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Spreads in $PG(3, q)$ admitting several homology groups of order $q+1$

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Abstract. The set of translation planes with spreads in $PG(3, q)$ admitting at least three homology groups with distinct axes of order $q+1$ is completely determined. Apart from the Desarguesian and Hall planes of order q^2, the only possible plane is the Heimbeck plane of order 7^2 admitting several quaternion homology groups of order 8. A classification is also given of all translation planes with spreads in $PG(3, q)$ that admit at least three distinct homology groups of order $q+1$. Recent results connecting translation planes with spreads in $PG(3, q)$ admitting cyclic affine homology groups of order $q+1$ with conical flocks spreads provide the background for applications showing how the associated collineation groups are interrelated.

Keywords: homology groups, translation planes, spreads, flocks of quadratic cones, hyperbolic fibrations.

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1 Introduction

A flock of a quadratic cone in $PG(3, q)$ is a covering of the cone minus the vertex by a set of q conics. There are equivalences between flocks of quadratic cones and translation planes with spreads in $PG(3, q)$ admitting certain elation groups of order q, the so-called 'regulus-inducing' elation groups; the axis and any component orbit is a regulus in $PG(3, q)$. This result may be stated more generally over any field K that admits a quadratic extension field. It is a surprise then that flocks of a quadratic cone in $PG(3, q)$ are also equivalent to translation planes with spreads in $PG(3, q)$ that admit cyclic affine homology groups of order $q+1$, where again the result may be stated more generally. Of fundamental importance is the work of Baker, Ebert and Penttila [1] that connects 'regular hyperbolic fibrations with constant back half' with flocks of quadratic cones

[i]The author is indebted to Ulrich Dempwolff for the analysis of the translation planes of order 49. In particular, the appendix arose out of a series of e-mail exchanges.

(the reader is referred either to [1] or [15] for the requisite definitions). These ideas provide the framework for the following general result.

1 Theorem. *(Johnson [15]) Let π be a translation plane with spread in $PG(3,K)$, for K a field. Assume that π admits an affine homology group H, so that some orbit of components is a regulus in $PG(3,K)$.*

(1) Then π produces a regular hyperbolic fibration with constant back half.

(2) Conversely, each translation plane obtained from a regular hyperbolic fibration with constant back half admits an affine homology group H, one orbit of which is a regulus in $PG(3,K)$.

The group H is isomorphic to a subgroup of the collineation group of a Pappian spread Σ, coordinatized by a quadratic extension field K^+, $H \simeq \langle t^{\sigma+1}; t \in K^+ - \{0\}\rangle$, where σ is the unique involution in $Gal_K K^+$.

(3) Let $\mathcal{H}$ be a regular hyperbolic fibration with constant back half of $PG(3,K)$. The subgroup of $\Gamma L(4,K)$ that fixes each hyperbolic quadric of a regular hyperbolic fibration $\mathcal{H}$ and acts trivially on the front half is isomorphic to $\langle \rho, \langle t^{\sigma+1}; t \in K^+ - \{0\}\rangle\rangle$, where ρ is defined as follows: If $e^2 = ef + g$, f, g in K and $\langle e, 1\rangle_K = K^+$, then ρ is $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$, where $P = \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}$.

In particular, $\langle t^{\sigma+1}; t \in K^+ - \{0\}\rangle$ fixes each regulus and opposite regulus of each hyperbolic quadric of $\mathcal{H}$ and ρ inverts each regulus and opposite regulus of each hyperbolic quadric.

When K is finite, it turns out due to work of Jha and Johnson [13] that cyclic homology groups of order $q+1$, when the plane has order q^2, have the property that each component orbit defines a derivable net and when the spread is in $PG(3,q)$, each orbit defines a regulus. Hence, the following theorem holds.

2 Theorem. *(Johnson [15]) Translation planes with spreads in $PG(3,q)$ admitting cyclic affine homology groups of order $q+1$ are equivalent to flocks of quadratic cones.*

A major question concerning the relationship with the flock and the translation plane is the determination of the collineation group of the associated translation plane admitting a cyclic homology group as related to the group of the flock. In particular, if there is a cyclic homology group of order $q+1$, is the set $\{$*axis,coaxis*$\}$ invariant by the full collineation group? Furthermore, if the set $\{$*axis,coaxis*$\}$ is invariant under the full collineation group is the cyclic homology group normal in the full collineation group? For example, could there be two distinct cyclic homology groups with the same axis and coaxis?

We consider the following general problem in this article: **Determine the translation planes of order q^2 with spread in $PG(3, q)$ that admit at least three affine homology groups of order $q+1$.** In the second section, we shall review the various examples. That these previous examples of translation planes are exceptionally rare, is made manifest, as this paper demonstrates. Our principal result is the following.

3 Theorem. *Let π denote a translation plane of order $q^2 = p^{2r}$, for p a prime, with spread in $PG(3, q)$ that admits an affine homology group H of order $q + 1$, in the translation complement.*

(1) Then either the set {axis,coaxis} *of H is invariant under the full collineation group or π is one of the following planes:*

- *(a) Desarguesian,*
- *(b) Hall plane,*
- *(c) Desarguesian,the Heimbeck plane of order* 49 *of type* III.

(2) If there exist at least three mutually distinct affine homology groups of order $q + 1$ but one or two axes, we have one of the following situations:

The plane is either

- *(a) the irregular nearfield plane of order* 25*,*
- *(b) the irregular nearfield plane of order* 49*,*
- *(c) $q = 11$ or 19 and admits $SL(2, 5)$ as an affine homology group (the irregular nearfield plane of orders 11^2 and 19^2 and the two exceptional Lüneburg planes of order 19^2 are examples),*
- *(d) $q \equiv -1 \bmod 4$, and there are two homology groups H_1, H_2 of order $q + 1$, with the same axis M and same coaxis L, exactly one of which is cyclic. Furthermore, in this last case, the group generated by the two groups H_1H_2 has order $2(q + 1)$. If K^* denotes the kernel homology group then $H_1H_2K^*$ induces the regular nearfield group of dimension two on the coaxis L.*

Also, there is a corresponding flock of a quadratic cone admitting a collineation g fixing one regulus and permuting the remaining $q - 1$ reguli in $(q - 1)/2$ pairs. In this case, there is an affine homology group of order 2 *acting on the flock plane.*

Indeed, the conical flock spread is either Desarguesian, Fisher or constructed from a Desarguesian spread by $3q$-double-nest construction and the conical flock plane is described in Theorem 8.

More generally, we may prove the following result when the assumption is that there are but two homology groups of order $q+1$.

4 Theorem. *Let π be a translation plane with spread in $PG(3,q)$ that admits at least two homology groups of order $q+1$. Then one of the following occurs:*

(1) $q \in \{5,7,11,19,23\}$ (the irregular nearfield planes and the exceptional Lüneburg planes are examples),

(2) π is André,

(3) q is odd and π is constructed from a Desarguesian spread by $(q+1)$-nest replacement (actually $q = 5$ or 7 for the irregular nearfield planes also occur here),

(4) q is odd and π is constructed from a Desarguesian spread by a combination of $(q+1)$-nest and André net-replacement,

(5) q is odd and $q \equiv -1 \bmod 4$ and the axis/coaxis pair is invariant under the full collineation group (furthermore, there is a non-cyclic homology group of order $q+1$),

(6) $q = 7$ and the plane is the Heimbeck plane of type III with 10 homology axes of quaternion groups of order 8.

These general results applied to cyclic homology groups then provide the following theorem connecting the collineation groups of the translation planes associated with a regular hyperbolic fibration with constant back half and the collineation groups of the associated translation planes arising from flocks of quadratic cones.

5 Theorem. *Let π_H be a translation plane of order q^2 with spread in $PG(3,q)$ admitting a cyclic affine homology group H of order $q+1$. Let $\mathcal{H}$ denote the regular hyperbolic fibration obtained from π_H and let π_E be the corresponding conical flock spread.*

Then one of the following occurs.

(1) H is normal in the collineation group F_H of π_H. In this case, the collineation group of π_H is a subgroup of the group of the regular hyperbolic fibration and induces a permutation group F_H/H on the associated q reguli sharing a component, fixing one, of the corresponding conical flock spread π_E.

If Ker is the subgroup of π_E that fixes each regulus then π_H is either

(a) *Desarguesian or*

(b) *Kantor-Knuth of odd order or*

(c) $Ker = K^*$, *the kernel homology group of order* $q-1$.

