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Razumikhin–type theorems on stability in terms of two measures for impulsive functional differential systems

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Abstract. This paper studies the stability problems for impulsive functional differential equations with finite delay and fixed moments of impulse effect. By using the Lyapunov functions and Razumikhin technique sufficient conditions for stability in terms of two different piecewise continuous measures of such equations are found.

Keywords: Stability in terms of two measures, Razumikhin technique, Impulsive functional differential equations.

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1 Introduction

In the last few decades, the theory of impulsive ordinary differential equations marked a rapid development. See, for example, [1–3, 11–13] and the references cited therein. Now there also exists a well-developed qualitative theory of functional differential equations [7–10, 12, 16]. Systems of impulsive functional differential equations provide mathematical models of many natural processes and phenomena in the field of natural sciences and technology. Their theory is considerably richer than the theory of ordinary differential equations (without delay) and of functional differential equations (without impulses). Recently stability problems on some linear and nonlinear impulsive functional differential equations are investigated in several papers [1, 4–6, 15, 17, 18].

This paper studies the stability of the solutions of impulsive systems of functional differential equations with fixed moments of impulse effect in terms of two different piecewise continuous measures. The priorities of this approach are useful and well known in the investigations on the stability and boundedness of the solutions of differential equations, as well as in the generalizations obtained by this method [7, 13, 14].

In order to study the stability of the solutions of impulsive systems under considerations, we are concerned with the application of the concept of

Lyapunov–Razumikhin functions and with differential inequalities on piecewise continuous functions. It is well known that Lyapunov-Razumikhin function method have been widely used in the treatment of the stability of functional differential equations without impulses [7, 8, 12, 16]. Such a method applied to the investigation of various type of stability of impulsive functional differential equations can be found in [1, 4–6, 15, 17, 18].

2 Preliminary notes and definitions

Let R^n be the n–dimensional Euclidean space with norm $|.|$; $R_+ = [0, \infty)$.

Let $r > 0$ and $E = \{\phi : [-r, 0] \to R^n,\ \phi(t)$ is continuous everywhere except at finite number of points $t = \tau_k \in [-r, 0]$ at which $\phi(\tau_k - 0)$ and $\phi(\tau_k + 0)$ exist and $\phi(\tau_k - 0) = \phi(\tau_k)\}$. If $t > t_0$, $t_0 \in R_+$ we define $x_t \in E$ by $x_t = x(t + s)$, $-r \leq s \leq 0$.

Consider the system of impulsive functional differential equations

$$\begin{cases} \dot{x}(t) = f(t, x_t),\ t > t_0,\ t \neq \tau_k, \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k - 0)),\ \tau_k > t_0, \end{cases} \tag{1}$$

where $f : (t_0, \infty) \times E \to R^n$; $I_k : R^n \to R^n$; $\tau_k < \tau_{k+1}$ with $\lim_{k\to\infty} \tau_k = \infty$.

Let $\phi \in E$. Denote by $x(t) = x(t; t_0, \phi)$, $x \in R^n$ the solution of system (1) satisfying the initial conditions:

$$\begin{cases} x(t; t_0, \phi) = \phi(t - t_0),\ t_0 - r \leq t \leq t_0, \\ x(t_0 + 0; t_0, \phi) = \phi(0) \end{cases} \tag{2}$$

The solution $x(t) = x(t; t_0, \phi)$ of the initial value problem (1), (2) is characterized by the following:

a. For $t_0 - r \leq t \leq t_0$ the solution $x(t)$ satisfied the initial conditions (2).

b. In the interval $[t_0, \infty)$ the solution $x(t; t_0, \phi)$ of problem (1),(2) is a piecewise continuous function with points of discontinuity of the first kind $t = \tau_k, \tau_k \in [t_0, \infty)$ at which it is continuous from the left.

