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A note on some homology spheres which are 2-fold coverings of inequivalent knots

Agnese Ilaria Telloni
Dipartimento di Matematica, Università di Modena e Reggio E.,
Via Campi 213/B, 41100 Modena, Italy.
agneseilaria.telloni@unimo.it

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Abstract. We construct a family of closed 3–manifolds $M_{\alpha,r}$, which are homeomorphic to the Brieskorn homology spheres $\Sigma(2, \alpha+1, q+2r-1)$, where $q = \alpha(r-1)$ and both $\alpha \geq 1$ and $q \geq 3$ are odd. We show that $M_{\alpha,r}$ can be represented as 2–fold covering of the 3–sphere branched over two inequivalent knots. Our proofs follow immediately from two different symmetries of a genus 2 Heegaard diagram of $\Sigma(2, \alpha+1, q+2r-1)$, and generalize analogous results proved in [BGM], [IK], [SIK] and [T].

Keywords: 3–manifold, branched covering, orbifold, fundamental group, homology 3–sphere, $(1,1)$-knot, torus knot.

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1 Introduction

An interesting class of closed connected 3–manifolds is constituted by the 2–fold coverings of the 3–sphere branched over knots. It is well-known that such a representation is not unique, in general. Several examples of inequivalent knots with the same 2–fold branched covering space were given in [BGM], [IK], [Mo], [SIK], [T] and [V]. If the 2–fold branched covering is a spherical Seifert manifold, then the representation is unique (see, for example, [Mo]). In the non-spherical case, the representation is not unique. If a non-spherical Seifert manifold is the 2–fold covering of $\mathbb{S}^3$ branched over a knot K, then either K is a Seifert knot (i.e., $\mathbb{S}^3 \setminus K$ admits a Seifert fibration by circles), or K is a Montesinos knot, that is, $\mathbb{S}^3 \setminus K$ admits a Seifert fibration by circles and intervals (see, for example, [BZ], Chapter 12, p.195). The two situations can occur simultaneously for the same manifold (see the papers quoted above). In [BGM] and [SIK] it was independently shown that the Brieskorn homology 3–sphere $\Sigma(2, 3, 7)$ is the 2–fold covering of $\mathbb{S}^3$ branched over two inequivalent knots, i.e., the torus knot $T(3, 7)$ and the Montesinos knot $\mathbf{m}(-1; 1/2; 1/3; 1/7)$. In [IK] it was constructed combinatorially a class of 3–manifolds M_q which are homeomorphic to the Brieskorn homology 3–spheres $\Sigma(2, 3, q)$, where $(2, 3, q)$ are relatively prime. See [Mi] for the definitions and basic results on such manifolds. Then it was shown in [IK] that M_q is the 2–fold covering of $\mathbb{S}^3$ branched over a knot K_q which is inequivalent with the torus knot $T(3, q)$ for $q \geq 7$. Moreover, those authors showed that two inequivalent Heegaard splittings of genus 2, which represent $\Sigma(2, 3, q)$, and are associated with $T(3, q)$

and K_q, become equivalent after a single stabilization (see Section 2 for more definitions). In [IK] the proofs of the above results are performed by using the combinatorial representation of 3–manifolds via special classes of edge-colored graphs, known as crystallizations (see the references of the quoted paper). In this paper we extend the results of [BGM], [IK], [SIK], and [T] for the class of the Brieskorn homology spheres $\Sigma(2, \alpha+1, q+2r-1)$, where $q = \alpha(r-1)$ and both $\alpha \geq 1$ and $q \geq 3$ are odd. Our proofs are very fast and arise immediately from two different symmetries of a genus 2 Heegaard diagram $D(\alpha, r)$, constructed in the next section, which represents the above homology sphere.