(2) H *is not normal in* F_H *and there is a collineation inverting the axis and coaxis of* H. *In this case,* π_H *is one of the following types of planes:*

(a) *André,*

(b) q *is odd and* π_H *is constructed from a Desarguesian spread by* $(q+1)$-*nest replacement, or*

(c) q *is odd and* π_H *is constructed from a Desarguesian spread by a combination of* $(q+1)$-*nest and André net-replacement.*

2 The known examples

2.1 The exceptional Lüneburg planes admitting $SL(2,3) \times SL(2,3)$

There are eleven translation planes of order p^2 admitting $SL(2,3)\times SL(2,3)$ generated by affine homologies, admitting a collineation group G, such that G_L is doubly transitive on L for any component L, and such that the component orbits of the two homology groups isomorphic to $SL(2,3)$ are identical. These collectively are known as the exceptional Lüneburg planes of type $F*p$ (see section 19 of Lüneburg [20]). In this case, $p \in \{5,7,11,23\}$.

We are interested in when $SL(2,3)$ contains two distinct groups of order $p+1$.

Consider A_4 and note that there is a unique Sylow 2-subgroup S_2 such that any 3-group acts transitively on $S_2-\{1\}$ by conjugation. Hence, there are no subgroups of A_4 of order 6. Since there is a unique involutory affine homology with a given axis in a translation plane, it follows that there are not two groups of order 12 in $SL(2,3)$. Hence, p cannot be 11. We know that the affine homology group induced on the coaxis is a subgroup of $SL(2,3)Z_{p-1}$, implying that there cannot be two homology subgroups of order 24 with a given axis. Hence, when $p=23$, there are two homology groups of order $23+1$ but not three.

However, when $p=5$, there are four Sylow 3-subgroups and hence four groups of order 6 in $SL(2,3)$, implying that the irregular nearfield plane of order 5^2 has at least three affine homology groups of order $5+1$.

However, when $p=7$, there are two distinct quaternion groups of order 8 in $SL(2,3)\times SL(2,3)$, and two cyclic groups of order 8 in $GL(2,3)\times GL(2,3)$. Hence, in the irregular nearfield plane there are at least four homology groups of order 8. In the exceptional Lüneburg plane, there are two distinct quaternion groups of order 8.

Summary.

(1) When $p = 5$ or 7 the irregular nearfield planes of order p^2 admit at least three affine homology groups of order $p+1$.

(2) When $p = 7$, the exceptional Lüneburg plane of order 7^2 admits at least two affine homology groups of order $7+1$.

(3) When $p = 23$, the irregular nearfield plane admits at least two affine homology groups of order $23+1$.

2.2 The exceptional Lüneburg planes admitting $SL(2,5) \times SL(2,5)$

There are 14 translation planes of order p^2 admitting $SL(2,5) \times SL(2,5)$, generated by affine homologies, where there is a group G such that the stabilizer G_L of a non-axis, non-coaxis component L is transitive on the non-zero points of L and the orbits of the two homology groups isomorphic to $SL(2,5)$ are identical (see Lüneburg [20], section 18). Among these planes are the irregular nearfield planes. When $p = 11^2$, there is only the irregular nearfield plane and when $p = 19^2$, there is the irregular nearfield plane and three others. The question is are there three affine homology groups of order $p+1$?

Note that in $SL(2,5)$, there are six Sylow 5-subgroups of order 5, normalized by a group of order 4. Hence, there are at least two affine homology groups of order $20 = 19+1$ with the same axis. Hence, the irregular nearfield plane and the three exceptional Lüneburg planes of order 19^2 admit at least three affine homology groups of order $19+1$.

In this setting $p \in \{11, 19, 29, 59\}$.

Consider A_5, and the subgroup $\langle (123), (45)(23) \rangle$ and note that

$$(45)(23)(123)(45)(23) = (132).$$

Hence, A_5 has subgroups of order 6 (i. e., the normalizer of a Sylow 3-subgroup), implying that $SL(2,5)$ has at least two subgroups of order $11+1$. Thus, the irregular nearfield plane of order 11^2 admits at least three homology groups of order $11+1$.

Summary.

(1) The irregular nearfield plane of order 11^2 with non-solvable homology groups admit at least three homology groups of order $11+1$.

(2) The irregular nearfield plane and the three exceptional Lüneburg planes of order 19^2 admit at least three homology groups of order $19+1$.

2.3 The regular nearfield planes of odd order q^2 and kernel $GF(q)$

Since a homology group is generated from the Frobenius automorphism of $GF(q)$ and the cyclic group of order $(q^2-1)/2$ of $\Gamma L(1,q^2)$, and is sharply transitive on the non-zero vectors of the coaxis, the regular nearfield planes of odd order q^2, for $q \equiv -1 \bmod 4$, admit homology subgroups of order $2(q+1)$, containing a cyclic subgroup of order $(q+1)$ and a non-Abelian subgroup of order $(q+1)$, containing a cyclic subgroup of order $(q+1)/2$.

Hence, there are at least three homology groups of order $q+1$.

2.4 Flocks of quadratic cones; two groups

Now consider a conical flock plane; a translation plane corresponding to a flock of a quadratic cone, when q is odd and $q \equiv -1 \bmod 4$. Assume that there is an affine homology of order 2 acting on the conical flock plane. If the corresponding spread is

$$x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}; u,t \in GF(q),$$

where f and g are functions on $GF(q)$, then the collineation takes the form:

$$(x,y) \longmapsto (x,-y).$$

From the calculations of Baker, Ebert and Penttila [1, see p. 6], it follows that there is an associated collineation of the corresponding hyperbolic fibration of the form

$$(x,y) \longmapsto (x,yA),$$

where A is the matrix of an associated field involutory automorphism of an associated field times an element of the field:

$$\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u,t \in K.$$

Note that this mapping normalizes the cyclic homology group of order $(q+1)$ arising from the construction of the hyperbolic fibration. So, the orbits of this homology group of order $(q+1)$ are permuted by the collineation in question.

By a proper choice of reguli, there are translation planes admitting both groups as affine homology groups. Hence,

$$A=\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}.$$

For example, if $g=0$, and $u^2+t^2=-1$, then $A^2=-I_2$. Note that $A=\begin{bmatrix} u & t \\ t & -u \end{bmatrix}$ does not have an eigenvalue of 1 and hence is fixed-point-free on the coaxis. Then the group generated by A, the field elements M of determinant 1 and the scalar group K^* of order $q-1$ will induce a group on the coaxis $x=0$ which is regular and corresponds to the regular nearfield group of dimension 2. We note that $AM=M^{-1}A$, for field elements of determinant 1, as $M^q=M^{-1}$. Hence, we obtain an affine homology group of order $2(q+1)$ admitting two subgroups of order $q+1$, $\langle M\rangle$, and $\left\langle A,M^2\right\rangle$, exactly one of which is cyclic (which is, of course, normal).

We have proved the following result.

6 Theorem. *Let ρ be a finite conical flock plane of odd order q^2, $q\equiv -1 \bmod 4$ admitting an affine homology group of order 2; the spread has the following form:*

$$x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}; u,t\in GF(q),$$
$$f(-t)=-f(t), g(-t)=-g(t).$$

Then there are associated translation planes of order q^2 admitting a cyclic homology group of order $q+1$ that also admit an affine homology group of order $2(q+1)$, with the following presentation:

$$\left\langle g,h; h^{q+1}=g^4=1, g^2=h^{(q+1)/2},\ gh=h^{-1}g\right\rangle,$$

admitting two distinct affine homology groups of order $q+1$:

$$\langle h\rangle, \left\langle g,h^2\right\rangle.$$

2.5 Flocks of quadratic cones; three groups

In this section, we shall be considering three types of translation planes: translation planes π_H admitting cyclic affine homology groups of order $q+1$, conical flock planes π_E, corresponding to these previous planes and Desarguesian planes Σ that are related as noted below.

Concerning translation planes π_H of order q^2 admitting cyclic affine homology groups of order $q+1$ and their associated conical flock translation planes π_E, we see that if G is the group of a plane π_H and H is a cyclic homology group of order $q+1$ then G/H is a group that acts on the set of reguli of the associated conical flock spread π_E. This group fixes one regulus and if π_H is not an André plane, then π_E is not Desarguesian so that G/H arises from a group G^+ of the translation plane π_E such that action on the set of reguli is G/H. It is possible that there is a collineation in G^+ that is not in the kernel homology group K^* of order $q-1$ but fixes each regulus. However, the possible planes can only be the Kantor-Knuth or Desarguesian planes by the theory of 'rigidity' of Jha and Johnson [12] The Kantor-Knuth spread is monomial and all monomial conical flock spreads correspond to j-planes (see Johnson [14]). Hence, either the plane π_E is Kantor-Knuth or the group G^+ may be assumed to contain K^* and G^+/K^* is isomorphic to G/H. Furthermore, note that the Kantor-Knuth planes do not admit groups of order $(q+1)$.

