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Collineation groups of translation planes constructed by multiple hyper-regulus replacement

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Abstract. New classes of mutually disjoint hyper-reguli of order q^n, for $n > 2$, have been determined by Jha and Johnson, which are not André hyper-reguli when $n > 3$. When n is odd, each hyper-regulus permits at least two replacements and when q is odd and $(n, q-1) = 1$, there are $(q-1)/2$ mutually disjoint hyper-reguli each of which may be replaced at least two ways. Each of the possible $2^{(q-1)/2}$ translation planes constructed is not André or generalized André. In this article, the full collineation group of these planes is completely determined.

Keywords: hyper-regulus, multiple replacement, André hyper-reguli

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1 Introduction

In three related articles (see [1–3]), the authors have constructed hyper-reguli of order q^n and degree $(q^n-1)/(q-1)$ that produce new translation planes which are not generalized André translation planes. When $n = 3$, each hyper-regulus is André and there are constructions of sets of $(q-1)/2$, $(q-3)/2$ and $q/2-1$ mutually disjoint hyper-reguli, each of which has exactly two distinct replacements. However, each subset of cardinality > 1 is a non-linear set, thus producing completely new translation planes. When $n > 3$, these hyper-reguli are not André hyper-reguli, thus again constructing classes of non-André and non-generalized André planes, even when but a single hyper-regulus is replaced. It is known that André and generalized André planes may be recognized by their collineation groups. In particular, there are always large affine homology groups in generalized André planes. In contrast, the planes constructed by hyper-regulus replacement often have trivial affine homology groups. Furthermore, there are constructions of sets of mutually disjoint hyper-reguli of various sizes. There are, of course, at most $(q-1)$ mutually disjoint hyper-reguli of order q^n and degree $(q^n-1)/(q-1)$. In our constructions the number of hyper-reguli used in the construction can be quite large. For example, from the authors' work:

1 Theorem (Jha and Johnson [1]). *Let Σ be the Desarguesian plane of order q^n. Assume that $(n, q-1) = 1$. Let $k \neq 1$ be any divisor of $(q-1)$ and let $C_{(q-1)/k}$ be the cyclic subgroup of order $(q-1)/k$ of $GF(q)^*$. Let $\lambda \cup \{1\}$ be a proper subset of coset representatives for $C_{(q-1)/k}$ whose elements give proper cosets of $C_{(q-1)/k}$ with the property that if ρ and γ are distinct in λ then $\rho\gamma^{\pm 1} \in \lambda \cup \{1\}$. Now choose b so that $b^{(q^n-1)/(q-1)} \notin \bigcup_{g \in \lambda \cup \{1\}} gC$.*

Then

$$\left\{ y = x^q \alpha \rho d^{1-q} + x^{q^{n-1}} \alpha^{-1} \rho^{-1} b d^{1-q^{n-1}};\ d \in GF(q^n)^*, \alpha \in C_{(q-1)/k}, \rho \in \lambda \right\}$$

is a set of $|\lambda \cup \{1\}|\,(q-1)/k$ mutually disjoint hyper-reguli.

In all of our examples, the number of hyper-reguli is $\leq (q-1)/2$ when q is odd and $\leq q/2 - 1$, when q is even.

In this article, we determine the complete collineation group of the translation planes obtained by the replacement of any subset of these hyper-reguli. More generally, we obtain the following characterization of André nets and develop general theorems regarding the replacement of a set of hyper-reguli by homology-type replacement from a Desarguesian plane ('homology-type replacement' means that the replacement net for the hyper-regulus is defined by the orbit of a subspace lying across the hyper-regulus in 1-dimensional $GF(q)$-subspaces under the kernel homology group of order $q^n - 1$). Since we now have non-André hyper-reguli, there is the problem of providing criteria to show that a given hyper-regulus cannot be André. In this regard, we prove the following characterization theorem.

2 Theorem. *Suppose a hyper-regulus of order q^n and degree $(q^n-1)/(q-1)$ in a Desarguesian affine plane Σ of order q^n for $n > 3$ has a homology-type replacement which has a Desarguesian subset D of cardinality*

$$> ((q^n-1)/(q-1) + (q^{[n]} + q))/2.$$

Then the hyper-regulus is André.

Concerning the collineation group of the translation planes, we provide the following results.

3 Theorem. *Let $\mathcal{H}$ be any set of hyper-reguli of degrees $(q^n-1)/(q-1)$ and order q^n in a Desarguesian affine plane Σ of order q^n, $q > 3$, each of which has a homology-type replacement.*

(1) *If the cardinality of* $\mathcal{H}$ *is* $< (q-1)/2$ *then the full collineation group of a translation plane obtained by the homology-type replacement of any subset is the group inherited from the Desarguesian plane* Σ.

(2) *Assume the cardinality of* $\mathcal{H}$ *is* $(q-1)/2$ *and the union of the replacement hyper-reguli is not Desarguesian. Then the full collineation group of a translation plane obtained by the homology-type replacement of any subset is the group inherited from the Desarguesian plane* Σ.

There are various constructions of mutually disjoint hyper-reguli given in Jha–Johnson [2], of which the group-constructed sets are most easily described. The collineation group of any translation plane obtained by the replacement of a subset of a group-constructed set of hyper-reguli is the group inherited from the associated Desarguesian plane and is usually quite small, containing essentially only the cyclic kernel homology group of order $(q^n - 1)$ that fixes each hyper-regulus.

4 Theorem. *Let* π *be a translation plane of order* q^n, *and kernel* $GF(q^{(n,k)})$, *for* $n/(n,k) > 3$, *obtained by the replacement of a subset of hyper-reguli of type* (b,k,t) *of order* q^n *and degrees* $(q^n-1)/(q^{(n,k)}-1)$ *from a Desarguesian affine plane* Σ.

Then the full collineation group of π *is a subgroup of* $\Gamma L(2,q^n)$ *and the intersection with* $GL(2,q^n)$ *is either the kernel homology group of* Σ *or* $(q^{(k,n)}-1)/t$ *is even and there is an affine homology of order* 2 *permuting the subset of hyper-reguli used in the replacement in orbits of length* 2.