2 Heegaard diagrams

Let M be a closed connected orientable 3–manifold. A *Heegaard splitting* of M is a pair (V, W) of two homeomorphic orientable compact cubes with handles V and W such that $M = V \cup W$ and $V \cap W = \partial V = \partial W$ (see for example [R], Chapter 9, Section C, and [FM], Chapter 5). The closed connected orientable surface $F = \partial V = \partial W$ is called the *Heegaard surface* of the splitting (V, W) of M. It is known that every closed connected orientable 3–manifold M admits a Heegaard splitting. The *Heegaard genus* $g(M)$ of M is the smallest integer g such that M has a Heegaard surface of genus g. Two Heegaard splittings (V, W) and (V', W') of M are said to be *equivalent* if there is a homeomorphism $h : M \to M$ such that $h(V) = V'$ or W', and $h(W) = W'$ or V'. Given a splitting (V, W) of M, let $D_1, \ldots, D_g$ be a collection of pairwise disjoint properly embedded discs in W which cut W into a 3–cell. The pairwise disjoint simple closed curves $\mathbf{w}_i = \partial D_i$ cut $F = \partial W$ into a 2–sphere with $2g$ holes. We say that $\mathbf{w} = \{\mathbf{w}_1, \ldots, \mathbf{w}_g\}$ is a set of *meridians* of the handlebody W. Let $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_g\}$ be a set of meridians of the handlebody V. Then the triple $(F, \mathbf{v}, \mathbf{w})$ is called a *Heegaard diagram* associated to the splitting (V, W) of M (or, briefly, a *Heegaard diagram* of M). Following [BH], we recall that a Heegaard diagram associated to the splitting (V, W) of M is said to be *2-symmetric* if it satisfies the following conditions: (1) there is an orientation–preserving involution ρ of M which sends V onto V (resp. W onto W); (2) the orbit space V/ρ (resp. W/ρ) of V (resp. W) under the action of ρ is a 3–ball; (3) the image of the fixed point set of ρ is an unknotted set of arcs in the ball V/ρ (resp. W/ρ). A Heegaard diagram can be drawn in a plane by flattening the above 2–sphere with $2g$ holes (whose quotient space is the Heegaard surface F). In this case, a set of meridians can be re–obtained by identifying in pairs the boundaries of the holes, while the other one gives rise to a set of pairwise disjoint simple arcs connecting the boundaries of the holes. Of course, there exist many different Heegaard diagrams representing the same manifold. The equivalence problem was solved by Singer in [S]: two different Heegaard diagrams of the same 3–manifold are related by a finite sequence of certain elementary moves (and/or their inverses), called *Singer's moves*. For every couple of odd integers $\alpha \geq 1$ and $r \geq 3$, set $q = \alpha(r-1)$, and let us consider the family of planar graphs $D(\alpha, r)$ depicted in Figure 1. Eachone of the circles F_1, F_2, F_1' and F_2' has exactly $2q + 2r - 1$ vertices which are connected by $2q + 2r - 1$ arcs between the circles. These arcs give rise to exactly two simple closed disjoint curves in the Heegaard surface F. In the planar representation shown in Figure 1, the arcs forming one of these curves are labelled by an arrow, while the arcs forming the other curve are labelled by two arrows. As depicted in Figure 1, there is a first sequence of $r-1$ alternate 1– and 2–arrow horizontal arcs from the circle F_2 (resp. F_1') to F_1 (resp. F_2'), which is followed by a second sequence of $r-1$ alternate 1– and 2–arrow horizontal arcs from the circle F_1 (resp. F_2') to F_2 (resp. F_1'). Furthermore, there are $q+1$ alternate 2– and 1–arrow (resp. 1– and 2–arrow) vertical arcs from F_1' (resp. F_2') to F_1 (resp. F_2). Finally, there are q arcs joining F_1 (resp. F_2) with F_2' (resp. F_1'). In particular, they are an alternance of $(r-1)/2$-tuple of pairs formed by one 2-arrow line and one 1-arrow line and $(r-1)/2$-tuple of pairs formed by one 1-arrow line and one 2-arrow line. The constructed planar graph $D(\alpha, r)$ admits two different symmetries: there is a rotational symmetry of order two which interchanges the circles F_1 and F_2 and there is an orientation-preserving involution which fixes two symmetry axes on the circles F_1 and F_2 and the axis connecting the vertices $3q/2+1$ and $7q/2+2$ on the circle obtained from the horizontal line plus infinity (see Figure 1). In particular, the fixed axis of the circle F_1 has one end on the vertex $-t$ (i.e. $2r+2q-1-t$ mod $2r+2q-1$) and the other one in the middle point (on F_1) of the vertices labelled by $2r-2+t$ and $2r-1+t$, while the axis of F_2 connects the point $q+1-t$ with the middle point (on F_2) of the vertices labelled by $q+2r-1+t$ and $q+2r+t$, where $t=(q-r+1)/2$.