Let $\tau_0 = t_0 - r$. Introduce the following notations:

$$I_0 = [t_0 - r, \infty);$$
$$G_k = \{(t, x) \in I_0 \times R^n : \tau_{k-1} < t < \tau_k\},\ \ k = 1, 2, \ldots;$$
$$G = \bigcup_{k=1}^{\infty} G_k.$$

1 Definition. We shall say that the function $V : I_0 \times R^n \to R_+$ belongs to the class V_0 if:

1. The function V is continuous in G and locally Lipschitz continuous with respect to its second argument in each of the sets $G_k,\ k = 1, 2, \ldots$.
2. For each $k = 1, 2, \ldots$ and $x \in R^n$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{\substack{t\to\tau_k \\ t<\tau_k}} V(t, x),\ V(\tau_k + 0, x) = \lim_{\substack{t\to\tau_k \\ t>\tau_k}} V(t, x).$$

3. The equality $V(\tau_k - 0, x) = V(\tau_k, x)$ is valid.

In the sequel we will use the next classes of functions:

$$\begin{aligned} K &= \{a \in C[R_+, R_+] : a(r) \text{ is strictly increasing and } a(0) = 0\,\}\,; \\ CK &= \{a \in C[I_0 \times R_+, R_+] : a(t, .) \in K \text{ for any fixed } t \in I_0\,\}\,; \\ \Gamma &= \{h \in V_0 : \inf_{x\in R^n} h(t, x) = 0 \text{ for each } t \in I_0\,\}\,. \end{aligned}$$

2 Definition. Let $h, h^0 \in \Gamma$ and define for $\phi \in E$

$$\begin{cases} h_0(t, \phi) = \sup\limits_{-r\le s\le 0} h^0(t + s, \phi(s)), \\ \bar{h}(t, \phi) = \sup\limits_{-r\le s\le 0} h(t + s, \phi(s)). \end{cases} \tag{3}$$

Then:

(a) h_0 is *finer* than $\bar{h}$ if there exist a number $\delta > 0$ and a function $\varphi \in K$ such that $h_0(t, \phi) < \delta$ implies $\bar{h}(t, \phi) \le \varphi(h_0(t, \phi))$.

(b) h_0 is *weakly finer* than $\bar{h}$ if there exist a number $\delta > 0$ and a function $\varphi \in CK$ such that $h_0(t, \phi) < \delta$ implies $\bar{h}(t, \phi) \le \varphi(t, h_0(t, \phi))$.

3 Definition. Let $h, h^0 \in \Gamma$ and $V \in V_0$. The function V is said to be:

(a) *h–positively definite* if there exist a number $\delta > 0$ and a function $a \in K$ such that $h(t, x) < \delta$ implies $V(t, x) \ge a(h(t, x))$.

(b) *h_0–decrescent* if there exist a number $\delta > 0$ and a function $b \in K$ such that $h_0(t, \phi) < \delta$ implies $V(t + 0, x) \le b(h_0(t, \phi))$.

(c) *weakly h_0–decrescent* if there exist a number $\delta > 0$ and a function $b \in CK$ such that $h_0(t, \phi) < \delta$ implies $V(t + 0, x) \le b(t, h_0(t, \phi))$.

We will use the following definitions of stability of the system (1) in terms of two different measures, that generalize various classical notions of stability.

4 Definition. Let h, $h^0 \in \Gamma$ and h_0 is defined by (3). The system (1) is said to be:

(a) (h_0, h)–*stable* if

$$(\forall t_0 \in R_+)(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$
$$(\forall \phi \in E : h_0(t_0, \phi) < \delta)(\forall t > t_0) : \quad h(t, x(t; t_0, \phi)) < \varepsilon.$$

(b) (h_0, h)–*uniformly stable* if the number δ from (a) does not depend on t_0.

(c) (h_0, h)–*equiattractive* if

$$(\forall t_0 \in R_+)(\exists \delta = \delta(t_0) > 0)(\forall \varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)$$
$$(\forall \phi \in E : \quad h_0(t_0, \phi) < \delta)(\forall t > t_0 + T) : \quad h(t, x(t; t_0, \phi)) < \varepsilon.$$

(d) (h_0, h)–*uniformly attractive* if the numbers δ and T from (c) are independent on t_0.

(e) (h_0, h)–*equiasymptotically stable* if it is (h_0, h)–stable and (h_0, h)–equiattractive.