Now assume that there are at least three distinct affine homology groups with affine axes acting in π_H. The axes could be distinct also, but assume that they are not. Then there must be two groups with the same axis and coaxis. It turns out that normally (and it is the object of this paper to prove this) the third group must have axis and coaxis equal to the coaxis and axis, respectively, of the preceding two homology groups. In this case, except for a few sporadic cases, we are able to show that there is a cyclic homology group of order $q+1$ on the two group axis, where the group generated by the two homology groups has order $2(q+1)$. In this case, $q \equiv -1 \bmod 4$. Furthermore, one of the groups of order $q+1$ is cyclic and there is an associated flock of a quadratic cone by the above Theorem 2. If there are two cyclic homology groups of order $(q+1)$ then the translation plane admitting the homology groups is determined as either an André plane, a plane constructed by $(q+1)$-nest replacement or a combination of André replacement and $(q+1)$-nest replacement. Let H denote the cyclic homology group of order $q+1$. Then, $N_G(H)/H$ induces a collineation group on the associated flock and this group leaves one of the conics invariant. Considering what this says about the corresponding translation plane, we have a group of order $2(q+1)$ that normalizes the 'regulus-inducing' elation group E of order q (assuming that the associated flock is not linear or equivalently that the translation plane is not Desarguesian). Hence, we obtain a collineation group acting on the translation plane of order $q(2(q+1))t$, where there is a subgroup of order $2t$ that fixes each of the q reguli of the spread, so by the above remarks, this is a subgroup of the kernel homology group K^* of order $q-1$. Since this group arises from a subgroup of $GL(4,q)$, it follows from the action on the conical flock that the corresponding group is a subgroup of $GL(4,q)$ that fixes a regulus and normalizes E. Hence, there is a subgroup of order $2(q+1)t$ that fixes the axis $x=0$ of E and a second component $y=0$. Since this group is a subgroup of $GL(4,q)$ that leaves invariant a regulus, it is also in $GL(2,q)*GL(2,q)$, where the $*$ denotes a central product by the center of either group. There is then a subgroup of $GL(2,q)*GL(2,q)$ that fixes two components of the regulus net and order $2(q+1)t$. We know that in the affine homology translation plane, any subgroup of odd prime power order of the homology group of order $(q+1)$ (that is not cyclic) is cyclic. Moreover, the product of the Sylow t-subgroups of odd order forms a normal and cyclic subgroup. Assume that $q+1$ has an odd prime factor u. Then there is an element τ_u of order u that fixes at least three components. Note that τ_u must centralize E, implying that τ_u fixes the regulus linewise. Hence, there is an associated Desarguesian affine plane Σ admitting a group G of order $2(q+1)t$, there a subgroup of order $(q+1)_{2'}$ is a cyclic homology group (since the cyclic group is in G/K^*, and $(q+1,q-1)=2$). Therefore, the intersection with $GL(2,q^2)$, the linear collineation group of Σ, has order divisible by $(q+1)t$ and this group fixes two components of the conical flock plane which are then components of Σ as well. Moreover, the regulus of the conical flock plane also becomes a regulus of Σ fixed by G, so the stabilizer in $G\cap GL(2,q^2)$ of three components is a kernel homology group. Hence, we have a kernel homology group of order divisible by $(q+1)/2$ of the Desarguesian plane Σ. Now assume that q is odd and $q+1=2^a$, so that $q=p$ is prime. In the original affine homology group plane π_H, there is a homology group of order $(q+1)$ and since we are assuming that we don't have two cyclic homology groups of order $(q+1)$, this group must be generalized quaternion. Therefore, there is a cyclic sub homology group of order $(q+1)/2$. So, there is a cyclic subgroup acting on the conical flock, so there is a group G of order $2(q+1)t$, normalizing the elation group E, such that the group induced on the set of q reguli contains a cyclic group of order $(q+1)/2$. Thus, it follows that G contains a cyclic group C of order divisible by $(q+1)/2$. If $q>3$, the same ideas show that there is an associated Desarguesian affine plane Σ admitting the normalizer of C as a collineation group. Since the groups originate from a direct product of affine homology groups, it follows that the cyclic affine homology group of order $(q+1)/2$ is normalized by the group of order $2(q+1)^2$ of the affine homology group plane. This means that the group of order $2(q+1)$ induced (not necessarily faithfully) on the set of reguli of the conical flock plane also normalizes the corresponding cyclic group. Since $q=p$, the conical flock plane cannot be Kantor-Knuth without being Desarguesian. Hence, G^+/K^* contains a normal cyclic subgroup of order $(q+1)/2$, so let N be a normal subgroup of G^+ containing K^* such that N/K^* is cyclic of order $(q+1)/2$. Let $N/K^*=\langle gK^*\rangle$, such that $g^{(q+1)/2}\in K^*$. Since we may assume that $q\equiv -1 \bmod 4$ (as otherwise all affine homology groups of order $(q+1)$ are cyclic), it follows that there is an element g of order either $(q+1)/2$ or $(q+1)$ such that gK^* generates N/K^*. Since all of these groups arise from subgroups of $GL(4,q)$, it follows that $\langle g\rangle$ is normal in N and N is Abelian. Therefore, there exists a unique Sylow 2-subgroup of N, implying that $\langle g\rangle$ is characteristic in N of order $(q+1)$, so that we have a normal subgroup of order divisible by $(q+1)/2$ in a subgroup of order $2(q+1)t$. Furthermore, since we have a cyclic group of order $(q+1)/2$ and if $q>3$, the same ideas show that there is an associated Desarguesian affine plane Σ admitting as a collineation group the normalizer of this cyclic group of order $(q+1)/2$. Hence, we again have a subgroup of $\Gamma L(2,q^2)$ of order $2(q+1)t$, that fixes two components and normalizes an elation group E of order q. So, this group intersects $GL(2,q^2)$ is a group of order divisible by $(q+1)t$, implying there is a kernel homology group of order at least $(q+1)/2$.

Hence, we have shown the following theorem.

7 Theorem. *Let π_H be a translation plane of order q^2 with spread in $PG(3,q)$ that admits at least three affine homology groups, where one is assumed cyclic, so q is odd. Then there is an associated Desarguesian affine plane Σ such that the conical flock plane π_E associated with π_H admits a Desarguesian collineation group of order $q(q+1)/2$, which is a product of a regulus-inducing elation group of order q and a kernel homology group of order $(q+1)/2$ of Σ.*

Such planes have been classified by Jha and Johnson [9], [10], [11].

8 Theorem. *(Jha and Johnson [9]) Let π be a translation plane of order q^2 with spread in $PG(3,q)$ that admits a linear group G with the following properties:*

(i) G has order $q(q+1)/2$,

(ii) there is an associated Desarguesian affine plane Σ of order q^2 such that $G=EZ$ where E is a normal, regulus-inducing elation group of Σ and Z is a kernel homology group of order $(q+1)/2$ of Σ.

Then

(1) π is either a conical flock plane or a derived conical flock plane.

(2) π is either Desarguesian or Hall or

(3) if 4 does not divide $(q+1)$ then π is Fisher or derived Fisher.

(4) If π is of odd order q^2, $4\mid(q+1)$, then either π is one of the planes of part (2) or (3), or π may be either

(a) constructed from a Desarguesian plane by double-nest replacement of a $3q$-double-nest or

(b) derived from a plane which may be so constructed, by a base regulus net fixed by the group of order $q(q+1)/2$.

(5) If π is constructed by $3q$-double-nest replacement, the replacement net consists of a set of exactly $3(q+1)/4$ base reguli (E-orbits of components of Σ). This set is replaced by $\{\pi_i EZ; i = 1, 2, 3\}$ where π_i are Baer subplanes of Σ that intersect exactly $(q+1)/2$ base reguli in two components each.

The sets $\mathcal{B}_i$ of $(q+1)/2$ base reguli of intersection pairwise have the property that $|\mathcal{B}_i \cap \mathcal{B}_j| = (q+1)/4$ for $i \neq j$, $i, j = 1, 2, 3$ and $\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 = \varnothing$.

Hence, the type of associated conical flock plane that is obtained from a translation plane of order q^2 admitting at least three affine homology groups of order $(q+1)$ is determined. The reader is referred to Jha and Johnson [9] or [11] for the definition of a double-nest of reguli. We shall see later that if there are three affine homology groups of order $q+1$ then either the order is in $\{5, 7, 11, 19\}$ or admit homology groups of order $q+1$ with symmetric axes (the axis and coaxis of one group is the coaxis and axis of the second group, respectively), at least one of which is cyclic. Hence, we have the following theorem.