In particular, the full group of affine homologies is either trivial or of order 2.

These results show that there are vast numbers of mutually non-isomorphic planes that many be constructed in this manner.

5 Theorem. *Let* π *be a translation plane of order* $q^n = p^{nd}$, *for* p *a prime, and kernel* $GF(q^{(n,k)})$, *for* $n/(n,k) > 3$, *obtained by the replacement of a subset of a group-constructed set of hyper-reguli of type* (b,k,t) *of order* q^n *and degrees* $(q^n-1)/(q^{(n,k)}-1)$ *from a Desarguesian affine plane* Σ *(note that we require* $((q^{(n,k)}-1)/t, n/(n,k)) = 1$*).*

(1) *Then there are at least*

$$\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t} \begin{pmatrix} (q^{(n,k)}-1)/t \\ i \end{pmatrix}}{\left((q^{(n,k)}-1)/t, 2\right) |GalGF(q^n)|}$$

mutually non-isomorphic translation planes constructed.

(2) *If* π *is constructed using a subset of* (b,k,t) *and* ρ *is constructed using a*

subset of (b^*, k^*, t)*, where* $(n,k) = (n,k^*)$*, then*

$$k^* \in \{k, n-k\},\ modulo\ n,$$

and

(a) *when* $(n,2k) = (n,k)$ *there are at least*

$$(q^{(n,k)} - 2)/(|GalGF(q^n)|, (q^{(n,k)} - 2))$$

ways of choosing the term b *or* b^* *so that no hyper-regulus of one set is isomorphic to any hyper-regulus of the second set, while*

(b) *when* $(n,2k) = 2(n,k)$*, there are at least*

$$(q^{(n,k)} + 1)(q^{(n,k)} - 2)/(|GalGF(q^n)|, (q^{(n,k)} + 1)(q^{(n,k)} - 2))$$

ways of choosing the term b *or* b^* *so that no hyper-regulus of one set is isomorphic to any hyper-regulus of the second set.*

Let $\theta(k) = \{j; 1 \le j \le n; (k,n) = (j,n)\}$.

(3) *Hence, when* $(n,2k) = (n,k)$*, there are at least*

$$\left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t} \binom{(q^{(n,k)}-1)/t}{i}}{\left((q^{(n,k)}-1)/t, 2\right)|GalGF(q^n)|}\right)\left(\frac{(q^{(n,k)}-2)}{\left(|GalGF(q^n)|, (q^{(n,k)}-2)\right)|\theta(k)|/2}\right)$$

mutually non-isomorphic translation planes obtained using the same group, i. e., the same t*, and when* $(n,2k) = 2(n,k)$*, there are at least*

$$\left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t} \binom{(q^{(n,k)}-1)/t}{i}}{\left((q^{(n,k)}-1)/t, 2\right)|GalGF(q^n)|}\right)(q^{(n,k)}+1)(q^{(n,k)}-2)/\Delta,$$

where

$$\Delta = \left(|GalGF(q^n)|, (q^{(n,k)}+1)(q^{(n,k)}-2)\right)|\theta(k)|/2,$$

mutually non-isomorphic translation planes, all with kernel $GF(q^{(n,k)})$.

(4) *If* $(n,k) \neq (n,k^*)$ *a plane obtained using the group* $C_{(q^{(n,k)}-1)/t}$ *and a plane obtained using the group* $C_{(q^{(n,k^*)}-1)/t^*}$ *cannot be isomorphic. Let* $\delta(n)$ *denote the number of divisors of* n *not including the integer* n*. For each* $\delta(n)$*, we may choose a corresponding* t_n*. Furthermore, let* $\delta_2(n)$ *denote the divisors* z *of* n *so that* $2z$ *is also a divisor of* n *and let* $\delta_{2'}(n)$ *denote the complement of*

$\delta_2(n)$ in $\delta(n)$. In the following summations, we assume that (n,k) is appropriate to either $\delta_2(n)$ or $\delta_{2'}(n)$.

Then there are at least

$$\sum_{(n,k)=1}^{\delta_2(n)} \left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t_n} \binom{(q^{(n,k)}-1)/t_n}{i}}{\left((q^{(n,k)}-1)/t_n, 2\right) |GalGF(q^n)|} \right) (q^{(n,k)}+1)(q^{(n,k)}-2)/\Delta$$

$$+ \sum_{(n,k)=1}^{\delta_{2'}(n)} \left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t} \binom{(q^{(n,k)}-1)/t}{i}}{\left((q^{(n,k)}-1)/t, 2\right) |GalGF(q^n)|} \right) \cdot$$

$$\cdot \left(\frac{(q^{(n,k)}-2)}{\left(|GalGF(q^n)|, (q^{(n,k)}-2)\right) |\theta(k)|/2} \right),$$

where

$$\Delta = \left(|GalGF(q^n)|, (q^{(n,k)}+1)(q^{(n,k)}-2) \right) |\theta(k)|/2,$$

mutually disjoint translation planes of order q^n and kernel containing $GF(q)$ that may be obtained from a Desarguesian affine plane by the replacement of a subset of a group-constructed set of hyper-reguli.

2 Characterization of the André nets

Let a field $K^n \simeq GF(q^n)$, and consider the Desarguesian affine plane with spread

$$x = 0, y = xm; m \in K^n \simeq GF(q^n).$$

The sets

$$A_\delta = \left\{ y = xm; m^{(q^n-1)/(q-1)} = \delta \right\}, \text{ for } \delta \in F \simeq GF(q),$$

are called the 'André' nets of degree $(q^n-1)/(q-1)$. Here, there are corresponding sets that cover these André nets, defined as follows:

$$A_\delta^{q^k} = \left\{ y = x^{q^k} m; m^{(q^n-1)/(q-1)} = \delta \right\}, \text{ for } \delta \in F \simeq GF(q),\ k = 1, 2, \ldots, n-1.$$

Clearly, each of the nets $A_\delta^{q^k}$ is Desarguesian, and the union of any set using the same k is also Desarguesian. We have constructed hyper-reguli that are not André, but the question remains: could these nets have Desarguesian homology-type replacements?

