Note di Matematica **24,** n. 2, 2005, 85–95.

On some combinatorial properties of finite linear spaces

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Received: 20/3/2003; accepted: 12/1/2005.

Abstract. In this note we investigate some properties of lines of maximal size in a finite linear space. Also, we give a new and unified proof of two theorems by Hanani [6], (and Varga [10]) and Melone [9].

Keywords: Linear spaces, affine planes, projective planes

MSC 2000 classification: 05B25, 51A45

1 Introduction

In the literature on finite linear spaces one can find a number of papers which show that lines of maximal size are a tool to investigate finite linear spaces.

For example in [4–7, 9, 10] characterization results on finite linear spaces are obtained when the number of lines meeting a line of maximal size[1] is given.

In this paper we study some properties of finite linear spaces using lines of maximal size.

Also, we give a new and unified proof for two theorems of Hanani (and Varga) [6,10] and Melone [9]. Our proof is based on the analysis of the difference $|k-m|$, where k and m denote the maximum line length and the minimum point degree, respectively.

1.1 Definitions and preliminary results

A *finite linear space* on *v points* and with *b lines* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of v points and $\mathcal{L}$ is a family of b subsets (the lines) of $\mathcal{P}$ such that: *any two points are on a unique line, each line contains at least two points and there are at least two lines.*

[i]This research was partially supported by G.N.S.A.G.A. of INdAM and the MIUR project Strutture Geometriche Combinatoria e loro Applicazioni

[1]Notice that such number is greater or equal to the number of points of the linear space less one [6, 10].

The *degree* of a point p is the number $[p]$ of lines on p and the *length* of a line ℓ is its size $|\ell|$ [11].

The *order* of a finite linear space $(\mathcal{P}, \mathcal{L})$ is the integer n such that $n+1 = \max_{p \in \mathcal{P}}[p]$.

Denote by k the maximal line length and by m the minimum point degree.

Two lines ℓ and ℓ' of a linear space are *parallel* if $\ell = \ell'$ or $\ell \cap \ell' = \emptyset$.

If ℓ is a line then δ_ℓ will denote the number of lines parallel to ℓ and different from ℓ, and i_ℓ will denote the number of lines meeting ℓ and different from ℓ.

The *near–pencil* on v points is the linear space on v points with a line of length $v-1$ [11].

A (h,k)*–cross*, $3 \leq h \leq k$, is the linear space on $h+k-1$ points, with a point of degree 2 on which there are two lines of length h and k respectively [11].

A linear space is *irreducible* if every line has length at least three.

A *projective* plane is an irreducible linear space such that any two lines meet in a point [11].

The projective plane of order two is also called the *Fano plane*.

An *affine plane* is a linear space such that for every point–line pair (p, ℓ), with $p \notin \ell$, the number of lines on p missing ℓ is 1 [11].

The *Fano quasi–plane* is the affine plane of order two with the near–pencil on 3 points at infinity.

If $(\mathcal{P}, \mathcal{L})$ is a finite linear space and X is a subset of $\mathcal{P}$, such that $\mathcal{P} \setminus X$ contains at least three non–collinear points, the linear space $(\mathcal{P}', \mathcal{L}')$, where

$$\mathcal{P}' = \mathcal{P} \setminus X$$

$$\mathcal{L}' = \{\, \ell \setminus \{\ell \cap X\} \mid \ell \in \mathcal{L} \text{ and } |\ell \setminus \ell \cap X| \geq 2 \,\},$$

is called the *complement of X in $(\mathcal{P}, \mathcal{L})$* [1].

The complement of a line in a projective plane is an affine plane.

A *punctured* (*doubly–punctured*) linear space $(\mathcal{P}, \mathcal{L})$ is the complement of a point (two points) in $(\mathcal{P}, \mathcal{L})$.

1 Theorem (de Bruijn and Erdős, 1948). *Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space. Then $b \geq v$. Moreover, equality holds if and only if $(\mathcal{P}, \mathcal{L})$ is a projective plane or a near–pencil.*

2 Theorem (Hanani [6], Varga [10]). *Let $(\mathcal{P}, \mathcal{L})$ a finite linear space on v points, and let ℓ be a line of maximal length, then $i_\ell \geq v-1$, and the equality holds if and only if $(\mathcal{P}, \mathcal{L})$ is a (possibly degenerate) projective plane.*

3 Theorem (Melone [9]). *Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space such that each line of maximal length k meets exactly v other lines, then one of the following cases occurs.*

(i) $(\mathcal{P}, \mathcal{L})$ *is a finite affine plane of order* k.