9 Theorem. *Let π be a translation plane of order q^2, with spread in $PG(3, q)$, for $q \notin \{5, 7, 11, 19\}$ that admits at least three affine homology groups of order $(q + 1)$. Then π corresponds to a flock of a quadratic cone and the corresponding conical flock spread is either*

(1) Desarguesian,

(2) Fisher or

(3) $q \equiv -1 \bmod 4$ and the plane may be constructed from a Desarguesian plane by $3q$-double-nest replacement.

2.6 The Heimbeck planes

Heimbeck [8] classifies all translation planes of order 7^2 that admit a quaternion affine homology group of order 8. There are exactly ten planes, of which there is a unique plane, type III, that admits at least three quaternion homology groups. The set of orbit lengths of the full collineation group on the components is $\{10, 40\}$.

2.7 The Hall planes

Of course, the very exceptional Hall plane of order 9 admits 10 homology axes of groups of order 4. The Hall planes of order q^2, $q > 3$, admit $(q^2 - q)$ homology axes of cyclic groups of order $q + 1$.

Hence, we have examples in each of the possibilities listed in the main theorem when there are at least three affine homology groups of order $q + 1$. The specific examples are the irregular nearfield planes of orders $5^2, 7^2, 11^2, 19^2$ and the exceptional Lüneburg planes of order 19^2, the Heimbeck planes of order 7^2, the Hall and Desarguesian planes and planes arising from flocks of quadratic cones admitting involutory affine homologies and arising from the Desarguesian, Fisher, or $3q$-double-nest replacements.

3 Translation planes of order q^2 admitting homology groups of order $q + 1$

We assume that π is a translation plane of order q^2 with spread in $PG(3, q)$ that admits an affine homology group H of order $q+1$ in the translation complement. In two papers ([19], [18]), Johnson and Pomareda completely classified the translation planes with spreads in $PG(3, q)$ that admit 'many' homology axes of groups of prime odd order u. The term 'many' is defined to be $> q + 1$ axes. The main theorem is as follows.

10 Theorem. *(Johnson and Pomareda [19, Theorem 2]) Let π be a translation plane of order q^2 with spread in $PG(3, q)$ that admits $> q + 1$ axes (or $> q + 1$ coaxes) of homologies of odd order $u \neq 1$.*

Then one of the following situations occurs:

(i) all of the homology groups have the same axis or all have the same coaxis and there is an elation group of order $> q + 1$ with affine axis,

(ii) π is Desarguesian,

(iii) π is Hall,

(iv) π is Ott-Schaeffer of order 2^{2r} where r is odd and the order $u = 3$,

(v) π is a Hering plane of order p^{2r} where r and p are both odd and $u = 3$,

(vi) G is $SL(2, 9)$, $u = 3$ and $q = 7, 11, 13, 17$ and the planes are enumerated in Biliotti-Korchmáros [2],

(vii) G is $GL(2, 3), u = 3$ and $q = 5$, and the plane is determined as in Johnson and Ostrom [16].

In Johnson and Pomareda [19], there is also an analysis of translation planes admitting two cyclic homology groups of order $q+1$.

11 Theorem. *(Johnson and Pomareda [19, Theorem 18]) Let π be a translation plane with spread in $PG(3,q)$ that admits at least two cyclic homology groups of order $q+1$. Then π is one of the following types of planes:*

(1) André,

(2) q is odd and π is constructed from a Desarguesian spread by $(q+1)$-nest replacement or

(3) q is odd and π is constructed from a Desarguesian spread by a combination of $(q+1)$-nest and André net-replacement.

However, the case when q is prime p and $p+1=2^a$ was considered in Johnson and Pomareda [19] only in the case when there might be p-elements in the group generated by the affine homology groups. We argue this case more generally here for arbitrary homology groups of order $p+1$ $(q+1)$. (See $(*)$ below.) Furthermore, this theorem remains valid even for this case. Note that it has been previously pointed out in Johnson and Pomareda [19] that the irregular nearfield planes of order 5^2 and 7^2 may be obtained from a Desarguesian plane by $q+1$-nest replacement.

Regarding the proof of the above theorem, we make the following points, stated in the form of lemmas.

12 Lemma. *Assume that there are two affine homology groups of order $q+1$ that are symmetric in the sense that the axis and coaxis of one group is coaxis and axis, respectively of the second group.*

Then, if the groups are cyclic, the planes may be classified as in Theorem 11.

13 Lemma. *Let π be a translation plane of order q^2 with spread in $PG(3,q)$, admitting an affine homology group H of order $q+1$, H in the translation complement. If q is even or $q \equiv 1 \bmod 4$, then H is cyclic.*

Proof. This is a result of Johnson [15]. $\boxed{QED}$

14 Lemma. *Let π be a translation plane of order q^2 with spread in $PG(3,q)$, admitting an affine homology group H of order $q+1$ in the translation complement. If q is even and π is non-Desarguesian and non-Hall then* {axis,coaxis} *of H is invariant under the full translation complement.*

Proof. Apply Theorem 11 above, to obtain an André plane. But, by Foulser [7], the full collineation group leaves invariant {*axis,coaxis*}, when the plane is a non-Desarguesian, non-Hall André plane. $\boxed{QED}$

Hence, in the following, we may assume that q is odd.

15 Lemma. *Let π be a non-Desarguesian, non-Hall translation plane of odd order q^2 with spread in $PG(3, q)$, admitting an affine homology group H of order $q + 1$, H in the translation complement. Let L and M denote the axis and coaxis on H. Let G denote the group generated by the set of affine homology groups of π.*

If $\{L, M\}$ is not invariant under the full translation complement of π, then the order of G is not divisible by p, for $q = p^r$.

Proof. In the argument of Theorem 18 [19], there is absolutely no use of the additional hypothesis that H is cyclic when dealing with the question of p-elements. The argument is separated into the parts when $q^2 - 1$ does and does not admit a p-primitive divisor but, in either case, no use of the cyclic assumption is used. $\boxed{QED}$

16 Lemma. *Under the assumptions of the previous lemma, if $\{L, M\}$ is not invariant under the full translation complement of π, $q + 1$ cannot have an odd prime factor u.*

Proof. If so, then we may apply Theorem 10, since we now are forced to have $> q + 1$ homology axes. Then, noting that Sylow u-subgroups of homology groups are cyclic for odd u, the argument given in Theorem 11 applies directly (the only outstanding cases left from the possibilities given in Theorem 10 is then G is $SL(2, 9)$, $u = 3$ and $q = 7, 11, 13, 17$). $\boxed{QED}$

$(*)$: **So, we are left with the following possibilities when $\{L, M\}$ is not invariant under the full translation complement of π : $q = p$ and $p + 1 = 2^a$, and $p \nmid |G|$.**

17 Lemma. *The Hall plane of order* 9 *admits* 10 *cyclic homology groups of order* 4*.*

Proof. Note that any translation plane of order 9 is Desarguesian or the Hall plane of order 9. In the latter case, π does admit homology groups of order 4. To see this, note that we may consider the Hall plane as constructed from the Desarguesian plane by the derivation of a single regulus net. So, the Hall plane is an André plane that then admits an affine homology group of order $3 + 1$. Since the full collineation group of the Hall plane is transitive on the line at infinity, the result follows (see, e. g., Lüneburg [20]). $\boxed{QED}$

18 Lemma. *The Hall plane of order q^2, $q > 3$, admits $(q^2 - q)$ cyclic homology groups of order $q + 1$.*

Proof. The Hall plane is an André plane obtained by the derivation of a single regulus net in a Desarguesian affine plane. Since the group $GL(2, q)$ fixes the derived net, it becomes a collineation group of the Hall plane. $\boxed{QED}$

Note that any affine homology group is automatically in the linear translation complement, implying that G is a subgroup of the linear translation complement $GL(4,q)$.

19 Theorem. *If π has order 7^2 and* {axis,coaxis} *is not invariant then π is Desarguesian, Hall or the Heimbeck plane of class* III.

Proof. All translation planes and their collineation groups have been determined by computer by Charnes and Dempwolff [3], as well as Mathon and Royle [22]. In particular, on p. 1211 of Charnes and Dempwolff [3], there are two tables listing first the planes with involutory homologies and second the Heimbeck planes with quaternion group of homologies. Also listed in this second table are the orbits on the line at infinity. It is somewhat difficult to check, as a computer check of the groups of the associated spreads is required, but there is exactly one non-Desarguesian plane, the Heimbeck plane of class III of order 49, with at least three homology group of order 8.

More details on the order-49 planes are given in the appendix. $\boxed{QED}$

If the axis-coaxis set is not invariant, we have noted that the collineation group cannot have order divisible by p, where the order of the plane is q^2, $p^r = q$ (and in this case, $r = 1$). Hence, we may apply Ostrom's main theorem of [21], which we include for convenience.