6 Theorem. *Let Σ be a Desarguesian plane of order q^n, $n > 3$, and let $\mathcal{H}$ be a hyper-regulus of degree $(q^n - 1)/(q - 1)$ admitting a replacement $\mathcal{H}^*$ admitting the kernel homology group Z of order $q^n - 1$. If $\mathcal{H}^*$ is Desarguesian then $\mathcal{H}$ is André.*

Proof. If $\mathcal{H}^*$ is Desarguesian, let Σ_1 denote the unique Desarguesian spread containing $\mathcal{H}^*$. Let Z^* denote the kernel homology group of Σ_1. We note that Z is a collineation group of Σ_1 and, as such, since Z is cyclic but not a kernel homology subgroup then fixes exactly two components of Σ_1, say L and M. But if L and M are fixed by Z, they must be components of Σ. Hence, Σ and Σ_1 have at least two common components L and M. Moreover, acting on Σ_1, Z is generated by $(x, y) \longmapsto (xa, yc)$, where both c and a have order $q^n - 1$, for if not, there would be an affine homology in Z, but Z is fixed-point-free. Note that we have taken L and M as $x = 0, y = 0$, common to both Σ_1 and Σ. Let N be a component of $\mathcal{H}^*$, so a 1-dimensional K-space for some field K isomorphic to $GF(q^n)$. The $(q^n - 1)/(q - 1)$ 1-dimensional $GF(q)$-subspaces of N lie on components of $\mathcal{H}$. Similarly, the $(q^n - 1)/(q - 1)$ 1-dimensional $GF(q)$-subspaces of J also lie on components of $\mathcal{H}$.

But the lying-over subspaces look like $y = \sum \alpha_i x^{q^i}$ relative to Σ_1, and Z fixes each such element of this group will fix each element of $\mathcal{H}^*$. Note that the orders of a and c are both $q^n - 1$, as there are no affine homologies Also, we know the order of $a^{-1}c$ is $(q^n - 1)/(q - 1)$ so $a^{(q^n-1)/(q-1)} = c^{(q^n-1)/(q-1)}$. Since $y = \sum \alpha_i x^{q^i}$ maps to $y = \sum \alpha_i x^{q^i} a^{-q^i} c = \sum \alpha_i x^{q^i}$, true for all x, this implies that

$$\alpha_i = \alpha_i a^{-q^i} c.$$

So

$$c = a^{q^i}.$$

Now suppose that there are least two indices $k = i$ and $k = j$ such that $\alpha_k \neq 0$. Then it follows that $a^{q^i} = a^{q^j}$, which implies that $a^{q^i(1-q^{j-i})} = 1$, so that $j = i$ mod n. Thus, every lying-across subspace must look like $y = x^{q^j}\alpha_j$, for various j's and α_j's. Note that the above reasoning also demonstrates that the j's must be the same while the α_j's can vary. That is, the lying-over subspaces look like

$$y = x^{q^i}\alpha_j,$$

for j fixed and α_j varying over a set of $(q^n - 1)/(q - 1)$ elements of a field $GF(q^n)$. Each lying-over subspace must lie over $(q^n - 1)/(q - 1)$ components of Σ_1. If $y = xm$ of Σ_1 intersects $y = x^{q^j}\alpha_j$ in a 1-dimensional $GF(q)$-subspace, then we have shown that

$$m = x^{q^j-1}\alpha_j$$

has a unique solution. This implies that

$$m^{(q^n-1)/(q-1)} = \alpha_j^{(q^n-1)/(q-1)}.$$

This means that

$$\alpha_j^{(q^n-1)/(q-1)} = \beta_j^{(q^n-1)/(q-1)}.$$

Hence, we have established that the lying-over subspaces look like

$$y = x^{q^i}\alpha_0\omega,$$

for all ω such that $\omega^{(q^n-1)/(q-1)} = 1$.

Now consider the collineation group of Σ_1 which fixes all components of Σ_1, the kernel homology group. Then, $y = x^{q^j}\alpha_0\omega$ maps to $y = x^{q^j}d^{1-q^j}\alpha\omega$, which is one of the lying-over components. Hence, the set of lying-over components is permuted by the kernel homology group of Σ_1, so is a collineation group of Σ. This proves that the hyper-regulus is an André hyper-regulus, which completes the proof of the theorem. $\boxed{QED}$

7 Theorem. *Suppose a hyper-regulus in a Desarguesian affine plane Σ of order q^n for $n > 3$ has a homology-type replacement which has a Desarguesian subset D of cardinality*

$$> ((q^n-1)/(q-1) + (q^{[n]} + q))/2,$$

where $q^{[n]}$ is the order of the largest proper subfield of $GF(q^n)$. Then the replacement hyper-regulus is Desarguesian so the hyper-regulus is André.

Proof. Let g be a collineation of Z (kernel homology group of Σ) and let D be in Σ_1. We claim that Σ_1 is unique: this is true if

$$((q^n-1)/(q-1) + (q^{[n]}+1))/2 \geq (q^{[n]}+1),$$

and this is true if and only if

$$(q^n-1)/(q-1) \geq (q^{[n]}+1)$$

and

$$1 + q + q^2 + \cdots + q^n - 1 \geq (q^{[n]}+1),$$

unless possibly $n - 1 = [n]$, implying that $n = 2$.

Hence, Σ_1 is unique. Now consider Dg. The cardinality of $\mathcal{H}^* - D$ is

$$\leq (q^n-1)/(q-1) - \Big((q^n-1)/(q-1) + (q^{[n]}+1)\Big)/2$$
$$= \Big((q^n-1)/(q-1) - (q^{[n]}+1)\Big)/2.$$

So, even if Dg is mapped onto $\mathcal{H}^* - D$, then $Dg \cap D$ still has cardinality larger than

$$\Big((q^n-1)/(q-1)+(q^{[n]}+1)\Big)/2 - \Big((q^n-1)/(q-1)-(q^{[n]}+1)\Big)/2 = (q^{[n]}+1).$$

Thus,

$$|\Sigma_1 g \cap \Sigma_1| > (q^{[n]}+1),$$

implying that $\Sigma_1 g = \Sigma_1$. Hence, Z is a collineation group of Σ_1. However, Z is an orbit of $\mathcal{H}^*$, so that $\mathcal{H}^*$ is Desarguesian. $\boxed{QED}$

We now consider the collineation group of a translation plane constructed from a Desarguesian plane by multiple hyper-regulus replacement.