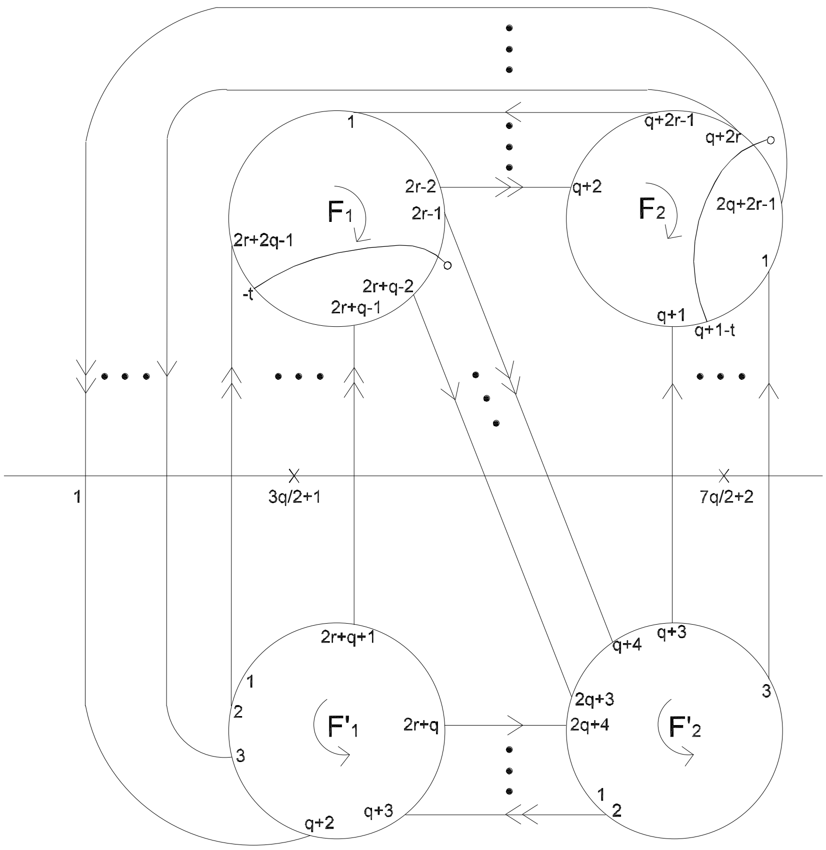

Figure 1. The planar graph $D(\alpha, r)$

Let F be the closed orientable surface of genus 2 obtained by identifying the holes F_1 and F_2 with F_1' and F_2', respectively, so that equally labelled vertices are identified. Then the arcs with one (resp. two) arrow (s) connect together into a closed curve v_1 (resp. v_2) on F. Let u_i, $i=1,2$, be the boundary curve of the hole F_i. The triple (F, u, v), $u=(u_1, u_2)$, and $v=(v_1, v_2)$, satisfies the condition that the surfaces $F \setminus u$ and $F \setminus v$ be connected, and any connected component of $F \setminus (u \cup v)$ is an open 2-cell. So we get (see for example [FM], Proposition 5.2, p.130)

Theorem 2.1. *For every couple of odd integers $\alpha \geq 1$ and $r \geq 3$, the planar graph $D(\alpha, r)$ in Figure 1 together with the above-mentioned pairings of (F_1, F_1') and (F_2, F_2') is a genus 2 Heegaard diagram, also denoted by $D(\alpha, r)$, of a closed connected orientable 3–manifold $M(\alpha, r)$.*

From the above Heegaard diagram, we immediately obtain the following result.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, the fundamental group $\Pi(\alpha, r)$ of the closed 3–manifold $M(\alpha, r)$ can be cyclically presented by*

$$\begin{aligned}\Pi(\alpha, r) &= \langle x, y \colon [(xy)^{(r-1)/2}(yx)^{(r-1)/2}]^{(\alpha+1)/2} = (yx)^{(\alpha+1)(r-1)/2}y, \\ &\qquad [(yx)^{(r-1)/2}(xy)^{(r-1)/2}]^{(\alpha+1)/2} = (xy)^{(\alpha+1)(r-1)/2}x\rangle \\ &\cong \langle u, v \colon \{[(uv)^{(\alpha+1)/2}(u^{-1})^{\alpha+1}][(vu)^{(\alpha+1)/2}(v^{-1})^{\alpha+1}]\}^{(r-1)/2}u^{-1} = 1, \\ &\qquad \{[(vu)^{(\alpha+1)/2}(v^{-1})^{\alpha+1}][(uv)^{\alpha+1}(u^{-1})^{\alpha+1}]\}^{(r-1)/2}u^{-1} = 1\rangle.\end{aligned}$$