(ii) $(\mathcal{P}, \mathcal{L})$ *is a punctured projective plane of order* $k-1$.

(iii) $(\mathcal{P}, \mathcal{L})$ *is the Fano quasi–plane.*

(iv) $(\mathcal{P}, \mathcal{L})$ *is the doubly–punctured Fano plane.*

In this paper a unified proof of Theorems 2 and 3 is given, and moreover the following result is proved.

4 Theorem. *Let* $(\mathcal{P}, \mathcal{L})$ *a finite linear space with* v *points and* b *lines. Let* k *and* m *denote the maximum line length and the minimum point degree, respectively, and let* $s = b - v$. *Then* $k \geq m + s$ *if, and only if,* $(\mathcal{P}, \mathcal{L})$ *is either a near–pencil, or a projective plane or the* $(3, s+2)$*–cross.*

2 Lines of maximal length

In this section $(\mathcal{P}, \mathcal{L})$ is a finite linear space with v points and $b-v = s$,($s \geq 0$ by Theorem 1).

Let L be a line of maximal length k, then $b = 1+i_L+\delta_L \geq k(m-1)+1+\delta_L$. Counting v via the lines passing through a point of degree m gives $v \leq m(k-1)+1$ and so $b \leq m(k-1)+1+s$. It follows that

$$k(m-1)+1+\delta_L \leq b \leq m(k-1)+1+s,$$

and so

$$k \geq m + \delta_L - s. \tag{1}$$

If $k = m - s$ (and so $m - s \geq 2$), then $\delta_L = 0$ for each line L of length k, hence $s = 0$, $b = v$ and from the de Bruijn-Erdős theorem it follows that $(\mathcal{P}, \mathcal{L})$ is a projective plane, or a near–pencil on 3 points.

So if $s \geq 1$, we have that $k \geq m + 1 - s$.

Finite linear spaces with constant line length k, have constant point degree (i. e. they are $2-(v, k, 1)$–designs), thus it follows that each line has a constant number δ of parallel lines, and so $k = m + \delta - s$.

The following proposition says when this property makes a linear space a $2-(v, k, 1)$–design.

Let

$$\delta = \min\{\, \delta_\ell \mid \ell \text{ line of maximal length} \,\},$$

then

5 Proposition. *Let* $(\mathcal{P}, \mathcal{L})$ *be a finite linear space with* v *points and* $b = v+s$ *lines. If* $k = m + \delta - s (\geq 2)$, *then* $(\mathcal{P}, \mathcal{L})$ *is a* $2-(v, m+\delta-s, 1)$*–design.*

Proof. Let L be a line of length k with $\delta_L = \delta$. From $k = m + \delta - s$ it follows that $b = v + s \leq m(k-1) + 1 + s = m(m + \delta - s - 1) + 1 + s$, and so by $b \geq k(m-1) + 1 + \delta = m(m + \delta - s - 1) + 1 + s$ we have

$$b = m(m + \delta - s) - m + 1 + s$$

and

$$v = m(m + \delta - s) - m + 1.$$

Thus, on a point of degree m there are all lines of length k and all points of L have degree m.

Let H be a line of length k different from L, then by definition of δ we have $\delta \leq \delta_H$, but $m + \delta - s = k \geq m + \delta_H - s$, and so $\delta_H = \delta$. Hence lines of length k have all points of degree m. Since on a point of degree m there are all lines of length k, it follows that a point of degree at least $m + 1$ cannot be connected with a point of L. Thus each point has degree m and each line has length k, and so the assertion follows. QED

6 Proposition. *$\delta_L \leq s$ for each line L of length k.*

Proof. Assume on the contrary that there is a line L with $\delta_L \geq s + 1$. From (1) it follows that $k \geq m + 1$. Hence points of degree m are on lines of length k.