8 Theorem. *If the set of hyper-reguli of order q^n, for $n > 3$ and $q > 3$, has cardinality $\leq (q-1)/2$ and contains a non-André regulus then the full collineation group is the group inherited from the Desarguesian affine plane Σ.*

Proof. Let g be a collineation of π. Consider $\Sigma g \cap \Sigma$ and assume that $|\Sigma g \cap \Sigma| < (q^{[n]}+1)$. If the cardinality of the set of hyper-reguli is $< (q-1)/2$ then $(\pi - \Sigma)$ has cardinality at most $((q-1)/2-1)(q^n-1)/(q-1) = (q-3)/2(q^n-1)/(q-1)$, so that $\Sigma - \pi$ has cardinality at least $2+(q+2)/2(q^n-1)/(q-1)$, which means that $(\Sigma-\pi)g \cap (\pi - \Sigma)$ has cardinality at most $(q-3)/2(q^n-1)/(q-1)$ so that $(\Sigma - \pi)g \cap (\Sigma - \pi)$ has cardinality at least

$$2 + 5(q^n-1)/(q-1) > q^{[n]}+1,$$

implying that $\Sigma g = \Sigma$. Hence, assume that the cardinality of the set of hyper-reguli is exactly $(q-1)/2$ and assume that $|\Sigma g \cap \Sigma| < (q^{[n]}+1)$. Then $(\Sigma - \pi)g \cap (\pi - \Sigma)$ has cardinality strictly larger than $(q^n-1)/2 - (q^{[n]}+1)$. This means that each hyper-regulus has a Desarguesian subset of cardinality at least

$$\frac{\big((q^n-1)/2 - (q^{[n]}+1)\big)}{(q-1)/2} = (q^n-1)/(q-1) - 2(q^{[n]}+1)/(q-1).$$

Now

$$(q^n-1)/(q-1) - 2(q^{[n]}+1)/(q-1) > \Big((q^n-1)/(q-1)+(q^{[n]}+1)\Big)/2$$

if and only if

$$q^n > q^{[n]+1} + 3q^{[n]} + q + 4.$$

But

$$5q^{n/2+1} + 4 > q^{[n]+1} + 3q^{[n]} + q + 4.$$

Now $q^n \geq 5q^{n/2+1} + 4$. Since $q > 3$, then $q^{n/2+2} \geq 5q^{n/2+1}$. So, if $n \geq n/2 + 2$, i. e., $n \geq 4$, the inequality holds, unless possibly $q = 5$ and $n = 4$. But then $[n] = 2$, and clearly

$$5^4 > 5^3 + 3 \cdot 5^2 + 5 + 4.$$

Hence, by the previous corollary it follows that each of the hyper-reguli is an André hyper-regulus, a contradiction to our assumptions. $\boxed{QED}$

9 Theorem. *Assume that π has order q^3, for $q > 3$. If a set of hyper-reguli of cardinality $\leq (q-1)/2$ is not linear then the full collineation group is the inherited group.*

Actually, if π has order q^n, for $q > 3$, $n \geq 3$, constructed by homology-type replacement of a non-Desarguesian set of hyper-reguli of cardinality $\leq (q-1)/2$, then the full group is the inherited group.

Proof. The argument above shows that we are finished or the set has cardinality exactly $(q-1)/2$ and all of the hyper-reguli are Desarguesian in the same Desarguesian plane. $\boxed{QED}$

10 Remark. Every generalized André plane of order q^3 and kernel $GF(q)$ is an André plane.

Proof. See (12.5) of Lüneburg [4]. $\boxed{QED}$

11 Corollary. *Assume that π has order q^3, for $q > 3$. If a set of hyper-reguli of cardinality $\leq (q-1)/2$ is not linear then π is not a generalized André plane.*

Proof. If the plane is a generalized André plane, it is also an André plane. Hence, there is an affine homology group of order $(q^3-1)/(q-1)$. Since the full collineation group is the inherited group, this group is also a Desarguesian group—from the same Desarguesian plane used in the construction of π. However, the orbits of this group define André nets. Thus, at least one of the nets of the set of hyper-reguli is not fixed by the group H of order $(q^3-1)/(q-1)$. Represent the group generated by $g\colon (x, y) \longmapsto (x, ya)$, for a of order $(q^3-1)/(q-1)$. Then a replacement for a non-linear André hyper-regulus has the form $y = \sum_{i=0}^{2} \alpha_i x^{q^i}$. Suppose an element g^i fixes the non-linear hyper-regulus net. Then this means that $\alpha_i a^i = \alpha_i d^{1-q^i}$. If there are two non-zero α_i's then $d^{1-q^i} = d^{1-q^k}$, so that $d^{q^i(q^{k-i}-1)} = 1$. Since we have order q^3, it follows that $d^{q-1} = 1$ or $d^{q^2-1} = 1$, again implying that d^{1-q} has order dividing $(q+1)$. But this says that a^i has order 1 or order dividing $(q+1)$. But $(1+q+q^2, q+1) = 1$. Therefore, if g^i fixes the non-linear André net (that is, not in the linear set defined by the homology group), then it is itself André, a contradiction. Hence, there are at least $1+q+q^2$ nets, a contradiction. $\boxed{QED}$

3 Collineation groups of the hyper-regulus planes of Jha–Johnson

The authors [2] have given a variety of constructions of mutually disjoint hyper-reguli. Choosing any subset of such produces a translation plane. For simplicity, any such plane constructed in this manner shall be called a 'Jha–Johnson hyper-regulus plane'.

In previous articles, we have constructed hyper-reguli of the following general form:

$$\left\{y = x^{q^k}\alpha d^{1-q^k} + x^{q^{n-k}}\alpha^{-1}\rho d^{1-q^{n-k}}b; d \in GF(q^n) - \{0\}\right\},$$

where α, ρ are in $GF(q^{(n,k)}) - \{0\}$, and where the condition for the existence of this hyper-regulus is

$$b^{(q^n-1)/(q^{(k,n)}-1)} \neq \rho^n \alpha^{-n/(k,n)}.$$

Furthermore, in Jha–Johnson [1], we have determined when two hyper-reguli of this general type could be isomorphic.