These presentations are geometric, that is, they correspond to spines of the considered manifold.

From the above presentations, one can easily verify that the first integral homology group of $M(\alpha, r)$ is trivial.

3 Covering properties

As a consequence of the symmetries of the Heegaard diagrams $D(\alpha, r)$, we can determine some covering properties of the homology spheres $M(\alpha, r)$. In particular, the rotational symmetry of order two which interchanges the circles F_1 and F_2 of $D(\alpha, r)$ corresponds in a natural way to the involution τ of $\Pi(\alpha, r)$ which sends x to y, and viceversa. So we can construct the split extension group $E(\alpha, r)$ of $\Pi(\alpha, r)$ by the action of the cyclic group $\langle \tau \colon \tau^2 = 1\rangle$.

Theorem 3.1. *With the above notation, the split extension group $E(\alpha, r)$ has the finite presentation*

$$E(\alpha, r) = \langle x, \tau \colon \tau^2 = 1,\ [(x\tau)^{r-2}x]^{\alpha+2} = (x\tau)^{(\alpha+2)(r-1)+1}\rangle.$$

Proof. Setting $y = \tau^{-1}x\tau$ in the cyclic relations of $\Pi(\alpha, r)$ (see Theorem 2.2), we obtain

$$[(x\tau^{-1}x\tau)^{(r-1)/2}(\tau^{-1}x\tau x)^{(r-1)/2}]^{(\alpha+1)/2} = (\tau^{-1}x\tau x)^{(\alpha+1)(r-1)/2}\tau^{-1}x\tau,$$

that is

$$[(x\tau)^{r-2}x]^{\alpha+1} = \tau(x\tau)^{(\alpha+1)(r-1)+1}.$$

Multiplying by $(x\tau)^{r-2}x$ on left of both sides, we get just the second relation in the group presentation of the statement. □

We recall that a knot K in a lens space $L(h, \ell)$ (possibly $\mathbb{S}^3$) is said to be a *(1,1)-knot* if there exists a genus one Heegaard splitting $(L(h, \ell), K) = (V_1, K_1) \cup (V_2, K_2)$, where V_i is a solid torus and $K_i \subset V_i$ is a properly embedded trivial arc, for $i = 1, 2$, and $\phi : (\partial V_2, \partial K_2) \to (\partial V_1, \partial K_1)$ is an attaching homeomorphism. An arc K properly embedded in a solid torus V is said to be *trivial* if there is a disk D in V with $K \subset \partial D$ and $\partial D \setminus K \subset \partial V$ (see, for example, [CK]). Set $W_i = (V_i, K_i)$, $i = 1, 2$. We call the pair (W_1, W_2) a *(1,1)-splitting* of $(L(h, \ell), K)$.

Theorem 3.2. *For every odd integers $\alpha \geq 1$ and $r \geq 3$, the genus 2 homology 3–sphere $M(\alpha, r)$ is homeomorphic to the 2–fold cyclic covering of $\mathbb{S}^3$ branched over the torus knot $T(\alpha+2, q+2r-1)$, where $q = \alpha(r-1)$.*