20 Theorem. *(Ostrom [21, (2.17)]) Let π be a translation plane of order q^2 with spread in $PG(3,q)$. Let G be a subgroup of the linear translation complement and assume that $(p,|G|) = 1$ where $p^r = q$. Let $\overline{G} = GK/K$ where K is the kernel homology group of order $q-1$.*

Then at least one of the following holds.

(a) G is cyclic.

(b) $\overline{G}$ has a normal subgroup of index 1 or 2 which is cyclic or dihedral or isomorphic to one of $PSL(2,3)$, $PGL(2,3)$ or $PSL(2,5)$.

(c) G has a cyclic normal subgroup H such that G/H is isomorphic to a subgroup of S_4.

(d) G has a normal subgroup H of index 1 or 2. H is isomorphic to a subgroup of $GL(2,q^a)$, for some a, such that the homomorphic image of H in $PSL(2,q^a)$ is one of the groups in the list given under (b).

(e) There are five pairs of points on ℓ_∞ such that if (P,Q) is any such pair, there is an involutory homology with center P (or Q) whose axis goes through Q (or P). $\overline{G}$ has a normal subgroup $\overline{E}$ which is elementary Abelian of order 16. For each pair (P,Q) each element of $\overline{E}$ either fixes P and Q or interchanges them. $\overline{G}$ induces a transitive permutation group on these ten points.

(f) $\overline{G}$ *has a subgroup isomorphic to* $PSL(2,9)$ *and acts in the following manner: Each Sylow* 3*-subgroup has exactly two fixed points on* ℓ_∞*. If* (P,Q) *is such a pair,* G *contains* (P,OQ) *and* (Q,OP)*-homologies of order* 3*. There are ten such pairs and* $\overline{G}$ *is transitive on these ten pairs.*

(g) G *has a reducible normal subgroup* H *not faithful on its minimal subspaces and satisfying the following conditions:*

Either the minimal H*-spaces have dimension two and* H *has index* 2 *in* G *or the minimal* H*-spaces have dimension* 1*. In the latter case, if* H_0 *is a subgroup fixing some minimal* H*-space pointwise, then* H/H_0 *is cyclic and* G/H *is isomorphic to a subgroup of* S_4*.*

We may now prove the following two main results.

21 Theorem. *Let* π *denote a translation plane of order* $q^2 = p^{2r}$*, for* p *a prime, with spread in* $PG(3,q)$ *that admits an affine homology group* H *of order* $q+1$*, in the translation complement.*

(1) Then either the set {axis,coaxis} *of* H *is invariant under the full collineation group or* π *is one of the following planes:*

(a) Desarguesian,

(b) the Hall plane,

(c) the plane is the Heimbeck plane of order 49 *of type* III.

(2) If there exist at least three mutually distinct affine homology groups of order $q+1$ *but one or two axes, we have one of the following situations:*

The plane is either

(a) the irregular nearfield plane of order 25*,*

(b) the irregular nearfield plane of order 49*,*

(c) $q = 11$ *or* 19 *and admits* $SL(2,5)$ *as an affine homology group (the irregular nearfield plane of orders* 11^2 *and* 19^2 *and the two exceptional Lüneburg planes of order* 19^2 *are examples),*

(d) $q \equiv -1 \bmod 4$*, and there are two homology groups* H_1*,* H_2 *of order* $q+1$*, with the same axis* M *and same coaxis* L*, exactly one of which is cyclic. Furthermore, in this last case, the group generated by the two groups* H_1H_2 *has order* $2(q+1)$*. If* K^* *denotes the kernel homology group then* $H_1H_2K^*$ *induces the regular nearfield group of dimension two on the coaxis* L*.*

Also, there is a corresponding flock of a quadratic cone admitting a collineation g fixing one regulus and permuting the remaining $q-1$ reguli in $(q-1)/2$ pairs. In this case, there is an affine homology group of order 2 acting on the flock plane.

Indeed, the conical flock spread is either Desarguesian, Fisher or constructed from a Desarguesian spread by $3q$-double-nest construction and the conical flock plane is described in Theorem 8.

22 Theorem. *Let π be a translation plane with spread in $PG(3,q)$ that admits at least two homology groups of order $q+1$. Then one of the following occurs:*

(1) $q \in \{5, 7, 11, 19, 23\}$ (the irregular nearfield planes and the exceptional Lüneburg planes are examples),

(2) π is André,

(3) q is odd and π is constructed from a Desarguesian spread by $(q+1)$-nest replacement (actually $q = 5$ or 7 for the irregular nearfield planes also occur here),

(4) q is odd and π is constructed from a Desarguesian spread by a combination of $(q+1)$-nest and André net-replacement,

(5) q is odd and $q \equiv -1 \bmod 4$ and the axis/coaxis pair is invariant under the full collineation group (furthermore, there is a non-cyclic homology group of order $q+1$),

(6) $q = 7$ and the plane is the Heimbeck plane of type III *with* 10 *homology axes of quaternion groups of order* 8.

We give the proof to both theorems as a series of lemmas. If there is an affine homology group of order $q+1$ and the pair $\{axis, coaxis\}$ is not invariant, we may assume by the above lemmas and remarks that $q = p$ and $p+1 = 2^a$, and there are no p-collineations. Hence, there are at least $q+2$ axes of homology groups of order $q+1$. Let G denote the group generated by the homology groups. Note that it is clear that G is irreducible and primitive. We refer to parts O(a) through O(g) of Ostrom's Theorem 20. In the following, we assume the above conditions and the hypothesis in the statement of the result and assume that the $\{axis, coaxis\}$ is not invariant.

23 Lemma. *Case $O(a)$ does not occur.*

Proof. Assume case O(a) above, that G is cyclic. Then G fixes each axis and coaxis, contrary to assumption. $\boxed{QED}$

24 Lemma. *Case O(b) does not occur.*

Proof. In case O(a), $\overline{G}$ has a normal subgroup $\overline{H}$ of index at most 2. $\overline{G}$ contains a subgroup $\overline{C}$ of order $(q+1)$, arising from an affine homology group of order $q+1$, so $\overline{C} \cap \overline{H}$ has order at least $(q+1)/2$. If $\overline{H}$ is isomorphic to A_4, S_4 or A_5, then $(q+1)/2$ divides 12, 24, 60, and one of the groups has a subgroup of order 2^{a-1}. Thus, simply by order, $p = 3$ or 7.

So, there is a cyclic normal subgroup $\overline{B}$ of index 2 or 4 of $\overline{G}$. Then $\overline{C} \cap \overline{B}$ of order divisible by $(q+1)/4$. Assuming that $q > 3$, we have a non-trivial intersection, then $\overline{B}$ fixes or interchanges the axis and coaxis of C, so $\overline{B}$ fixes two infinite points, and fixes all centers and cocenters of all homology groups, a contradiction, since there are at least $q+2$ centers and cocenters. $\boxed{QED}$

25 Lemma. *Case O(c) does not occur.*

Proof. In case O(c), G/H divides $2^3 \cdot 3$, implying that $C \cap H$ is a cyclic subgroup of order dividing $(q+1)/8$, where C is an homology group of order $q+1$. Assume that q is not 7. Then H is a cyclic normal group properly containing at least $q+2$ affine homologies with distinct axes, a contradiction. $\boxed{QED}$

26 Lemma. *Case O(d) does not occur or one of the listed possibilities occurs.*

Proof. In case O(d) G has a normal subgroup H of index 1 or 2. H is isomorphic to a subgroup of $GL(2, q^a)$, for some a, such that the homomorphic image of H in $PSL(2, q^a)$ is one of the groups in the list given under O(b). So H s a subgroup of $GL(2, q^a)$, containing no p-elements and contains a homology subgroup of order divisible by $(q+1)/2$. The quotient modulo the kernel homology group K^* contains a subgroup of order divisible by $(q+1)/4$. By Theorem 19, we may assume that $q > 7$. Then we need to avoid A_4, S_4 and A_5, which we will if there is a cyclic 2-subgroup of order > 4. Hence, if $(q+1) > 32$, we have a 2-group of order at least 16. Except for $p = 3$ or 7, the only outstanding case is $q = 31$, and we are forced into the S_4 case; HK^*/K^*, in $PGL(2, q^a)$ intersected with $PSL(2, q^a)$ is S_4. Hence, the 2-groups of H have order 2^5. So H is transitive on the set of homology axes. But modulo the kernel the only elements of odd order have order 3. Since there is a homology group of order 16, and there are at least $q+1 \geq 33$ axes, we have a contradiction and this case does not arise.