If there are at least two points of degree m, then there is a single line of length k so $v \leq k + (m-1)^2$. It follows that

$$1 + s + k(m-1) + 1 \leq 1 + \delta_L + k(m-1) \leq b \leq k + (m-1)^2 + s,$$

and so

$$k(m-2) \leq m(m-2) - 1.$$

Thus being $k \geq m + 1$, it follows that

$$m(m-2) + m - 2 \leq m(m-2) - 1,$$

i. e. $m \leq 1$, contradicting the fact that on each point there are at least two lines.

Hence there is a single point of degree m. Then $b \geq km + \delta_L \geq km + s + 1$.

Thus

$$km + s + 1 \leq b \leq m(k-1) + 1 + s,$$

that is a contradiction.

Hence the assertion follows. QED

7 Corollary (Hanani [6], Varga [10]). *Each line of length k meets at least other $v - 1$ lines.*

PROOF. Let L be a line of length k, then from Proposition 6 it follows that $v+s=b=1+i_L+\delta_L\leq 1+i_L+s$, that is $i_L\geq v-1$. QED

8 Proposition. *Let $1\leq u\leq m-2$ be an integer. If $k=m-u$ then $s\geq 3u-1$.*

PROOF. Let L be a line of size $k=m-u$, and let h be a line parallel to L. Let x and y be two points of h, since they have degree at least m, we have that on x (respectively on y) there are $m-k-1$ lines parallel to L and different from L, so $\delta_L\geq 2(m-k)-1$. Since $k\geq m+\delta_L-s$, it follows that $k\geq m+2(m-k)-1-s$, thus $3(m-u)\geq 3m-(s+1)$ from which it follows that $s\geq 3u-1$. QED

2.1 Finite linear spaces with $k\geq m+s$

Finite projective planes, near–pencils and $(3,s+2)$–crosses fulfill the inequality $k\geq m+s$. In this section we are going to show that there is no other finite linear space with this property, i. e. we prove Theorem 4.

If $s=0$ by the fundamental theorem $(\mathcal{P},\mathcal{L})$ is a near–pencil or a finite projective plane. So one has to consider the case $s\geq 1$.

9 Theorem. *If $s\geq 1$ and $k\geq m+s$ then $m=2$ and $(\mathcal{P},\mathcal{L})$ is the $(3,s+2)$–cross.*

The following series of propositions gives the proof of Theorem 9.

10 Proposition. *If $k\geq m+s$, $(s\geq 1)$, then there is a single point of degree m, and either $s=1$ and $(\mathcal{P},\mathcal{L})$ is the $(3,3)$–cross or there is a single line of length k.*

PROOF. Assume on the contrary that there are at least two points of degree m. Then there is a single line L of length k, and each point of degree m is on L.

Counting v via the lines on a point of degree m, we have $v\leq k+(m-1)^2$. Thus,

$$k(m-1)+1+\delta_L\leq b\leq k+(m-1)^2+s,$$

from which

$$k(m-2)+\delta_L\leq m(m-2)+s,$$

that is

$$(k-m)(m-2)\leq s-\delta_L.$$

If $m>3$ then $s<s-\delta_L$, a contradiction. So $m\leq 3$.

If $m=2$, then $v=k+1$ and $(\mathcal{P},\mathcal{L})$ is the near–pencil on $k+1$ points, a contradiction since $s\geq 1$.

If $m=3$, then from the previous equation it follows that $k=m+s$ and $\delta_L=0$. Each line different from L has length at most 3, so $v\leq k+4$.

¿From $m = 3$ it follows that $v \geq k + 2$.

If $v = k + 2$, then from $\delta_L = 0$ it follows that the line connecting the two points outside of L meets L in a point p with $[p] = 2 < m$, that is impossible.

If $v = k + 3$, then from $\delta_L = 0$ it follows that there are exactly three points of degree 3 on L and exactly three lines of length 3. The points of L have degree either 3 or 4, and so $b = 3k - 2$. Since $b = v + s = k + 3 + s$ we have

$$3k - 2 = v + s = k + 3 + s$$

hence $2k = 5 + s$.

Thus $6 + 2s = 2m + 2s = 5 + s$, a contradiction.