Proof. Let $\mathcal{O}_2(\alpha, r)$ denote the 3–dimensional orbifold obtained from $M(\alpha, r)$ under the rotational symmetry which interchanges the circles F_1 and F_2. Then $\mathcal{O}_2(\alpha, r)$ has the 3–sphere as topological underlying space, and its singular set is a knot, $L(\alpha, r)$ say, with branching index 2. The 2-fold covering $M(\alpha, r) \to \mathcal{O}_2(\alpha, r)$ induces the exact sequence of groups $1 \to \Pi(\alpha, r) \to E(\alpha, r) \to \mathbb{Z}_2 \to 1$, hence the split extension group $E(\alpha, r)$ is the fundamental group of $\mathcal{O}_2(\alpha, r)$. Using a geometric algorithm described in [GH], it is possible to draw explicitly the knot $L(\alpha, r)$ which arises from the symmetry axis of the involution τ on $D(\alpha, r)$. We see that $L(\alpha, r)$ is a (1,1)-knot, hence, a two-generator knot (see, for example, [CK]). Thus it is a prime knot on $\mathbb{S}^3$ by [N]. This implies that $L(\alpha, r)$ is completely determined by its group (see, for example, Theorem 6.1.12 of [K]). From Theorem 3.1 the group of $L(\alpha, r)$ can be presented by $\langle x, \tau \colon [(x\tau)^{r-2}x]^{\alpha+2} = (x\tau)^{(\alpha+2)(r-1)+1} \rangle$. Moreover, (τ, x) is a meridian-longitude pair of the knot. Setting $a = (x\tau)^{r-2}x$ and $b = x\tau$ with inverse relations $x = b^{2-r}a$ and $\tau = a^{-1}b^{r-1}$, the above presentation becomes $\langle a, b : \ a^{\alpha+2} = b^{q+2r-1} \rangle$. Since the transformation matrix between the pairs (τ, x) and (a, b) is $\begin{pmatrix} -1 & 1 \\ r-1 & 2-r \end{pmatrix}$ with determinant -1, we see that (a, b) is also a meridian-longitude pair of the knot. Since the relation of the knot group is $a^{\alpha+2} = b^{q+2r-1}$, it just defines the torus knot of type $(\alpha+2, q+2r-1)$, where $q = \alpha(r-1)$. □

The Heegaard diagram $D(\alpha, r)$ in Figure 1 admits a further orientation-preserving involution which fixes the above-mentioned symmetry axes on the circles F_1 and F_2 and the axis connecting the points $3q/2+1$ and $7q/2+2$ of the circle given by the horizontal axis closed at infinity (see Figure 1). By a construction described in [BH], [KV] and [T], we get the following result

Theorem 3.3. *The genus 2 homology 3–sphere $M(\alpha, r)$, $\alpha \geq 1$ and $r \geq 3$ odd, is the 2–fold cyclic covering of $\mathbb{S}^3$ branched over the 3-bridge knot $K(\alpha, r)$ depicted in Figure 2 (the case $\alpha = 1$ is drawn in Figure 3).*

Theorem 3.4. *For every couple of odd integers $\alpha \geq 1$ and $r \geq 3$, the knot $K(\alpha, r)$ is inequivalent with the torus knot $T(\alpha+2, q+2r-1)$, where $q = \alpha + 2r - 1$.*

Proof. It is well-known that the bridge number of the torus knot of type (p, q), with (p, q)=1, is $b = min(|p|, |q|)$ ([Mu], Theorem 7.5.3). So, the torus knot $T(\alpha+2, q+2r-1)$ has exactly $min(\alpha+2, q+2r-1) = \alpha+2$ bridge, while the bridge number of $K(\alpha, r)$ is 3. Hence, these knots are inequivalent for every $\alpha > 1$. In case α=1, we prove that the knots have different genus, and hence they are distinct. Following [Mu], the genus of the torus knot of type (p, q) is $(p-1)(q-1)/2$. So, if $\alpha = 1$, then the genus of the torus knot $T(\alpha+2, q+2r-1)$ is $3(r-1)$. We can determine the genus of the knot $K(\alpha, r)$ by constructing a Seifert surface $F(\alpha, r)$ having $K(\alpha, r)$ as its boundary (for more details on Seifert surfaces see [R]). If d and c denote the number of Seifert circles and the number of crossings of $K(\alpha, r)$, respectively, then we have $g(K(\alpha, r)) = (1-d+c)/2$. Under the condition $\alpha = 1$, we obtain $c = 4+6(r-1)$ and $d = 3(r-1)/2+1$ or $d = (r-1)/2+6$ if 4 divides q or not. Thus $g(K(\alpha, r)) = 9(r-1)/4+2$ (resp. $11(r-1)/4-1/2$) if 4 divides q (resp. 4 does not divide q). But eachone of these numbers can not be equal to the genus of the torus knot $T(\alpha+2, q+2r-1)$. This completes the proof. □