So, otherwise, we know that HK^*/K^* is a subgroup of a dihedral group of order dividing $2(q+1)$, which contains a characteristic cyclic subgroup; the cyclic T/K^*. Hence, we have a normal subgroup T containing K^* of index $1, 2$ or 4 such that T/K^* is cyclic. Note that T/K^* is the group induced on the line at infinity. Hence, $C \cap T$ has order at least $(q+1)/4$, for any homology group C

of order $q+1$. But, then this means that T fixes all axis-coaxis pairs of points, pointwise, a contradiction, if there are at least $q+2$ such pairs. $\boxed{QED}$

27 Lemma. *Case* O(e) *does not occur, or we have one of the listed possibilities.*

Proof. There are five pairs of points on ℓ_∞ such that if (P,Q) is any such pair, there is an involutory homology with center P (or Q) whose axis goes through Q (or P). $\overline{G}$ has a normal subgroup $\overline{E}$ which is elementary Abelian of order 16. For each pair (P,Q) each element of $\overline{E}$ either fixes P and Q or interchanges them. $\overline{G}$ induces a transitive permutation group on these ten points. This means that $q+2 \leq 10$. Since $q=p$ is odd, then $p \leq 7$ and since p is not 7, $p=3$ or 5, so $p=3$ since $p+1=2^a$, and the plane is Hall as noted above. $\boxed{QED}$

28 Lemma. *Case* O(f) *implies that* $q=7$.

Proof. Hence, again there are 10 centers, implying that $q+1 \leq 10$, so $q=7$. $\boxed{QED}$

The final case is O(g).

29 Lemma. *Case* O(g) *does not occur.*

Proof. If O(g) occurs, then G has a reducible normal subgroup H not faithful on its minimal subspaces and satisfying the following conditions: Either the minimal H-spaces have dimension two and H has index 2 in G or the minimal H-spaces have dimension 1. In the latter case, if H_0 is a subgroup fixing some minimal H-space pointwise, then H/H_0 is cyclic and G/H is isomorphic to a subgroup of S_4. However, since G is primitive, this cannot occur. $\boxed{QED}$

So, what we have proved is that the set $\{$*axis,coaxis*$\}$ is invariant. It remains to consider if we have more than three homology groups of order $q+1$.

Now assume that there are at least three homology groups of order $q+1$. If there are at least three axes of homologies, we may appeal to the previous arguments. Hence, assume that there are at least two distinct homology groups H_1 and H_2 of order $q+1$ with the same axis L and let M denote the coaxis. Let F denote the full homology group with axis L and coaxis M. Then, F is in $GL(4,q)$, so that $F \mid M$ is a subgroup of $GL(2,q)$. Since F is a homology group, the Sylow v-subgroups for v odd are cyclic, and cyclic or generalized quaternion if $v=2$ (see, e. g., [20, p. 11 (3.5)]). Let K^* denote the kernel homology group of order $q-1$. Acting on M, FK^*/K^* in $PGL(2,q)$ is a subgroup of a dihedral group of order $2(q\pm 1)$ or is A_4, S_4, A_5, and has order divisible by $(q+1)/(2,q-1)$. Moreover, if q is even then H is cyclic, as noted below.

30 Theorem. *(Johnson [15]) Let π be a translation plane with spread in $PG(3,q)$ that admits an affine homology group H of order $q+1$ in the translation complement. If any of the following conditions hold, π constructs a regular* *hyperbolic fibration with constant back half and hence a corresponding flock of a quadratic cone.*

(1) q is even,

(2) q is odd and $q \equiv 1 \bmod 4$,

(3) H is Abelian,

(4) H is cyclic.

What the previous result shows is that we may assume that H_i are cyclic when q is even or $q \equiv -1 \bmod 4$.

31 Lemma. *If H_1 and H_2 are both cyclic then $H_1 \cap H_2$ has order dividing 2.*

Proof. Then the component orbits of H_i are reguli by Theorem 1 and the following theorem for the finite case, for $i = 1, 2$. Assume that $H_1 \cap H_2$ has order > 2. Then there is an orbit of components of H_1 with at least 3 elements common to an orbit of H_2. Since both components orbits are reguli, the two reguli are equal, implying that the two orbits are equal. This implies that $H_1 = H_2$, since these groups have the same axis and coaxis. $\boxed{QED}$

32 Lemma. *If H_1 and H_2 are both cyclic then F, the full homology group with axis L, has order at least $(q+1)^2/(2, q-1)$, and the order of F divides $q^2 - 1$. Then one of the following occurs:*

(1) $q \in \{3, 7\}$ and FK^*/K^* is a subgroup of a dihedral group of order $2(q+1)$,

(2) $q \in \{3, 5, 7\}$ and FK^*/K^* is isomorphic to A_4,

(3) $q \in \{3, 5, 7, 9, 11\}$ and FK^*/K^* is isomorphic to S_4 or

(4) $q \in \{3, 7, 9, 11\}$ and FK^*/K^* is isomorphic to A_5.

Proof. The first part follows from the previous lemma. Hence, FK^*/K^* has order at least $(q+1)^2/(2, q-1)^2$ and contains a cyclic subgroup of order $(q+1)/(2, q-1)$. Assume that q is even. Then FK^*/K^* is a group of order divisible by $(q+1)^2$ and contains a cyclic subgroup of order $q+1$. Since the group is a subgroup of a dihedral group of order $2(q \pm 1)$ or A_4, S_4 or A_5, this cannot occur. Hence, q is odd and there is a subgroup of FK^*/K^* of order divisible by $(q+1)^2/4$, containing a cyclic subgroup of order $(q+1)/2$. If the group is a subgroup of a dihedral group of order $2(q+1)$, then $q+1$ divides 8. Hence, $q = 3$ or 7. So, assume that FK^*/K^* is A_4, S_4 or A_5. Therefore, $(q+1)^2 \leq 4\,|A_4, S_4$ or $A_5|$. Then, $(q+1) \leq 4\{\sqrt{3}, \sqrt{6}, \sqrt{15}\}$, implying that

$q < 15$. Thus, $q \in \{3, 5, 7, 9, 11, 13\}$. Note that $13 + 1$ cannot divide any of the orders of the groups. Hence, q cannot be 13. The exact situation is then given in the statement of the lemma. $\boxed{QED}$

33 Lemma. *In general, if F is non-solvable then F contains a subgroup isomorphic to $SL(2, 5)$ and has order dividing $q^2 - 1$, so 120 divides $q^2 - 1$.*

Proof. By Lüneburg [20] (3.6)(b), p. 10, there is a unique subgroup of F isomorphic to $SL(2, 5)$. $\boxed{QED}$

Assume that F is non-solvable and $q = 9$. Then the 3-elements must be planar and since the spread is in $PG(3, 9)$, the 3-elements are Baer collineations. However, the 3-elements in F are affine homologies. If $q = 11$ and F is non-solvable, we would require that $SL(2, 5)$ contains a subgroup of order 12, since $120(q - 1)/z$, the order of F, must divide $11^2 - 1$, implying that $(q - 1)/z = 1$. However, this cannot occur.

We note that all translation planes of order q^2 for $q = 5, 7$ are completely known by computer (see [4], [5], [3], [22]). When $q = 3$ the plane is Hall or Desarguesian, when $q = 5$ and we have an affine homology group of order 6, the group must be cyclic and two distinct groups will generate $SL(2, 3)$. By [5], the only plane of order 25 to admit $SL(2, 3)$ is the irregular nearfield plane of order 25. When $q = 7$ the only planes admitting homology groups of order 8 may be determined from [3] as is sketched in the appendices.

If there are cyclic homology groups of orders $q + 1$ and we have situation (3) in the previous lemma, assume $q = 9$ or 11. When $q = 9$, the 3-elements are Baer, a contradiction. When $q = 11$, we would require two distinct subgroups of order 12 in F, which we have indicated do occur, as indicated in the section on examples.

Hence, we have shown that when there are cyclic homology groups of orders $q + 1$, the planes are determined as stated in the main theorem.

Hence, assume that at least one of the groups, say H_1, is not cyclic. This means that $q \equiv -1 \bmod 4$.

34 Lemma. *If F is non-solvable then $q \in \{3, 7, 11, 19, 23\}$.*

Proof. Since FK^*/K^* has a subgroup of order $(q + 1)/2$, then $(q + 1)/2$ divides 60. The possible numbers that occur when $q \equiv -1 \bmod 4$ are exactly as indicated (note of course that 39 is not a prime power). $\boxed{QED}$

We have considered $q = 11$ previously. If $q = 19$ then $120(18)/z$ must divide $19^2 - 1 = (18 \cdot 20)$, implying that $18/z$ is either 3 or 1. If 3, then the plane is a nearfield plane admitting a non-solvable group, a possibility. If $18/z = 1$ then there must be a subgroup of $SL(2, 5)$ of order 20, still this is a possibility. If $q = 23$ then 120 must divide $23^2 - 1 = 22 \cdot 24$, a contradiction.