12 Theorem (Jha–Johnson [1]). *Let*

$$\mathcal{H}_{b,k} = \left\{y = x^{q^k}d^{1-q^k} + x^{q^{n-k}}d^{1-q^{n-k}}b; d \in GF(q^n) - \{0\}\right\} \cup M.$$

(1) $\mathcal{H}_{b,k}$ *and* $\mathcal{H}_{b^*,k}$ *are isomorphic if and only if*

$$b^* = b^{p^s}a^{-q^{n-k}+q^k} = b^{p^s}a^{-q^k(q^{n-2k}-1)}$$

for some element a *of* $GF(q^n) - \{0\}$.

(a) *When* $(n, 2k) = (n, k)$, *there are at least*

$$(q^{(n,k)} - 2)/(|GalGF(q^n)|, (q^{(n,k)} - 2))$$

mutually non-isomorphic hyper-reguli.

(b) *When* $(n, 2k) = 2(n, k)$, *there are at least*

$$(q^{(n,k)} + 1)(q^{(n,k)} - 2)/(|GalGF(q^n)|, (q^{(n,k)} + 1)(q^{(n,k)} - 2))$$

mutually non-isomorphic hyper-reguli.

(2) $\mathcal{H}_{b,k}$ *and* $\mathcal{H}_{b^*,k^*}$ *are isomorphic if and only if*

$$k^* \in \{k, n-k\}, \text{ modulo } n.$$

In the previous section, we have shown that if we replace a set of $\leq (q-1)/2$ hyper-reguli of degree $(q^n-1)/(q-1)$ and order q^n then, since for $n>3$, actually none of our hyper-reguli are André, the full collineation group of the translation plane is the group inherited from the associated Desarguesian affine plane Σ. What this means is that we may use the above result on isomorphism when considering if there is a collineation of our plane mapping one hyper-regulus to the other. Note that the full collineation group must permute the set of hyper-reguli since it now follows that the associated kernel homology group Z of order (q^n-1) of Σ is a normal subgroup of the full collineation group of π. What the isomorphism result tells us is that the collineation group that acts on a set of hyper-reguli of the form indicated is a subgroup of

$$\left\langle \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}; a,c \in GF(q^n), ac \neq 0 \right\rangle GalGF(q^n).$$

Furthermore, we know that the group

$$\left\langle \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}; d \in GF(q^n), d \neq 0 \right\rangle = Z$$

is a collineation group of any such translation plane π.

Suppose we consider two such hyper-reguli

$$\left\{ y = x^{q^k}\alpha d^{1-q^k} + x^{q^{n-k}}\alpha^{-1}\rho d^{1-q^{n-k}}b; d \in GF(q^n)-\{0\} \right\},$$

for

$$b^{(q^n-1)/(q-1)} \neq \rho^n\alpha^{-n},$$

and

$$\left\{ y = x^{q^k}\beta d^{1-q^k} + x^{q^{n-k}}\beta^{-1}\rho^* d^{1-q^{n-k}}b^*; d \in GF(q^n)-\{0\} \right\},$$

for

$$b^{*(q^n-1)/(q-1)} \neq \rho^{*n}\beta^{-n}.$$

Assume that there is a collineation mapping one to the other. If we take the mapping $\sigma\colon (x,y) \mapsto (x^w, y^w)$, where w is p^s, where $q=p^s$, then

$$\left\{ y = x^{q^k}\alpha d^{1-q^k} + x^{q^{n-k}}\alpha^{-1}\rho d^{1-q^{n-k}}b; d \in GF(q^n)-\{0\} \right\},$$

$$b^{(q^n-1)/(q^{(k,n)}-1)} \neq (\rho\alpha^{-1})^{n/(k,n)},$$

maps onto

$$\left\{ y = x^{q^k}\alpha^w d^{1-q^k} + x^{q^{n-k}}\alpha^{-w}\rho^w d^{1-q^{n-k}}b^{\omega}; d \in GF(q^n)-\{0\} \right\},$$

$$b^{w(q^n-1)/(q^{(n,k)}-1)} \neq \rho^{wn/(n,k)}\alpha^{-wn/(n,k)}.$$

Now consider a mapping of the form $(x,y) \longmapsto (x,y)\begin{bmatrix} a & 0 \\ 0 & e \end{bmatrix}; ae \neq 0$, assuming that $n > 3$. Since we have the mappings $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ fixing each of these hyper-reguli, we may assume that $a = 1$ in the above collineation. Then the collineation maps

$$y = x^{q^k}\alpha^w d^{1-q^k} + x^{q^{n-k}}\alpha^{-w}\rho^w d^{1-q^{n-k}} b^\omega; d \in GF(q^n) - \{0\},$$

$$b^{w(q^n-1)/(q^{(n,k)}-1)} \neq \rho^{wn/(n,k)}\alpha^{-wn/(n,k)},$$

onto

$$y = x^{q^k} e\alpha^w d^{1-q^k} + x^{q^{n-k}} a^{1-q^{n-k}} e\alpha^{-w}\rho^w d^{1-q^{n-k}} b^\omega,$$

$$d \in GF(q^n) - \{0\},\ b^{w(q^n-1)/(q^{(n,k)}-1)} \neq \rho^{wn/(n,k)}\alpha^{-wn/(n,k)}.$$

We consider then

$$y = x^{qk}\alpha^w + x^{q^{n-k}}\alpha^{-w}\rho^w b^w \longmapsto y = x^{q^k}\alpha^w e + x^{q^{n-k}}\alpha^{-w} e\rho^w b^w.$$

In keeping with our general form, we assume that the coefficient on the x^{q^k} term is in $GF(q^{(n,k)}) - \{0\}$. Hence, we may assume that e is in $GF(q)$, let $e = e^{-1}\lambda$ so that the last subspace is $y = x^{q^k}\alpha^w e + x^{q^{n-k}}\alpha^{-w}e^{-1}\lambda\rho^w b^w$. Thus, since e must be in $GF(q^{(k,n)})$, then $e^2\rho^w b^w$ is what we have called $\rho^* b^*$.