If $v = k + 4$, from $\delta_L = 0$ it follows that there are exactly three points of degree 3 and six lines of length 3.

Since the points of L have degree 3 or 5, we have $b = 4k - 5$. From $b = v + s = k + 4 + s$ a contradiction follows.

Hence there is a single point of degree m.

If there are at least two lines of length k, then they intersect in the single point of degree m, and each other point has degree at least k. Thus, if L is a line of length k, $b \geq m + (k - 1)^2 + \delta_L$. So

$$m + (k - 1)^2 + \delta_L \leq b \leq km - m + 1 + s,$$

$$k^2 - 2k + \delta_L \leq km - 2m + s$$

$$(k - 2)(k - m) \leq s - \delta_L$$

$$s(k - 2) \leq s - \delta_L$$

$$s(k - 3) \leq -\delta_L$$

and so $k \leq 3$.

Since $k \geq m + s \geq 3$, it follows that $k = 3$, $m = 2$ and $s = 1$.

So $v = 5$, $b = 6$ and $(\mathcal{P}, \mathcal{L})$ is the $(3, 3)$–cross.

Thus, if $(\mathcal{P}, \mathcal{L})$ is not the $(3, 3)$–cross, then there is a single line of length k. QED

Thus, from now on we may assume $s \geq 2$ and that there is a single line of length k.

11 Proposition. *If $s \geq 2$, $k \geq m+s$ and $m = 2$, then $(\mathcal{P}, \mathcal{L})$ is a $(3, s+2)$–cross.*

PROOF. Since $m = 2$, it follows that $(\mathcal{P}, \mathcal{L})$ is a (h, k)–cross. If $h = 2$, then $(\mathcal{P}, \mathcal{L})$ is a near–pencil, contradicting the fact that $s \neq 0$. So $h \geq 3$. Furthermore $k \geq s + 2$.

From $v = h + k - 1$ and $b = (h-1)(k-1) + 2$, it follows that

$$(h-1)(k-1) + 2 - h - (k-1) = b - v = s,$$

and so

$$(h-2)(k-2) = s.$$

By $k - 2 \geq s > 0$, it follows that $h = 3$ and $k = s + 2$, and so the assertion follows. QED

12 Proposition. *There is no finite linear space with $b - v = s \geq 1$, $k \geq m + s$ and $m \geq 3$.*

PROOF. Since $m \geq 3$, by Proposition 2.4 we have that there is a single point of degree m, $s \geq 2$ and there is a single line of length k.

Thus

$$\sum_{p \in \mathcal{P}} [p] \geq m + (k-1)(m+1) + (v-k)k = km + vk + k - k^2 - 1$$

and

$$\sum_{\ell \in \mathcal{L}} |\ell| \leq k + (m-1)(k-1) + (v + s - m)m.$$

From

$$\sum_{p \in \mathcal{P}} [p] = \sum_{\ell \in \mathcal{L}} |\ell|$$

it follows that

$$v(k-m) \leq (k-m)(k+m) + sm - (k - 2 + m). \tag{2}$$

From (2) it follows that

$$v \leq k + m + \frac{m(s-1)}{k-m} - \frac{2}{k-m},$$

since $k - m > s - 1$ we have

$$v < k + m + m = k + 2m.$$

But $v = b - s \geq m + (k-1)m + \delta_L - s$, and so, since $m \geq 3$,

$$3k + \delta_L - s < k + 2m,$$

$$2k < 2m + s - \delta_L$$

$$2m + 2s < 2m + s - \delta_L,$$

a contradiction. QED

So Theorem 9 is completely proved.

3 A new proof of a structure theorem

In this section we are going to give a new proof of the Theorems 2 and 3. Actually we prove the following result.

13 Theorem. *Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space such that each line of maximal length k meets at most v other lines, then one of the following cases occurs.*

(i) $(\mathcal{P}, \mathcal{L})$ *is a projective plane or a near–pencil.*

(ii) $(\mathcal{P}, \mathcal{L})$ *is a finite affine plane of order k.*

(iii) $(\mathcal{P}, \mathcal{L})$ *is a punctured projective plane of order $k-1$.*

(iv) $(\mathcal{P}, \mathcal{L})$ *is the Fano quasi–plane.*

(v) $(\mathcal{P}, \mathcal{L})$ *is the doubly–punctured Fano plane.*

The proof we give is more geometric than the previous ones. Furthermore when each line of length k meets exactly v other lines, our proof is shorter than that contained in [9].