(f) (h_0, h)–*uniformly asymptotically stable* if it is (h_0, h)–uniformly stable and (h_0, h)–uniformly attractive.

For a concrete choice of the measures h_0 and h Definition 4 is reduces to the following particular cases:

1) Lyapunov's stability of the zero solution of (1) if

$$h_0(t, \phi) = ||\phi|| = \sup_{s \in [-r, 0]} |\phi(s)| \text{ and } h(t, x) = |x|.$$

2) stability by part of the variables of the zero solution of (1) if

$$h_0(t, \phi) = ||\phi||, \;\; h(t, x) = |x|_k = \sqrt{x_1^2 + \cdots + x_k^2}, \;\; 1 \le k \le n,$$
$$x = (x_1, \ldots, x_n), \;\; 1 \le k \le n.$$

3) Lyapunov's stability of the non-null solution $x_0(t) = x_0(t; t_0, \phi_0)$ of (1) if $h_0(t, \phi) = ||\phi - \phi_0||, \;\; h(t, x) = |x - x_0(t)|$.

4) stability of conditionally invariant set B with respect to the set A, where $A \subset B \subset R^n$ if

$$h_0(t,\phi) = \sup_{s\in[-r,0]} d(\phi(s), A), \quad h(t,x) = d(x,B),$$

d being the distance function.

5) eventual stability of (1) if $h(t,x) = |x|$ and $h_0(t,\phi) = ||\phi|| + \alpha(t)$, $\alpha \in K$ and $\lim\limits_{t\to\infty} \alpha(t) = 0$.

We will use also the following classes of functions:

$PC[I_0, R^n] = \{x : I_0 \to R^n : x$ is piecewise continuous with points of discontinuity of the first kind $\tau_k,\ \tau_k \in I_0$ at which it is continuous from the left $\}$;
$PC^1[[t_0,\infty), R^n] = \{x \in PC[[t_0,\infty), R^n] : x$ is continuously differentiable everywhere except the points $\tau_k,\ \tau_k \in [t_0,\infty)$ at which $\dot{x}(\tau_k - 0)$ and $\dot{x}(\tau_k + 0)$ exist and $\dot{x}(\tau_k - 0) = \dot{x}(\tau_k)\ \}$;
$\Omega_1 = \{\, x \in PC[[t_0,\infty), R^n] : V(s, x(s)) \le V(t, x(t)), t - r < s \le t, t \ge t_0, V \in V_0 \,\}$.

Let $V \in V_0,\ t > t_0 - r,\ t \ne \tau_k,\ k = 1, 2, \ldots$ and $x \in PC[I_0, R^n]$. Introduce the function

$$D_-V(t, x(t)) = \lim_{\theta\to 0^-} inf \sigma^{-1}[V(t+\theta, x(t) + \theta f(t, x_t)) - V(t, x(t))].$$

Together with the system (1), we consider the scalar impulsive differential equation

$$\begin{cases} \dot{u}(t) = g(t, u(t)), & t \ne \tau_k, \\ \Delta u(\tau_k) = B_k(u(\tau_k)), \end{cases} \tag{4}$$

where $g : R_+ \times R_+ \to R$ and $B_k : R_+ \to R$.

Assume $\rho > 0,\ h, h^0 \in \Gamma$, h_0 is defined by (3) and let

$$S(h,\rho) = \{(t,x) \in I_0 \times R^n :\ h(t,x) < \rho\};$$
$$S(h_0,\rho) = \{(t,\phi) \in [t_0,\infty) \times E :\ h_0(t,\phi) < \rho\}.$$

Introduce the following assumptions:

A1. The function $f : (t_0,\infty) \times E \to R^n$ is continuous in $(\tau_{k-1}, \tau_k] \times E$ and for every $x_t \in E,\quad k = 1, 2, \ldots\ f(\tau_k - 0, x_t)$ and $f(\tau_k + 0, x_t)$ exist and $f(\tau_k - 0, x_t) = f(\tau_k, x_t)$.

A2. $I_k \in C[R^n, R^n],\ k \in N$.

A3. $t_0 - r = \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

A4. $g \in PC[R_+ \times R_+, R]$ and $g(t, 0) = 0$ for $t \in R_+$.

