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The non-existence of desarguesian t-parallelisms, t an odd prime

N.L. johnson
Department of Mathematics, University of Iowa
norman-johnson@uiowa.edu

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Abstract. In this article, it is shown that when t is a prime, partial Desarguesian t-parallelisms in $PG(zt-1,q)$ of m t-spreads are equivalent to translation nets of order q^{zt} and degree $1+q+m(q^{zt}-q)$. Thus, a bound is established for the number of t-spreads in partial Desarguesian t-parallelisms, when t is an odd prime. This also shows that there cannot be Desarguesian t-parallelisms when t is an odd prime.

Keywords: t-parallelism, translation planes, rational Pappian partial spreads

MSC 2000 classification: primary 51E20, secondary 51A40

1 Introduction.

This article considers partial Desarguesian t-parallelisms in $PG(zt-1,q)$, where t is an odd prime. Recall that a t-spread of a vector space V of dimension zt over $GF(q)$, is a partition of the non-zero vectors of V by a set of mutually disjoint t-dimensional vector subspaces. A Desarguesian t-spread is a t-spread with the property that there exists a field L ismorphic to $GF(q^t)$ so that V is a z-dimensional vector space over L and the elements of the t-spread are 1-dimensional L-subspaces. A Desarguesian t-parallelism if a covering of the t-dimensional subspaces by a set of t-spreads that do not share a common t-space.

There are a number of Desarguesian 2-parallelisms in $PG(3,q)$, and in this case, the terminology 'regular parallelism' is used. In this case, there must be a set $1+q+q^2$ 2-spreads in a Desarguesian parallelism. The known Desarguesian parallelisms are follows: There are two mutually non-isomorphic parallelisms in $PG(3,2)$, two in $PG(3,8)$, due to Denniston [1], two in $PG(3,5)$, due to Prince [5] and an infinite class in $PG(3,q)$, where $q\equiv 2\mathrm{mod}3$, due to Penttila and Williams [4] and this class contains all of the previously mentioned examples.

The set of Desarguesian parallelisms in $PG(3,q)$ is equivalent to the set of translation planes of order q^4 with spread in $PG(7,q)$ covered by a set of derivable partial spreads of degree $1+q^2$ that mutually share a regulus partial spread of degree $1+q$. Furthermore, it is precisely this connection that we wish to generalize in this article.

Our main result shows that when t is a prime, there is an equivalence with partial Desargusian t-parallelisms in $PG(zt-1,q)$ of m t-spreads and vector space translation nets of degree $1+q+m(q^t-q)$ and order q^{zt}, which consist of m rational Desarguesian partial spreads of degree $1+q^t$ and order q^{zt} that share a regulus net of degree $1+q$. Since such translation nets are bounded by the degree $1+q+\frac{(q^{zt-1}-1)}{(q^{t-1}-1)})(q^t-q)=1+q^{zt}$, we obtain a strong bound on the cardinality of partial Desarguesian t-parallelisms, which also shows that Desarguesian t-parallelisms cannot exist with t is an odd prime.

2 Partial Desarguesian t-parallelisms, t an odd Prime.

In Jha and Johnson [2], there is a general study of the connection between Desarguesian t-spreads in $PG(zt-1,q)$ and rational Desarguesian partial spreads of degree $1+q^t$ and order q^{zt} in $PG(2zt-1,q)$. By a 'rational Desarguesian partial spread', it is means that there is a field L of $zt \times zt$ matrices isomorphic to $GF(q^t)$, such that the components of the partial spread may be represented in the following form:

$$x=0, y=xM; M \in L.$$

It is also assumed that L contains a subfield K, whose elements are αI_{zt}, for all $\alpha \in GF(q)$. This means that the partial spread is a K-regulus in $PG(2zt-1,q)$. We recall that a 'K-regulus' is a set of $q+1$ $zt-1$ dimensional projective subspaces that are covered by a set of lines, such that a line intersecting three such subspaces intersects all $q+1$ of the subspaces.