Hence, we obtain the following lemma.

35 Lemma. *If F is non-solvable the possibilities are as listed in the theorem.*

36 Lemma. *If FK^*/K^* is either A_4 or S_4 then $(q+1)/2$ divides $2^3 \cdot 3$, implying that $q \in \{3, 7, 11, 23\}$.*

Proof. Simply compute using the restriction that $q \equiv -1 \bmod 4$. $\boxed{QED}$

37 Lemma. *If there are at least three affine homology groups of order $q+1$ then $q \neq 23$.*

Proof. If $q = 23$, it follows that if there are three affine homology groups then acting on one coaxis L, the group is a subgroup of $GL(2,3) \times Z_{11}$ (the kernel homology group then is isomorphic to Z_{22}). But, this is the irregular nearfield group for the plane of order 23^2. However, there then cannot be two subgroups of order 24 with the same axis. Hence, $q \neq 23$. $\boxed{QED}$

Hence, either we have one of the situations listed in the main theorem or FK^*/K^* is a subgroup of a dihedral group of order $2(q+1)$, there are two homology subgroups of F of order $(q+1)$, at least one of which is not cyclic.

38 Lemma. *If FK^*/K^* is a subgroup of a dihedral group of order $2(q+1)$ then F contains a normal subgroup N of order $(q+1)/2$.*

Proof. We know that there is a subgroup of FK^*/K^*, which contains a normal cyclic subgroup of order $(q+1)/4$, since H_1K^*/K^* has order $(q+1)/2$. (Note that there is a unique involutory homology with a given axis.) Hence, there is a normal subgroup N of F of order $(q+1)/2$. $\boxed{QED}$

39 Lemma. *If H_1 and H_2 are distinct subgroups of F of order $(q+1)$ then $H_1 \cap H_2$ is a subgroup of order either $(q+1)/4$ or $(q+1)/2$.*

Proof. Let H_1, H_2 be two subgroups of F of order $q+1$. Consider NH_1 of order $(q+1)^2/2z$, where $z = |H_1 \cap N|$. We note that $(q+1)^2/2z$ must divide q^2-1, since homology groups are semi-regular, implying that $(q+1)/2z$ divides $(q-1)$. Since $(q+1, q-1) = 2$, it follows that $(q+1)/2z$ divides 2. Hence, $z = (q+1)/4$ or $(q+1)/2$. $\boxed{QED}$

Hence, both H_1 and H_2 contain a common subgroup of order divisible by $(q+1)/4$, normal in N and hence $H_1 \cap H_2$ contains a normal cyclic subgroup of order $(q+1)/4$. Also, H_1N has order dividing q^2-1 and is $4(q+1)$ or $2(q+1)$. Since $(q-1)/2$ is odd, it follows that H_1N has order $2(q+1)$. So, we obtain the following lemma.

40 Lemma. *$H_1N = H_2N$ is a group of order $2(q+1)$.*

Proof. Similarly, H_2N has order $2(q+1)$ and N is a subgroup of order $(q+1)/2$, whose quotient modulo K^* is cyclic of order $(q+1)/4$ or $(q+1)/2$. Since F has order dividing q^2-1, it follows that $H_1N = H_2N$. $\boxed{QED}$

Thus, letting F^- denote H_1N, it follows that H_1 and H_2 are normal subgroups of F^- of index 2. There are $q-1$ H_1-orbits of length $q+1$ that are permuted by H_2. Since 4 does not divide $q-1$, there is an H_2-orbit of H_1-orbits of length dividing 2. So, there is a subgroup H_2^- of H_2 of order $(q+1)/2$ that fixes some H_1-orbit. Since these groups have the same axis and coaxis, it follows that H_2^- is a subgroup of H_1. This means that $H_1 \cap H_2$ has order exactly $(q+1)/2$ since $H_1 \neq H_2$. Hence, $H_1H_2 = H_1N = H_2N$.

41 Lemma. *H_1H_2 is transitive on the 1-dimensional $GF(q)$-subspaces on the coaxis L.*

Proof. Since the order of H_1H_2 is $2(q+1)$ and $(q-1)/2$ is odd, and H_1H_2 is semi-regular on the coaxis, it follows that the stabilizer of a 1-dimensional subspace has order dividing $(q-1, 2(q+1)) = 2$. Hence, this proves the lemma. $\boxed{QED}$

42 Lemma. *$H_1H_2K^*$ is regular on $L - \{0\}$.*

Proof. Again, $(q-1, 2(q+1)) = 2$, implying that the order of $H_1H_2K^*$ acting on L is $2(q+1)(q-1)/2 = q^2-1$. Since K^* is transitive on each 1-dimensional $GF(q)$-subspace, we have the proof. $\boxed{QED}$

43 Lemma. *$H_1H_2K^*$ is a subgroup of $\Gamma L(1, q^2) \cap GL(2, q)$.*

Proof. Since $H_1H_2K^*$ is regular on L, we may identify L with $GF(q^2)$, implying that $H_1H_2K^*$ is a faithful subgroup of $\Gamma L(1, q^2)$ acting on L, by Zassenhaus [23]. Since H_1H_2 is an affine homology group, it is in $GL(4, q)$ and acting on L is a faithful subgroup of $GL(2, q)$. This completes the proof. $\boxed{QED}$

We have considered this situation in the section on examples and we know that there is a cyclic subgroup in H_1H_2. That is, let $G = H_1H_2K^*$. Hence, the intersection $C = G \cap GL(1, q^2)$ has order divisible by $(q^2-1)/2$, since the mappings in $\Gamma L(2, q)$ must be $GF(q)$-linear. Hence, there is a cyclic subgroup C of order $(q^2-1)/2$ in $H_1H_2K^*$ containing K^*. H_1H_2C then has order q^2-1, implying that $H_1H_2 \cap C$ has order $2(q+1)(q^2-1)/2/i$ divides q^2-1, where i is the order of the intersection. Thus, $q+1$ divides i. Thus, there is a cyclic subgroup of H_1H_2 of order $(q+1)$. Hence, without loss of generality, we may assume that H_1 is cyclic of order $(q+1)$. So, we have a group of order $2(q+1)$ admitting a cyclic subgroup of order $(q+1)$ that contains a non-cyclic group H_2 of order $(q+1)$ containing a cyclic subgroup of order $(q+1)/2$. We know the Sylow 2-subgroups of a homology group are cyclic are generalized quaternion. Assume that the latter, that a Sylow 2-subgroup of H_1H_2 is generalized quaternion. Let this group be generated by h and t such that $h^{2^n} = t^4 = 1$, $h^{2^{n-1}} = t^2$, and $tht^{-1} = h^{-1}$, and these are elements of $\Gamma L(2, q) \cap GL(2, q)$, suppose that $t : x \longmapsto x^\sigma w$, where σ is 1 or q and $h : x \longmapsto x^\tau z$, where τ is 1 or q. We may assume that H_1 is a cyclic subgroup of $GL(1,q^2)$ of order $q+1$. Hence, we may assume that h is in $GL(1,q^2)$. Therefore, $\tau = 1$ and $\sigma = q$.

We then have a regular group on L in $\Gamma L(1,q^2)$ generated by $t : x \longmapsto x^q w$ and $s : x \longmapsto xz$, such that the order of z is $(q^2-1)/2$. At any rate, the situation stated in Theorem 9, as explicated in the examples section, now completely determines the situation.

This completes the proof of our main result.

4 Applications to conical flock planes

We now are able to combine our results to prove the following theorem.

44 Theorem. *Let π_H be a translation plane of order q^2 with spread in $PG(3,q)$ admitting a cyclic affine homology group H of order $q+1$. Let $\mathcal{H}$ denote the regular hyperbolic fibration obtained from π_H and let π_E be the corresponding conical flock spread.*

Then one of the following occurs.

(1) H is normal in the collineation group F_H of π_H. In this case, the collineation group of π_H is a subgroup of the group of the regular hyperbolic fibration and induces a permutation group F_H/H on the associated q reguli sharing a component, fixing one, of the corresponding conical flock spread π_E.

If Ker is the subgroup of π_E that fixes each regulus then π_H is either

(a) Desarguesian or

(b) Kantor-Knuth of odd order or

(c) $Ker = K^*$, the kernel homology group of order $q-1$.

(2) H is not normal in F_H and there is a collineation inverting the axis and coaxis of H. In this case, π_H is one of the following types of planes:

(a) André,

(b) q is odd and π_H is constructed from a Desarguesian spread by $(q+1)$-nest replacement or

(c) q is odd and π_H is constructed from a Desarguesian spread by a combination of $(q+1)$-nest and André net-replacement.