A5. $B_k \in C[R_+, R]$, $B_k(0) = 0$ and $\psi_k(u) = u + B_k(u)$ are nondecreasing with respect to u, $k \in N$.

A6. There exists ρ_0, $0 < \rho_0 < \rho$, such that $h(\tau_k, x) < \rho_0$ implies $h(\tau_k + 0, x + I_k(x)) < \rho$, $k = 1, 2, \dots$

In the proofs of the main theorems we will use the following comparison results.

5 Lemma. *[17] Assume the following conditions hold:*

(1) Assumptions $A1 - A5$ are valid

(2) The function $V \in V_0$, $V : S(h, \rho) \cap S(h_0, \rho) \to R_+$ is such that for $t > t_0$ and $x \in \Omega_1$ we have

$$\begin{cases} D_- V(t, x(t)) \le g(t, V(t, x(t))),\ t \ne \tau_k, \\ V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le \psi_k(V(\tau_k, x(\tau_k))). \end{cases}$$

(3) For the solution $x(t; t_0, \phi)$ ofthesystem (1) we have $x \in PC[I_0, R^n] \cap PC^1[[t_0, \infty), R^n]$ and $(t, x(t + 0; t_0, \phi)) \in S(h, \rho)$ as $t \in I_0$.

(4) The maximal solution $r(t; t_0, u_0)$, $u_0 \ge V(t_0 + 0, \phi(0))$, of the equation (4) is defined on the interval $[t_0, \infty)$.

Then

$$V(t, x(t; t_0, \phi)) \le r(t; t_0, u_0) \text{ for } t \in [t_0, \infty).$$

6 Corollary. *Let the following conditions hold:*

1. Assumptions $A1 - A3$ are met.

2. The function $V \in V_0$, $V : S(h, \rho) \cap S(h_0, \rho) \to R_+$ is such that for $t > t_0$ and $x \in \Omega_1$ we have

$$D_- V(t, x(t)) \le 0,\ t \ne \tau_k,$$
$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le V(\tau_k, x(\tau_k)).$$

3. Condition 3 of Lemma 5 holds.

Then

$$V(t, x(t; t_0, \phi)) \le V(t_0 + 0, \phi(0)),\ t \in [t_0, \infty).$$

3 Main results

7 Theorem. *Assume the following conditions hold:*

(1) Assumptions $A1 - A6$ *are valid.*

(2) $h, h^0 \in \Gamma$ *and* h_0 *is finer than* $\bar{h}$*, where* $h_0, \bar{h}$ *are defined by (3).*

(3) The function $V \in V_0,\ V : S(h,\rho) \cap S(h_0,\rho) \to R_+$ *is* h*-positively definite and* h_0*-decrescent.*

(4) For $t > t_0$ *and* $x \in \Omega_1$ *we have*

$$D_- V(t, x(t)) \le g(t, V(t, x(t))),\ t \ne \tau_k,$$
$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le \psi_k(V(\tau_k, x(\tau_k))),\ k = 1, 2, \ldots$$

Then the stability properties of the trivial solution of the equation (4) imply the corresponding (h_0, h)*-stability properties of system (1).*

Proof. Let us first prove (h_0, h)–stability.

Since V is h- positively definite on $S(h,\rho) \cap S(h_0,\rho)$ then there exists a function $b \in K$ such that:

$$V(t, x) \ge b(h(t, x)), \quad as \quad h(t, x) < \rho. \tag{5}$$

Let $0 < \varepsilon < \rho_0$, $t_0 \in R_+$ be given and suppose that the trivial solution of the equation (4) is stable. Then for given $b = b(\varepsilon) > 0$ there exists $\delta_0 = \delta_0(t_0, \varepsilon) > 0$ such that

$$r(t; t_0, u_0) < b(\varepsilon) \quad as \quad 0 \le u_0 < \delta_0,\ t \ge t_0, \tag{6}$$

where $r(t; t_0, u_0)$ is the maximal solution of (4) satisfying $r(t_0 + 0; t_0, u_0) = u_0$.