Theorem 1. *(Jha and Johnson [2],* (2.6)*). There is a* $1-1$ *correspondence between Desarguesian t-spreads of a zt-dimensional vector space over $GF(q)$ and rational Desarguesian nets of degree $1+q^t$ and order q^{zt} that contain a K-regulus in a vector space of dimension* $2zt$ *over $GF(q)$.*

There is also a general correspondence between partial Desarguesian 2-parallelisms and translation planes covered by rational Desarguesian nets, also established by the authors in [2] that generalizes unpublished work of Prohaska and Walker, and is inspired by the work of Walker [6], and Lunardon [3].

Theorem 2. *(Jha and Johnson [2]* (2.8)*). Let V be a vector space of dimension* $4z$ *over $GF(q)$, and let $\mathcal{R}$ be a regulus of V (of $PG(4z-1,q)$). Let Γ be a set of rational Desarguesian nets isomorphic of degree $1+q^2$ and order q^{2z} containing $\mathcal{R}$. Then*

$$\cup(\Gamma-\mathcal{R})\cup\mathcal{R}$$

is a partial spread if and only if for any choice of component A of $\mathcal{R}$, considered as a $2z$*-dimensional $GF(q)$-space, Γ induces a partial* 2*-parallelism of A.*

We now generalize Theorem 2 , for partial Desarguesian t-parallelisms, when t is an odd prime.

Theorem 3. *Let V be a vector space of dimension* $2tz$ *over $GF(q)$, and let $\mathcal{R}$ be a regulus of V (of $PG(2tz-1,q)$). Let Γ be a set of rational Desarguesian nets isomorphic of degree $1+q^t$ and order q^{tz} containing $\mathcal{R}$. If t is a prime, then*

$$\cup(\Gamma-\mathcal{R})\cup\mathcal{R}$$

is a partial spread if and only if for any choice of component A of $\mathcal{R}$, considered as a tz-dimensional $GF(q)$-space, Γ induces a Desarguesian partial t-parallelism of A.

Proof. Let A be a zt-dimensional vector subspace of a $2zt$-dimensional vector space V over a field $GF(q)$. Let S_1 be a Desarguesian t-spread of A and by Theorem 1 form the associated rational partial zt-spread R_1 of degree $1+q^t$ in V that contains the q-regulus partial spread N (of degree $q+1$). Now take a second Desarguesian t-spread of A S_2 and form the rational partial zt-spread R_2 of degree $1+q^t$ containing N. The proof of our result is finished if it could be shown that the two rational partial spreads do not share any points outside of the regulus N. So, assume that Q is a common point of R_1 and R_2 that does not lie in N (on a component of N). Since rational Desarguesian partial spreads are subplane covered nets, covered by Desarguesian subplanes of order q^t, therefore, there is a subplane π_1 of R_1 of order q^t and a subplane π_2 of R_2 of order q^t that share the point Q. Since π_1 is a $2t$-dimensional $GF(q)$-subspace generated any two t-intersections of components of the regulus N, take $x = 0, y = 0, y = x$ are three components of N, say taking A as $x = 0$. Then there are points on $x = 0, y = 0, y = x$ in π_1 as follows: $P_{x=0} + P_{y=0} = Q = Z_{x=0} + Z_{y=x} = W_{y=0} + W_{y=x}$, where the subscripts indicate what component the points are located.

Take the subspace $\Lambda = \langle P_{x=0}, P_{y=0}, Z_{x=0}, Z_{y=x}, W_{y=0}, W_{y=x}\rangle$, which is at least 2-dimensional over $GF(q)$. If the subspace is 2-dimensional over $GF(q)$, then since N is a regulus and since Λ contains points of $x = 0, y = 0, y = x$ then Λ must intersect all of components of N, which means that Q cannot be in Λ. Therefore, Λ has dimension at least 3 over $GF(q)$. Let $U = \pi_1 \cap \pi_2$ as a subspace of dimension at least 3. We claim that the dimension is 4. Actually, no two of the subspaces $\langle P_{x=0}, P_{y=0}\rangle$, $\langle Z_{x=0}, Z_{y=x}\rangle$, $\langle W_{y=0}, W_{y=x}\rangle$ can be equal since otherwise a 2-dimensional subspace would non-trivially intersect three components of a regulus, and the same contradiction applies. By the construction given in [2] to establish Theorem 1, then π_1 and π_2 admit the group element $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which means that $\pi_1 \cap \pi_2$ also admits this group. Therefore, this implies that the intersection on $x = 0$ is at least two dimensional. Hence, Λ is 4-dimensional, as it is generated by two mutually disjoint 2-dimensional $GF(q)$-subspaces.