The following series of lemmas is the proof of Theorem 13. Before to start with the proof, we recall that $i_L \geq v-1$ for all lines of maximal length k (Corollary 2.1).

14 Lemma. *If there is a line L of length k such that $i_L = v-1$ then $(\mathcal{P}, \mathcal{L})$ is a projective plane or a near–pencil.*

Proof. From $v+s = b = 1+i_L+\delta_L = v+\delta_L$ it follows that $s = \delta_L$. Therefore $b \geq k(m-1)+1+\delta_L$, and so being $b = v+s \leq m(k-1)+1+s = m(k-1)+1+\delta_L$ there follows that $k \geq m$.

Assume now that $k \geq m+1$. If there is a single point of degree m, then $b \geq km + \delta_L$. On the other hand $b \leq m(k-1)+1+\delta_L$, and so comparing the two values of b one obtains $m \leq 1$, that is impossible.

Hence there are at least two points of degree m. Thus there is a single line of length k and $b = v + \delta_L \leq k + (m-1)^2 + \delta_L$. Since $b \geq k(m-1)+1+\delta_L$, we have

$$k(m-2) \leq m(m-2),$$

and so by $k \geq m+1$ it follows that $m = 2$. Hence $v = k+1$ and $(\mathcal{P}, \mathcal{L})$ is the near–pencil on $k+1$ points.

If $k = m$, then $b = v + \delta_L \leq m(m-1)+1+\delta_L$. On the other hand $b \geq m(m-1)+1+\delta_L$, and so each point of L has degree m, and on a point of degree m there are only lines of length m, since $v = m(m-1)+1$.

If there is a point outside of L of degree at least $m+1$, then $v \geq 2+m(m-1)$, a contradiction.

Hence $\delta_L = 0$, and so $b = v$ and from Theorem 1 the assertion follows. QED

So from now on we may assume that every line of length k meets exactly v other lines. So, from $v + s = b = 1 + i_L + \delta_L$ it follows that $s = 1 + \delta_L$, for each line L of length k. Thus every line L of maximal length has exactly $\delta_L = \delta$ parallel lines. Thus, by equation (1), $k \geq m + \delta - s = m - 1$.

15 Lemma. *If $k = m - 1$, then $(\mathcal{P}, \mathcal{L})$ is an affine plane of order $m - 1$.*

Proof. Let L be a line of length k, then $b = v+1+\delta \leq m(m-2)+1+1+\delta$. From $b \geq k(m-1)+1+\delta = (m-1)^2+1+\delta$ it follows that $b = (m-1)^2+1+\delta$ and $v = (m-1)^2$. Hence on a point of degree m there are all lines of length $m-1$, and each point of L has degree m. Since each line of maximal length has exactly δ parallel lines, it follows that each line of length $m-1$ has all points of degree m. Hence all the points of $(\mathcal{P}, \mathcal{L})$ have degree m and so all the lines have length $m - 1$. It follows that $(\mathcal{P}, \mathcal{L})$ is an affine plane of order $m - 1$. QED

16 Lemma. *If $k = m$, then $(\mathcal{P}, \mathcal{L})$ is a punctured projective plane of order $m - 1$ or the Fano quasi–plane.*

Proof. Let L be a line of maximal length $k = m$, then $b = v + 1 + \delta \leq m(m-1) + 1 + 1 + \delta$. On the other hand $b \geq m(m-1) + 1 + \delta$, so

$$b \in \{\, m(m-1) + 1 + \delta,\ m(m-1) + 2 + \delta \,\}.$$

If $b = m(m-1) + 1 + \delta$, then $v = m(m-1)$, and so on a point of degree m there are $m - 1$ lines of length m and one line of length $m - 1$. Moreover each point of L has degree m, so L meets all the lines of length m. Since each line of length m has $s - 1 = \delta_L$ parallel lines, it follows that each line of length m has all points of degree m. If there is a line h of length at most $m - 2$, then it is parallel to all the lines of length m, and so if p is a point of L, then the parallel lines on p to h are at least the $m - 1$ lines of length m on p, that is $[p] \geq m - 1 + |h| \geq m + 1$, a contradiction. Hence the lines have length $m - 1$ and m.