Suppose we consider the collineation subgroup of $GL(2,q^n)$. In this case $w = 1$. This implies that $e^2\rho b = \rho^* b^*$.

3.1 Group-constructed sets of hyper-reguli

Since we have a variety of constructions, we first consider the group-constructed sets of hyper-reguli. In this case, α is in a subgroup $C_{(q^{(n,k)}-1)/t}$, where t is a proper divisor of $(q^{(n,k)}-1)$ and $b^{(q^n-1)/(q^{(n,k)}-1)} \notin \left(C_{(q^{(n,k)}-1)/t}\right)^{n/(n,k)}$ (so $\rho = 1$ in our previously defined hyper-reguli). The only restriction that we actually have is that it is possible to find a b so that

$$b^{(q^n-1)/(q^{(n,k)}-1)} \notin \left(C_{(q^{(n,k)}-1)/t}\right)^{n/(n,k)} \text{ and } \alpha^{n/(n,k)} \neq \beta^{n/(n,k)} \text{ for } \alpha \neq \beta.$$

Since $\alpha^{n/(n,k)} \neq \beta^{n/(n,k)}$ for $\alpha \neq \beta$, we see that $((q^{(n,k)}-1)/t, n/(n,k)) = 1$. Hence, $t \neq 1$. So, we see that there are a number of possibilities, by varying k, b, t.

13 Definition. We call the group-constructed set of hyper-reguli of order q^n and degree $(q^n-1)/(q^{(k,n)}-1)$ of type '(b,k,t)' if the set of hyper-reguli is

$$\left\{ y = x^{q^k}\alpha d^{1-q^k} + x^{q^{n-k}}\alpha^{-1}d^{1-q^{n-k}}b; d \in GF(q^n) - \{0\} \right\},$$

$$b^{(q^n-1)/(q^{(k,n)}-1)} \notin (C_{(q^{(n,k)}-1)/t}),$$

$$((q^{(n,k)}-1)/t, n/(n,k)) = 1.$$

Note that there are exactly $(q^{(n,k)}-1)/t$, $t \neq 1$, mutually disjoint hyper-reguli.

We have

$$y = x^{q^k}\alpha + x^{q^{n-k}}\alpha^{-1}b \longmapsto y = x^{q^k}\alpha^w e + x^{q^{n-k}}\alpha^{-w}eb^w = x^{q^{n-k}}\alpha^{-w}e^{-1}e^2b^w.$$

We consider the collineation group within $GL(2,q^n)$, so $w = 1$. Therefore, $e^2b = b$, so $e = \pm 1$. Therefore, when $((q^{(n,k)}-1)/t)$ is odd, the collineation subgroup is exactly $Z_{(q^n-1)}$, the kernel homology group of Σ. We denote the hyper-regulus $\left\{ y = x^{q^k}\alpha d^{1-q^k} + x^{q^{n-k}}\alpha^{-1}d^{1-q^{n-k}}b; d \in GF(q^n) - \{0\} \right\}$ by $\mathcal{H}^*_{\alpha,b,k}$. Since we may choose subsets of the set of hyper-reguli, we see that we would only obtain the group for $e = -1$, when $((q^{(n,k)}-1)/t)$ is even and whenever $\mathcal{H}^*_{\alpha,b,k}$ is chosen then also $\mathcal{H}^*_{-\alpha,b,k}$ is chosen to be in the subset used to produce the translation plane. Note that if there is an element within $GL(2,q^2)$, it is an affine homology of order 2. Thus, we have the following theorem.

14 Theorem. *Let π be a translation plane of order q^n and kernel $GF(q^{(n,k)})$ obtained by the replacement of a subset of hyper-reguli of type (b,k,t) of order q^n (of cardinality $\leq (q-1)/2$) and degrees $(q^n-1)/(q^{(n,k)}-1)$ from a Desarguesian affine plane Σ.*

Then the full collineation group of π is a subgroup of $\Gamma L(2,q^n)$ and the intersection with $GL(2,q^n)$ is either the kernel homology group of Σ or $(q^{(k,n)}-1)/t$ is even and there is an affine homology of order 2 permuting the subset of hyper-reguli used in the replacement in orbits of length 2.

In particular, the full group of affine homologies is either trivial or of order 2.

When $(q^{(n,k)}-1)/t$ is odd and we have a collineation as above then

$$e^2b^w = b.$$

So, there is a possible non-linear collineation, depending on the choice of b. Therefore, the full collineation group of any translation plane constructed by replacing a subset of a set of $(q^{(n,k)}-1)/t$ has order dividing $2(q^n-1)nd$, where $q = p^d$.

Now assume that two translation planes obtained by replacing distinct subsets are isomorphic. Then an isomorphism must be restricted to the Galois group and possibly an element of $GL(2,q^n)$ of order 2. Thus, coupling the result on isomorphism of individual hyper-reguli of Jha–Johnson [1], listed above, with our arguments, we obtain the following isomorphism theorem. Note in the following result we may vary k, b and t.

15 Theorem. *Let π be a translation plane of order $q^n = p^{nd}$, for p a prime, and kernel $GF(q^{(n,k)})$ obtained by the replacement of a subset of a group-constructed set of hyper-reguli of type (b,k,t) of order q^n and degrees $(q^n - 1)/(q^{(n,k)}-1)$ from a Desarguesian affine plane Σ (note that we require $((q^{(n,k)}-1)/t, n/(n,k)) = 1$).*

(1) *Then there are at least*

$$\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t} \binom{(q^{(n,k)}-1)/t}{i}}{\left((q^{(n,k)}-1)/t, 2\right)\left|GalGF(q^n)\right|}$$

mutually non-isomorphic translation planes constructed.

