{n+1}}$ each of length $< 1/2^{n+1}$. Denote $A_n = \bigcup_{k=1}^{2^{n-1}} I_{nk}$, $n \geq 1$, $A = \bigcup_{n \geq 1} A_n$, and $B = [0,1] - A$. We have $\lambda(A_n) = 2^{n-1}\alpha^n$, $n \geq 1$, and so

$$\lambda(A) = \sum_{n \geq 1} \lambda(A_n) = \frac{\alpha}{1 - 2\alpha} < 1.$$

Therefore, the Cantor-like set $B$ has positive Lebesgue measure.

Further, we define recursively a sequence $(f_n)$ of continuous, piecewise linear, strictly increasing functions on $[0,1]$ ($f_n(0) = 0$, $f_n(1) = 1$) by way of the next algorithm:

(a) Set $J_{n1} = [0, a_n]$. The graph of $f_n$ on $J_{n1}$ is the straight line joining the points $(0,0)$ and $(a_n, \alpha^n)$, and the graph of $f_n$ on $I_{n1}$ is the straight line joining the points $(a_n, \alpha^n)$ and $(a_n + \alpha^n, \alpha^n + \alpha^n/\lambda(A))$;

(b) For $1 \leq m \leq n$, define

$$f_n(a_m + \alpha^m + x) = \alpha^m + \frac{\alpha^m}{\lambda(A)} + f_n(x), \qquad x \in J_{m2},$$

i.e. the graph of $f_n$ on $J_{m2}$ is a translation of the graph of $f_n$ on $J_{m1}$;

(c) For $1 \leq m < n$, $f_n = f_m$ on $I_{m1}$.

Notice that the graph of $f_n$ is symmetric about the center of the square $[0,1] \times [0,1]$. For any $n \geq 1$, on account of (b) and (c), we have

$$f_{n+1}(x) = f_n(x), \qquad x \in \bigcup_{m=1}^{n} A_m, \tag{2}$$

and so, for each $p \geq 1$,

$$f_{n+p}(x) = f_n(x), \qquad x \in \bigcup_{m=1}^{n} A_m. \tag{3}$$

From (a), (b) and (2), we see that

$$|f_{n+1} - f_n| < \alpha^n, \qquad n \geq 1,$$

and so $(f_n)$ is a Cauchy sequence of continuous, strictly increasing functions. Thus there exists a continuous, nondecreasing function $f : [0,1] \longrightarrow [0,1]$ such that $(f_n)$ converges uniformly to $f$ $(f(0)=0,\ f(1)=1)$. For any $n \geq 1$, upon letting $p \to \infty$ in (3), we get

$$f(x) = f_n(x), \qquad x \in \bigcup_{m=1}^{n} A_m. \tag{4}$$

As each $f_n$ is strictly increasing, (4) implies that $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in A$ and $x_1 < x_2$. Actually $f$ is strictly increasing. For, if $x, x' \in [0,1]$ and $x < x'$, then exists $[x_1, x_2] \subset A$ such that $x \leq x_1 < x_2 \leq x'$.

Finally, we show that $f$ is absolutely continuous, while $f^{-1}$ is not absolutely continuous. Whatever $n \geq 1$, in view of (a) and (b), we have

$$\lambda(f_n(I_{nk})) = \lambda(f_n(I_{n1})) = \frac{\alpha^n}{\lambda(A)}, \qquad 1 \leq k \leq 2^{n-1},$$

and so

$$\lambda(f_n(A_n)) = \sum_{k=1}^{2^{n-1}} \lambda(f_n(I_{nk})) = \frac{2^{n-1}\alpha^n}{\lambda(A)}.$$

Therefore, applying (4), we obtain

$$\lambda(f(A)) = \sum_{n\geq 1} \lambda(f_n(A_n)) = \frac{1}{\lambda(A)} \sum_{n\geq 1} 2^{n-1}\alpha^n = 1. \tag{5}$$

On the other hand, for each $n \geq 1$, we have

$$f_n'(x) = \frac{1}{\lambda(A)}, \qquad x \in A_n,$$

and so

$$f'(x) = \frac{1}{\lambda(A)}, \qquad x \in A. \tag{6}$$

From (5), (6) and part (i) of Zareckii's theorem, it follows that $f$ is absolutely continuous. Consequently, in view of (1) and (6), we may write

$$1 = \int_0^1 f'(x)\,\mathrm{d}x = \int_A f'(x)\,\mathrm{d}x + \int_B f'(x)\,\mathrm{d}x = 1 + \int_B f'(x)\,\mathrm{d}x,$$

and so $f' = 0$ a.e. on $B$. As $\lambda(B) > 0$, part (ii) of Zareckii's theorem shows that $f^{-1}$ is not absolutely continuous.

## References

[1] P. I. NATANSON: Theory of Functions of a Real Variable, Ungar, New York, 1955.