Proof. Since H is cyclic, we know that the full collineation group of π_H leaves invariant {*axis,coaxis*} of H. If there is a collineation that inverts the axis and coaxis, then by the above mentioned result (and reproved for the special case $q+1 = 2^a$ and the more general case of arbitrary affine homology groups of order $q+1$), the planes are either André, $(q+1)$-nest plane or a combination of the two as indicated in the statement of the theorem. The result about the kernel of the action on the reguli of the conical flock plane follows directly from [12]. $\boxed{QED}$

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Appendix A—Comments

In these two appendices, we offer some of the details of the program that shows there is no non-Desarguesian plane of order 49 that admits three cyclic homology groups of order 8. This information was provided to the author by Professor Ulrich Dempwolff through a series of e-mail exchanges.

By Charnes and Dempwolff [3], there are actually 1347 nonisomorhpic translation planes of order 49. Some of these planes are transposes of each other and, in particular, homology groups H_x with axis $x = 0$ and coaxis $y = 0$ in one plane will show up as an isomorphic homology group H_y with axis $y = 0$ and coaxis $x = 0$ in the transposed plane. Hence, the abstract group generated by cyclic homology groups of order 8 will be the same in either plane. There are exactly 972 translation planes unique up to isomorphism and transposition. Hence, to determine if a translation plane of order 49 has a cyclic homology group of order 8, we need only check these 972 planes. The program given in Charnes and Dempwolff [3] has been converted into GAP and the collineation group for the first 972 planes were searched for homologies of order 8.

We shall list among the 972 planes all planes that have either one, two, or at least three cyclic homology groups of order 8. The set of planes of order 49 that admit quaternion homology groups of order 8 is the set of Heimbeck planes. We refer to Heimbeck [8] for details on these planes.

45 Remark. According to Theorem 11, every translation plane of order q^2 that admits at least two cyclic homology groups of order $q + 1$ is one of the following:

(1) André,

(2) q is odd and π is constructed from a Desarguesian spread by $(q + 1)$-nest replacement or

(3) q is odd and π is constructed from a Desarguesian spread by a combination of $(q+1)$-nest and André net-replacement.

46 Theorem. *Hence, there are at most* 2^6 *André planes and* 2^6 $(7+1)$*-nest planes in combination with André planes of order* 49*.*

Appendix B—The planes admitting cyclic homology groups of order 8

The reader might also note that the information we obtain here is essentially also available from U. Dempwolff's website : `www.mathematik.uni-kl.de/~dempw/`, but it would be necessary to run the GAP programs for the spreads and their collineation groups.

Notation. The notation used in the tables is as follows:

Number of the spread set

"Spread" means that the Spread property was checked

"Autgr ok" means that the given group acted indeed as an aut. group

"Ornung G 960" means the group has order 960

"Ornung S 64" means that a S_2 group has order 64

"Anz der fpf 16" means that a S_2 group has 16 elements of order 8

"Anz der homog 8" among these elements are 8 homologies with center on the line at infinity

"Ordn von Group(homog^G) 2688" means that the group generated by the conjugacy class of the first element in homog has order 2688

"H Group(homog[1]^G) enthaelt alle Homol." means that this group contains all homologies thus all 8-homologies.

47 Remark. We are trying to show that there are no non-Desarguesian translation planes of order 49 that admit cyclic affine homologies of order 8 with at least three axes.

(1) It follows from [15] that either

   (a) any cyclic homology group of order 8 in a translation plane of order 49 is normal within the subgroup of all homologies with the same axis and coaxis or

(b) the plane is the irregular nearfield planes admitting $GL(2,3)\times GL(2,3)$ generated by homologies.

(2) If there is exactly one axis and coaxis, there will be exactly four homologies of order 8 showing in the output, and the group generated by all homologies will have order 8.

(3) If there are exactly two homology groups of order 8, the pair $\{axis, coaxis\}$ will be fixed by the full collineation group and there will be exactly eight homologies of order 8 and the group generated will have order 64.

(4) The planes with one cyclic homology group of order 8 will have four homologies 2-group of order 8, generated by homologies and the planes with two cyclic homology groups of order 8 will have eight homologies of order 8 and a 2-group of order 64 generated by homologies.

Furthermore, we shall list only the planes that admit at least one cyclic homology group of order 8.

By the following computer calculations, there are

(5) exactly 11 (ten listed below, of which exactly nine are self-transpose) translation planes of order 49 admitting exactly one cyclic homology group of order 8 and

(6) exactly 13 translation planes of order 49 admitting exactly two cyclic homology groups of order 8, and the irregular nearfield plane admitting six cyclic homology groups of order 8, all of which are self-transpose.

B.1 The 11 spreads with one cyclic group

(1) Spread Nummer 89 Spread Autgr ok Ornung G 192 Ornung S 64 Anz der fpf 16 **Anz der homog 4,** Ordn von Group(homog^G) 8, H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(2) Spread Nummer 319 Spread Autgr ok Ornung G 192 Ornung S 64 Anz der fpf 16 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol.**Self transpose.**

(3) Spread Nummer 613 Spread Autgr ok Ornung G 96 Ornung S 32 Anz der fpf 8 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(4) Spread Nummer 938 Spread Autgr ok Ornung G 96 Ornung S 32 Anz der fpf 8 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(5) Spread Nummer 939 Spread Autgr ok Ornung G 96 Ornung S 32 Anz der fpf 8 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(6) Spread Nummer 943 Spread Autgr ok Ornung G 96 Ornung S 32 Anz der fpf 8 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Not isomorphic to its transpose. The transposed spread number will be larger than** 972 (this counts for two spreads with homology groups of order 4).

(7) Spread Nummer 953 Spread Autgr ok Ornung G 192 Ornung S 64 Anz der fpf 16 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(8) Spread Nummer 956 Spread Autgr ok Ornung G 96 Ornung S 32 Anz der fpf 8 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(9) Spread Nummer 965 Spread Autgr ok Ornung G 384 Ornung S 128 Anz der fpf 16 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(10) Spread Nummer 970 Spread Autgr ok Ornung G 576 Ornung S 64 Anz der fpf 8 **Anz der homog 4,** Ordn von Group(homog^G) 8 H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**

B.2 The 13 spreads with at least two cyclic groups

(1) Spread Nummer 2 Spread, Autgr ok, Ornung G 32256= $2^9 3^2 7$, Ornung S 512= 2^9, Anz der fpf 112, **Anz der homog 8,** Ordn von Group(homog^G) 2688= $2^7 3 \times 7$, H Group(homog[1]^G) enthaelt alle Homol. This plane is the Hall plane admitting $SL(2,7)$, where the 7-elements are Baer. **Self transpose.**

(2) Spread Nummer 3 Spread, Autgr ok, Ornung G 3072, Ornung S 1024, Anz der fpf 432, **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(3) Spread Nummer 13 Spread, Autgr ok, Ornung G 1536, Ornung S 512, Anz der fpf 112, **Anz der homog 8,** Ordn von Group(homog^G) **64**, H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(4) Spread Nummer 22 Spread Autgr ok Ornung G 768 Ornung S 256 Anz der fpf 80 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(5) Spread Nummer 24 Spread Autgr ok Ornung G 1536 Ornung S 512 Anz der fpf 304 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(6) Spread Nummer 106 Spread Autgr ok Ornung G 768 Ornung S 256 Anz der fpf 80 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(7) Spread Nummer 131 Spread Autgr ok Ornung G 1536 Ornung S 512 Anz der fpf 272 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose.**

(8) Spread Nummer 191 Spread Autgr ok Ornung G 3072 Ornung S 1024 Anz der fpf 112 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(9) Spread Nummer 314 Spread Autgr ok Ornung G 9216 Ornung S 1024 Anz der fpf 432 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(10) Spread Nummer 860 Spread Autgr ok Ornung G 384= $2^7 3$ Ornung S 128 Anz der fpf 48 **Anz der homog 8**, Ordn von Group(homog^G) **8,** H Group(homog[1]^G) enthaelt **NICHT** alle Homol. **Self transpose**. (The group does not contain all homologies.)

(11) Spread Nummer 944 Spread Autgr ok Ornung G 1536= $2^9 3$, Ornung S 512 Anz der fpf 48 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(12) Spread Nummer 959 Spread Autgr ok Ornung G 768 Ornung S 256 Anz der fpf 80 **Anz der homog 8,** Ordn von Group(homog^G) **64,** H Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.

(13) Spread Nummer 969 Spread Autgr ok Ornung G 13824 Ornung S 512= 2^9 Anz der fpf 272 **Ans der Homog 8,** Ordn von Group(homog^G) **2304**= $2^8 3^2$ H, Group(homog[1]^G) enthaelt alle Homol. **Self transpose**.