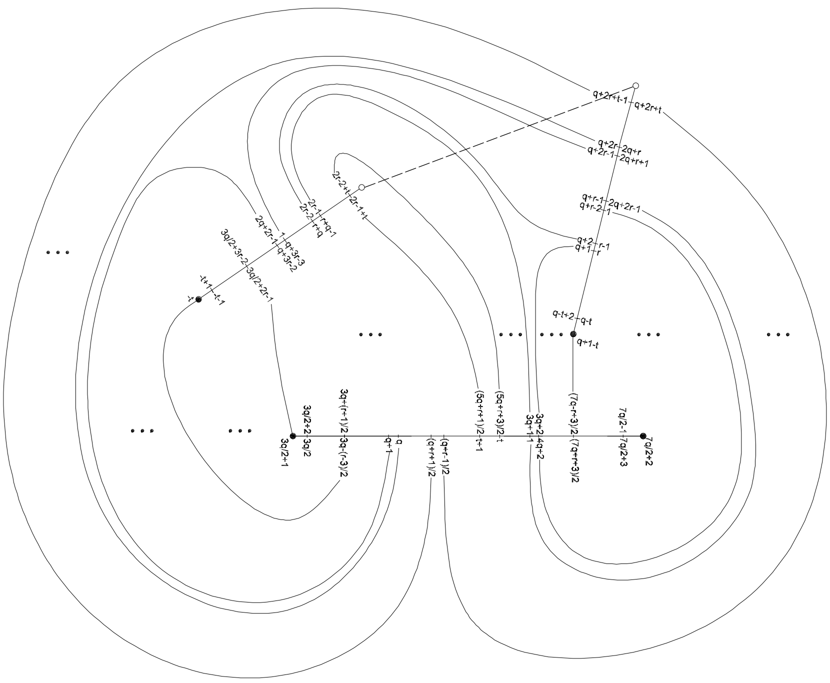

Figure 2. The knot $K(\alpha, r)$.

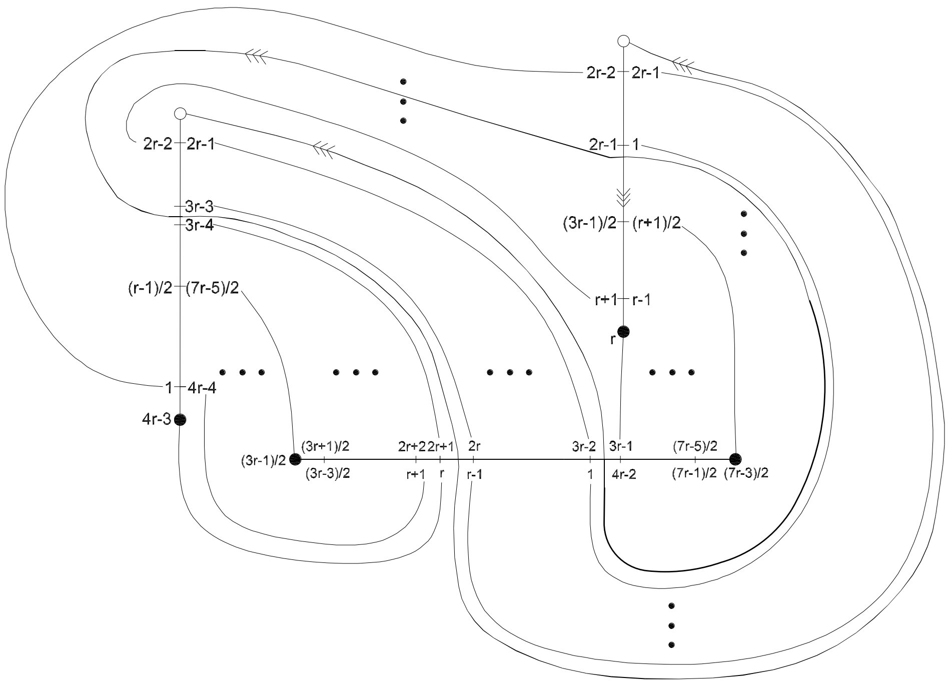

Figure 3. The knot $K(\alpha, r)$ for $\alpha = 1$.

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