We choose now $u_0 = V(t_0 + 0, \phi(0))$. Since V is h_0- decrescent there exist a number $\delta_1 > 0$ and a function $a \in K$ such that, for $h_0(t, \phi) < \delta_1$,

$$V(t + 0, x) \le a(h_0(t, \phi)). \tag{7}$$

On the other hand h_0 is finer than $\bar{h}$ and there exist a number $\delta_2 > 0$ and a function $\varphi \in K$ such that $h_0(t_0, \phi) < \delta_2$ implies

$$\bar{h}(t_0, \phi) \le \varphi(h_0(t_0, \phi)), \tag{8}$$

where $\delta_2 > 0$ is such that $\varphi(\delta_2) < \rho$. Hence by (3) we have

$$\begin{cases} h(t_0 + 0, \phi(0)) \le \bar{h}(t_0, \phi) \le \varphi(h_0(t_0, \phi)) < \varphi(\delta_2) < \rho, \\ h^0(t_0 + 0, \phi(0)) \le h_0(t_0, \phi) < \delta_2. \end{cases} \tag{9}$$

Setting $\delta_3 = \min(\delta_1, \delta_2)$. It follows from (5), (9) and (7) that $h_0(t_0, \phi) < \delta_3$ implies

$$b(h(t_0 + 0, \phi(0))) \leq V(t_0 + 0, \phi(0)) \leq a(h_0(t_0, \phi)). \tag{10}$$

Choose $\delta = \delta(t_0, \varepsilon) > 0$ such that $0 < \delta < \delta_3$, $a(\delta) < \delta_0$ and let $x(t) = x(t; t_0, \phi)$ to be such solution of the system (1) that $h_0(t_0, \phi) < \delta$. Then (10) shows that $h(t_0 + 0, \phi(0)) < \varepsilon$, since $\delta_0 < b(\varepsilon)$.

We claim that

$$h(t, x(t)) < \varepsilon \quad \text{as} \quad t > t_0.$$

If it is not true, then there would exists a $t^* > t_0$ such that $\tau_k < t^* \leq \tau_{k+1}$ for some fixed integer k and

$$h(t^*, x(t^*)) \geq \varepsilon \text{ and } h(t, x(t)) < \varepsilon, \quad t_0 < t \leq \tau_k.$$

Since $0 < \varepsilon < \rho_0$, condition A6 shows that

$$h(\tau_k + 0, x(\tau_k + 0)) = h(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) < \rho.$$

Therefore there exists t^0, $\tau_k < t^0 \leq t^*$, such that

$$\varepsilon \leq h(t^0, x(t^0)) < \rho \text{ and } h(t, x(t)) < \rho, \quad t_0 < t \leq t^0. \tag{11}$$

Applying now Lemma 5 for the interval $(t_0, t^0]$ and $u_0 = V(t_0 + 0, \phi(0))$ we obtain

$$V(t, x(t; t_0, \phi)) \leq r(t; t_0, V(t_0 + 0, \phi(0))), \quad t_0 < t \leq t^0. \tag{12}$$

So the implications (11), (5), (12) and (6) lead to

$$\begin{aligned} b(\varepsilon) &\leq b(h(t^0, x(t^0))) \leq V(t^0, x(t^0)) \\ &\leq r(t^0; t_0, V(t_0 + 0, \phi(0))) < b(\varepsilon). \end{aligned}$$

The contradiction we have already obtained shows that $h(t, x(t)) < \varepsilon$ for each $t > t_0$. Therefore the system (1) is (h_0, h)- stable.

If we suppose that the trivial solution of (4) is uniformly stable then it is clear that the number δ can be chosen independently of t_0 and thus we get the (h_0, h)- uniform stability of the system (1).

Let us suppose next that the trivial solution of (4) is equiasymptotically stable, which implies that the system (1) is (h_0, h)- stable. So, for each $t_0 \in R_+$ there exists a number $\delta_{01} = \delta_{01}(t_0, \rho) > 0$ such that if $h_0(t_0, \phi) < \delta_{01}$ then $h(t, x(t; t_0, \phi)) < \rho$ as $t > t_0$.