If x is a point of degree at least $m + 1$, then on x there is no line of length m. Thus a line t parallel to L is parallel to all the lines of length m. It follows that if p is a point of L, then $m = [x] \geq |t| + m - 1 \geq m + 1$,a contradiction.

Hence all points have degree m. It follows that $\delta = 0$, and so $b = m(m - 1) + 1 = v + 1$. Thus by Bridges theorem [2] $(\mathcal{P}, \mathcal{L})$ is a punctured projective plane[2] of order $m - 1$.

[2]Clearly one can prove directly that $(\mathcal{P}, \mathcal{L})$ is a punctured projective plane of order $m - 1$.

Let now $b = m(m-1)+1+1+\delta$. Then $v = m(m-1)+1$, and so on a point of degree m there are all lines of length m, and L has a point of degree $m+1$ and each line of length m has exactly one point of degree $m+1$ and $m-1$ points of degree m. So if there is a point of degree at least $m+2$, then on it there is no line of length m, and so it cannot be connected with points of degree m, that is impossible. So the maximum point degree is $m+1$.

Let p be a point of L, H a line on p different from L, and q the point of degree $m+1$ of L. Let t be the line on q parallel to H, and z be a point of t different from q, the parallel on z to L meets meets H, and so, since H has exactly one point of degree $m+1$, it follows that $t = \{q, z\}$, and also the parallel on z to L has length 2. It follows that $m = 3$, otherwise there is a line of length m, that has on z two parallel lines, and so $[z] \geq m+2$, a contradiction! So $v = 7$, and in $(\mathcal{P}, \mathcal{L})$ there are three points of degree $m+1 = 4$, on which there are two lines of length 2 and two of length 3, and four points of degree 3. So $b = 9$ and $\delta_L = 1$. Deleting the three points of degree $m+1$ we obtain the affine plane of order 2, and so $(\mathcal{P}, \mathcal{L})$ is the inflated affine plane of order 2 with a near–pencil on three points at infinity, that is the Fano quasi–plane. QED

17 Lemma. *If $k \geq m+1$, then $(\mathcal{P}, \mathcal{L})$ is the doubly punctured Fano plane.*

Proof. Let L be a line of length k. If there is a single point p of degree m, then $b \geq km + \delta$, so from $b \leq m(k-1)+2+\delta$ it follows that $m \leq 2$. Hence $m = 2$, $b = 2k + \delta$ and so all the points of $L \setminus \{p\}$ have degree $m+1 = 3$. If there is another line of length k, then $k \leq 3$, and so $k = m+1 = 3$ and $\delta = 0$. Hence all the points of $(\mathcal{P}, \mathcal{L})$ different from p have degree 3. Then $v = 5$, $b = 6$ and $(\mathcal{P}, \mathcal{L})$ is the doubly–punctured Fano plane.

If L is the unique line of length k, then counting v via the lines on a point of degree 3, we have $v = k+2$. From $i_L = 3(k-1)+2 = k+2$, it follows that $2k = 3$, a contradiction.

Finally, assume that there are at least two points of degree m, then L is the unique line of length k, so $v \leq k + (m-1)^2$. Hence $b \leq k + (m-1)^2 + 1 + \delta_L$. Since $b \geq k(m-1)+1+\delta$, it follows that $m \leq 3$ and $k = m+1$. If $m = 2$, then $v = 4$ and $(\mathcal{P}, \mathcal{L})$ is the near–pencil on 4 points, that is impossible! So $m = 3$, $k = 4$, all the points of L have degree 3 and $v = k+4 = 8$. On a point of degree m there are L and two lines of length 3. It follows that there are eight lines of length 3, that is impossible since the four points outside of L give rise to at most six lines of length 3. QED

Thus Theorem 13 is completely proved.

Actually, a line of length $m-1$ gives rise to a partition of $\mathcal{P}$, and so adding a new point to these lines one obtains a projective plane.

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