(2) *If π is constructed using a subset of (b,k,t) and ρ is constructed using a subset of (b^*,k^*,t), where $(n,k) = (n,k^*)$, then*

$$k^* \in \{k, n-k\},\ \textit{modulo } n,$$

and

(a) *when $(n,2k) = (n,k)$ there are at least*

$$(q^{(n,k)}-2)/(\left|GalGF(q^n)\right|, (q^{(n,k)}-2))$$

ways of choosing the term b or b^* so that no hyper-regulus of one set is isomorphic to any hyper-regulus of the second set, while

(b) *when $(n,2k) = 2(n,k)$, there are at least*

$$(q^{(n,k)}+1)(q^{(n,k)}-2)/(\left|GalGF(q^n)\right|, (q^{(n,k)}+1)(q^{(n,k)}-2))$$

ways of choosing the term b or b^* so that no hyper-regulus of one set is isomorphic to any hyper-regulus of the second set.

Let $\theta(k) = \{j; 1 \le j \le n; (k,n) = (j,n)\}$.

(3) *Hence, when $(n,2k) = (n,k)$, there are at least*

$$\left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t} \binom{(q^{(n,k)}-1)/t}{i}}{\left((q^{(n,k)}-1)/t, 2\right)\left|GalGF(q^n)\right|}\right)\left(\frac{(q^{(n,k)}-2)}{\left(\left|GalGF(q^n)\right|, (q^{(n,k)}-2)\right)\left|\theta(k)\right|/2}\right)$$

mutually non-isomorphic translation planes obtained using the same group, i. e., the same t, and when $(n,2k) = 2(n,k)$, there are at least

$$\left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t}\left(\begin{array}{c}(q^{(n,k)}-1)/t\\ i\end{array}\right)}{\left((q^{(n,k)}-1)/t,2\right)|GalGF(q^n)|}\right)(q^{(n,k)}+1)(q^{(n,k)}-2)/\Delta,$$

$$\Delta = \left(|GalGF(q^n)|,(q^{(n,k)}+1)(q^{(n,k)}-2)\right)|\theta(k)|/2,$$

mutually non-isomorphic translation planes, all with kernel $GF(q^{(n,k)})$.

(4) *If $(n,k) \neq (n,k^*)$ a plane obtained using the group $C_{(q^{(n,k)}-1)/t}$ and a plane obtained using the group $C_{(q^{(n,k^*)}-1)/t^*}$ cannot be isomorphic. Let $\delta(n)$ denote the number of divisors of n not including the integer n. For each $\delta(n)$, we may choose a corresponding t_n. Furthermore, let $\delta_2(n)$ denote the divisors z of n so that $2z$ is also a divisor of n and let $\delta_{2'}(n)$ denote the complement of $\delta_2(n)$ in $\delta(n)$. In the following summations, we assume that (n,k) is appropriate to either $\delta_2(n)$ or $\delta_{2'}(n)$.*

Then there are at least

$$\sum_{(n,k)=1}^{\delta_2(n)}\left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t_n}\left(\begin{array}{c}(q^{(n,k)}-1)/t_n\\ i\end{array}\right)}{\left((q^{(n,k)}-1)/t_n,2\right)|GalGF(q^n)|}\right)(q^{(n,k)}+1)(q^{(n,k)}-2)/\Delta$$
$$+\sum_{(n,k)=1}^{\delta_{2'}(n)}\left(\frac{\sum_{i=1}^{(q^{(n,k)}-1)/t}\left(\begin{array}{c}(q^{(n,k)}-1)/t\\ i\end{array}\right)}{\left((q^{(n,k)}-1)/t,2\right)|GalGF(q^n)|}\right)\cdot$$
$$\cdot\left(\frac{(q^{(n,k)}-2)}{\left(|GalGF(q^n)|,(q^{(n,k)}-2)\right)|\theta(k)|/2}\right),$$

where

$$\Delta = \left(|GalGF(q^n)|,(q^{(n,k)}+1)(q^{(n,k)}-2)\right)|\theta(k)|/2,$$

mutually disjoint translation planes of order q^n and kernel containing $GF(q)$ that may be obtained from a Desarguesian affine plane by the replacement of a subset of a group-constructed set of hyper-reguli.

16 Remark. Note in the statement above, we could have also varied t, that is varied the group used in the construction. However, since we are replacing subsets, it is not immediately obvious that different corresponding constructions would produce non-isomorphic planes, although certainly some would, since the groups would then be of different order.

Also, our count of mutually non-isomorphic planes is not as fine as could be considered, since it is possible to choose an element b not in a coset of b^w, thus reducing the collineation group to a subgroup of order $2(q^n-1)$.

4 Multiplicative sets and combinations

We have also constructed sets of hyper-reguli by the use of multiplicative sets and also by the combined use of multiplicative sets and group-constructed sets. Most of these planes also will have essentially only the kernel homology group of order (q^n-1), together with a subgroup isomorphic to $GalGF(q^n)$ as a subgroup, and hence will also produce vast numbers of mutually non-isomorphic planes.

For example, the following theorem gives such a construction.

17 Theorem (Jha and Johnson [2])**.** *Suppose that* $\{\alpha_i; \alpha_i \in GF(q)-\{0\}, i=1,2,\ldots,t\}$, *is a set of elements of* $GF(q)$ *such that*

$$\left(\frac{\alpha_i}{\alpha_j}\right)^{(n,q-1)} \neq 1,\ \alpha_j \neq \alpha_i,$$

and there exists an element b *in* $GF(q^n)-\{0\}$ *for which the following conditions hold:*

$$b^{(q^n-1)/(q-1)} \neq \left(\frac{\alpha_i}{\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_t}\right)^n,\ b^{(q^n-1)/(q-1)} \neq \left(\frac{1}{\alpha_1\cdots\widehat{\alpha}_i\cdots\widehat{\alpha}_j\cdots\alpha_t}\right)^n,$$

for $\alpha_i \neq \alpha_j$, *where* $\widehat{\alpha}_i$ *indicates that the element* α_i *is not in the product.*

Then

$$\mathcal{R}_t = \Big\{y = x^{q^k}\alpha_i d^{1-q} + x^{q^{n-k}}\alpha_1\alpha_2\cdots\widehat{\alpha}_i\cdots\alpha_t b d^{1-q^{n-1}};\ i=1,2,\ldots,t;$$
$$d \in GF(q^n)-\{0\}\Big\},\ \text{for } (k,n)=1,$$

is a partial spread of degree $t(q^n-1)/(q-1)$ *that lies over a set of* $t(q^n-1)/(q-1)$ *components in a Desarguesian affine plane* Σ *which is defined by a set of* t *hyper-reguli. If* $\mathcal{M}$ *denotes the set of components of* $\Sigma-\mathcal{R}_t$, *then*

$$\mathcal{R}_t \cup \mathcal{M}$$

is a spread with kernel $GF(q)$.