Let $0 < \varepsilon < \rho_0$ and $t_0 \in R_+$. The equiasymptotical stability of the null solution of the equation (4) implies that there exist $\delta_{02} = \delta_{02}(t_0) > 0$ and

$T = T(t_0, \varepsilon) > 0$ such that for $0 < u_0 < \delta_{02}$ and $t > t_0 + T$ the next inequality holds:

$$r(t; t_0, u_0) < b(\varepsilon). \tag{13}$$

Choosing $u_0 = V(t_0 + 0, \phi(0))$ as before, we find $\delta_{03} = \delta_{03}(t_0),\ 0 < \delta_{03} \leq \delta_{02}$ such that

$$a(\delta_{03}) < \delta_{02}. \tag{14}$$

It follows from (7) and (14) that if $h_0(t_0, \phi) < \delta_{03}$ then

$$V(t_0 + 0, \phi(0)) < a(h_0(t_0, \phi)) \leq a(\delta_{03}) < \delta_{02}.$$

In the case, by means of (13) we would have

$$r(t; t_0, V(t_0 + 0, \phi(0))) < b(\varepsilon), \quad t > t_0 + T. \tag{15}$$

Assume $\delta_0 = \min(\delta_{01}, \delta_{02}, \delta_{03})$ and let $h_0(t_0, \phi) < \delta_0$. Lemma 5 shows that if $x(t) = x(t; t_0, \phi)$ is an arbitrary solution of the system (1) then the estimate (12) holds for all $t > t_0 + T$. Therefore we obtain from (5), (12) and (15) that the inequalities

$$b(h(t, x(t))) \leq V(t, x(t)) \leq r(t; t_0, V(t_0 + 0, \phi(0))) < b(\varepsilon)$$

hold for each $t > t_0 + T$. Hence $h(t, x(t)) < \varepsilon$ as $t > t_0 + T$ which shows that the system (1) is (h_0, h)- equiattractive.

In case we suppose that the trivial solution of (4) is uniformly asymptotically stable we get that (1) is also (h_0, h)-uniformly asymptotically stable, since δ_0 and T will be independent of t_0.

Hence the proof of Theorem 7 is complete. QED

We have assumed in Theorem 7 stronger requirements on V, h, h_0 only to unify all the stability criteria in one theorem. This obviously puts burden on the comparison equation (4). However, to obtain only non-uniform stability criteria, we could weaken certain assumption of Theorem 7 as in the next result.

8 Theorem. *Assume the following conditions hold:*

(1) Assumptions $A1 - A6$ are valid.

(2) $h, h^0 \in \Gamma$ and h_0 is weakly finer than $\bar{h}$, where $h_0, \bar{h}$ are defined by (3).

(3) The function $V \in V_0,\ V : S(h, \rho) \cap S(h_0, \rho) \to R_+$ is h-positively definite and weakly h_0-decrescent.

(4) Condition 4 of Theorem 7 is valid.

Then the uniform and non-uniform stability properties of the trivial solution of the equation (4) imply the corresponding non-uniform (h_0, h)- stability properties of system (1).

The proof of Theorem 8 is analogous to the proof of Theorem 7; however Definition 2 (b) is used instead of Definition 2 (a), and Definition 3 (c) is used instead of Definition 3 (b).

9 Corollary. *Assume the following conditions hold:*

(1) *Assumptions $A1 - A3$ and $A6$ are met.*

(2) *Conditions 2 and 3 of Theorem 7 are valid.*

(3) *For each $t > 0$ and $x \in \Omega_1$ we have*

$$D_-V(t, x(t)) \leq 0,\ t \neq \tau_k,$$
$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \leq V(\tau_k, x(\tau_k)).$$

Then the system (1) is (h_0, h)-uniformly stable.

The proof of Corollary 9 could be done in the same way as in Theorem 7, using Corollary 6 now.