Let $P^*_{x=0} + P^*_{y=0} = Q$, where the points are in π_2. Then $P_{x=0} + P_{y=0} = P^*_{x=0} + P^*_{y=0} = Q$, clearly implies that $P_{x=0} = P^*_{x=0}$, $P_{y=0} = P^*_{y=0}$ so that Λ is common to $\pi_1 \cap \pi_2$.

Define a point-line geometry as follows: The 'points' are the points of U, the 'lines' are the lines PQ, of both π_1 and π_2, where, P, Q are distinct points of U. We claim then U becomes an affine plane and then an affine subplane of both π_1 and π_2. To see this, let P and Q be distinct points of U and form the unique line PQ common to both π_1 and π_2. Two lines of U are parallel if and only if they are parallel in π_1 and parallel in π_2. Now let PQ and RT be lines of U that are not parallel. Since they are both lines of π_1 and lines of π_2 then these two lines intersect in a common point of π_1 and π_2. Hence, two lines of U are either parallel or intersect uniquely. Let ℓ be a line PQ of U and let R be a point of U not incident with PQ. Form the line $(P-P)(Q-P)$ of U and note that $R-P$ is not incident with this line $0(Q-P)$, a common component of both π_1 and π_2, since 0 and $Q-P$ are in $\pi_1 \cap \pi_2$. So, we may assume that PQ contains 0, that is, it is a common component ℓ of π_1 and π_2. Assume without loss of generality that Q is not 0. Then $\ell + R = R(Q + R)$ is the unique common line of π_1 and π_2 parallel to ℓ and incident with R. Hence, $U = \pi_1 \cap \pi_2$ is an affine translation subplane and as a subspace is of dimension at least 4. But, π_1 is a Desarguesian affine plane of order q^t and U is an affine subplane of π_1 of order q^a, for $a \geq 2$. Therefore, a must divide t. Now assume that t is an odd prime. Then, $t = a$, since t is an odd prime.We have then shown that $R_1 \cup R_2$ is a net of degree $1 + q + 2(q^t - q)$, which is the union of two rational Desarguesian partial spreads of degrees $1 + q^t$, both of which share the regulus N. This completes the proof of the theorem. *QED*

So, partial Desarguesian t-spread in a vector space of dimenison zt over $GF(q)$ of cardinality m induces a partial spread of degree

$$1 + q + m(q^t - q).$$

The maximum partial spread has total degree $q^{zt} - q + 1 + q = 1 + q^{zt}$. Therefore,

$$m \leq (q^{zt-1} - 1)/(q^{t-1} - 1).$$

We then have the following corollary:

Corollary 1. *If t is a prime, the maximum cardinality of a Desarguesian partial t-spread in a vector space of dimension zt over $GF(q)$ is $\left[(q^{zt-1}-1)/(q^{t-1}-1)\right]$ and if this bound is taken on then $t-1$ must divide $zt-1$.*

Theorem 4. *Of course, if $t=2$, then $(q^{2r-1}-1)/(q-1)$ is the number of spreads of a 2-parallelism.*

If t is an odd prime, then to achieve a parallelism, we require

$$\frac{(q^{zt}-1)(q^{zt-1}-1)...(q^{zt-t-1}-1)}{(q^t-1)(q^{t-1}-1)...(q-1)}$$

t-spreads.

Therefore, a Desarguesian t-parallelism exists for t a prime if and only if $t=2$.

For example, suppose $t=3$ and $z=3$, we then are considering Desarguesian 3-spreads in 9-dimensional vector spaces over $GF(q)$.

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