4 Examples

10 Example. Consider the impulsive functional differential equation

$$\begin{cases} \dot{x}(t) = a(t)x^3(t) + b(t)x(t)x^2(t-r), & t \neq \tau_k, \\ x(t) = \phi_1(t), t \in [-r, 0], \\ \Delta x(\tau_k) = I_k(x(\tau_k)), \end{cases} \tag{16}$$

where $x \in PC[R_+, R]$; $a(t)$ and $b(t)$ are continuous in R_+, $b(t) \geq 0$, $a(t)+b(t) \leq -a < 0$; $r > 0$; $I_k(x), k = 1, 2, \ldots$ are continuous in R and such that $x+I_k(x) > 0$ and $|x + I_k(x)| \leq |x|$ for $x > 0$; $0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Let $h_0(t, \phi_1) = ||\phi_1||$ and $h(t, x) = |x|$. We consider the function

$$V(t, x) = \begin{cases} \alpha e^{-\frac{1}{x^2}}, & for\ x > 0, \\ 0, & for\ x = 0. \end{cases}$$

The set Ω_1 is defined by

$$\Omega_1 = \left\{x \in PC[R_+, R] :\ x^2(s) \leq x^2(t),\ t - r < s \leq t\right\}.$$

If $t > 0$ and $x \in \Omega_1$ we have

$$D_{-}V(t,x(t)) = \alpha e^{-\frac{1}{x^2(t)}}.\frac{2}{x^3(t)}[a(t)x^3(t) + b(t)x(t)x^2(t-r)]$$
$$\leq -2aV(t,x(t)), \quad t \neq \tau_k, \; k = 1,2,\ldots$$

Moreover

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) = \alpha e^{-\frac{1}{(x(\tau_k)+I_k(x(\tau_k)))^2}}$$
$$\leq V(\tau_k, x(\tau_k)), \; k = 1,2,\ldots, \quad x \in \Omega_1.$$

Since the trivial solution of the equation

$$\begin{cases} \dot{u}(t) = -2au(t), & t \neq \tau_k, \\ \Delta u(\tau_k) = 0, \end{cases}$$

is asymptotically stable [2], then Theorem 8 with $g(t,u) = -2au$ and $B_k(u) = 0$, $k = 1,2,\ldots$, shows that the equation (16) is $(h_0,h)-$asymptotically stable.

11 Example. Consider the impulsive functional differential equation

$$\begin{cases} \dot{x}(t) = -ax(t) + bx(t-r) - e(t)g(x(t)), & t \neq \tau_k, \\ x(t) = \phi_2(t), t \in [-r,0], \\ \Delta x(\tau_k) = -\alpha_k x(\tau_k), & k = 1,2,\ldots, \end{cases} \tag{17}$$

where a, b, $r > 0$; $e(t) \geq 0$ is a continuous function; $g(0) = 0$ and $xg(x) > 0$ if $x > 0$; $0 \leq \alpha_k \leq 2$, $k = 1,2,\ldots$; $0 < \tau_1 < \tau_2 < \cdots$ and $\lim\limits_{k\to\infty} \tau_k = \infty$.

Let $h_0(t,\phi_2) = ||\phi_2||$ and $h(t,x) = |x|$. We consider the function $V(t,x) = x^2$. The set Ω_1 is defined by

$$\Omega_1 = \left\{x \in PC[R_+,R] : \; x^2(s) \leq x^2(t), \; t-r < s \leq t\right\}.$$

If $t > 0$ and $x \in \Omega_1$ we have

$$D_{-}V(t,x(t)) = -2ax^2(t) + 2bx(t)x(t-r) - 2e(t)x(t)g(x(t))$$
$$\leq 2V(t,x(t))[-a+b], \quad t \neq \tau_k, \; k = 1,2,\ldots$$

Moreover

$$V(\tau_k + 0, x(\tau_k) - \alpha_k x(\tau_k)) = (1-\alpha_k)^2 V(\tau_k, x(\tau_k))$$
$$\leq V(\tau_k, x(\tau_k)), \; k = 1,2,\ldots, \quad x \in \Omega_1.$$

Assume the inequality $a \geq b$ holds. Then Corollary 9, shows that the equation (17) is $(h_0,h)-$uniformly stable.

Let the inequality $b \leq a - \varepsilon$ hold for some positive ε. Applying Theorem 7, we obtain that (17) is $(h_0,h)-$uniformly asymptotically stable.

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