18 Definition. Any set of mutually disjoint hyper-reguli obtained from a set λ as above shall be called a 'multiplicative' set of hyper-reguli. More precisely, we call this a 'multiplicative set of degree 1'.

4.1 The question of replacements

When n is odd and $n > 3$, we have shown in Jha–Johnson [2] that there are at least two replacements for each hyper-regulus net. Thus, for each hyper-regulus of a chosen subset of a set of mutually disjoint hyper-reguli, we may choose one of at least two replacements. Such choice of replacements produces an even larger number of mutually non-isomorphic planes.

5 Transformations on multiplicative sets and group-constructed sets

We have shown in a previous section that if we replace $\leq (q-1)/2$ mutually disjoint hyper-reguli then the full collineation group of any constructed translation plane of order q^n is the inherited group and within $GL(2, q^n)$, the kernel subgroup has index 1 or 2. Thus, for most of these constructed translation planes the full collineation group will fix each of the hyper-reguli. What this says is that the number of isomorphism classes of translation planes is enormous and it also says that using different groups or multiplicative sets will lead to non-isomorphic translation planes. In this context, we consider the following mapping from a multiplicative set to a set similar to a group-constructed set. As mentioned previously, consider $\{\alpha_i; \alpha_i \in GF(q) - \{0\},\ i = 1, 2, \ldots, t\}$, a set of elements of $GF(q)$ such that

$$\left(\frac{\alpha_i}{\alpha_j}\right)^{(n,q-1)} \neq 1,\ \alpha_j \neq \alpha_i,$$

and suppose there exists an element b in $GF(q^n) - \{0\}$ for which the following conditions hold:

$$b^{(q^n-1)/(q-1)} \neq \left(\frac{\alpha_i}{\alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_t}\right)^n,\ b^{(q^n-1)/(q-1)} \neq \left(\frac{1}{\alpha_1 \cdots \widehat{\alpha}_i \cdots \widehat{\alpha}_j \cdots \alpha_t}\right)^n,$$

for $\alpha_i \neq \alpha_j$, where $\widehat{\alpha}_i$ indicates that the element α_i is not in the product.

Then

$$\mathcal{R}_t = \Big\{ y = x^{q^k} \alpha_i d^{1-q} + x^{q^{n-k}} \alpha_1 \alpha_2 \cdots \widehat{\alpha}_i \cdots \alpha_t b d^{1-q^{n-1}};\ i = 1, 2, \ldots, t; \\ d \in GF(q^n) - \{0\} \Big\},\ \text{for } (k, n) = 1,$$

is a partial spread of degree $t(q^n - 1)/(q - 1)$. For the hyper-regulus

$$y = x^{q^k} \alpha_i d^{1-q} + x^{q^{n-k}} \alpha_1 \alpha_2 \cdots \widehat{\alpha}_i \cdots \alpha_t b d^{1-q^{n-1}},$$

we shall use the notation $(\alpha_i, \alpha_1\alpha_2 \cdots \widehat{\alpha}_i \cdots \alpha_t)$. Consider the mapping that takes $(\alpha_i, \alpha_1\alpha_2 \cdots \widehat{\alpha}_i \cdots \alpha_t)$ onto $(\alpha_i\alpha_j^{-1}, (\alpha_1\alpha_2 \cdots \widehat{\alpha}_i \cdots \alpha_t)(\alpha_1\alpha_2 \cdots \widehat{\alpha}_j \cdots \alpha_t)^{-1})$. However, this set is now simply $(\alpha_i\alpha_j^{-1}, (\alpha_i\alpha_j^{-1})^{-1})$, for all $i = 1, 2, 3, \ldots, t$, and j fixed but arbitrary. Thus, it is possible that certain multiplicative sets produce sets that are group-constructed under such a mapping. It would then appear that such transformations are actually isomorphisms. However, suppose that one such transformation σ is an isomorphism. Then (again assuming that we have $\leq (q-1)/2$ hyper-reguli) it would follow that σ must be a collineation of the associated Desarguesian affine plane. Now we have seen that normally σ must actually fix each hyper-regulus, that is, there is an enforced rigidity. Hence, such transformations are normally not isomorphisms. If one such transformation σ is an isomorphism and is in $GL(2, q^n)$, then since we are mapping one fundamental hyper-regulus to another, it follows that $\sigma = \begin{bmatrix} a^{-1} & 0 \\ 0 & c \end{bmatrix}$. Then we would have σ that maps

$$y = x^{q^k}\alpha_j + x^{q^{n-k}}\alpha_1\alpha_2 \cdots \widehat{\alpha}_j \cdots \alpha_t b$$

to

$$y = x^{q^k} + x^{q^{n-k}}b.$$

This implies that

$$a^{q^k}c\alpha_j = 1 = a^{q^{n-k}}c\alpha_1\alpha_2 \cdots \widehat{\alpha}_j \cdots \alpha_t.$$

For simplicity, take $k = 1$. Then $a^{q^{n-1}-q} \in GF(q)$. Note that the order of $a^{q^{n-1}-q}$ divides $q^{n-2}-1$, the same as q^2-1. Suppose that $(n, 2) = 1$, i. e., n odd. Then $((q-1)^2, q^n-1) = (q-1)(q-1, n)$. Suppose that $(q-1, n) = 1$. Then it follows that $a^{q^{n-1}-q} = 1$. However, this would then say that $\alpha_j = \alpha_1\alpha_2 \cdots \widehat{\alpha}_j \cdots \alpha_t$, normally a contradiction. For example, if the α_i's are all elements of a group of order $(q-1)/k$, then $\alpha_1\alpha_2 \cdots \widehat{\alpha}_j \cdots \alpha_t = (-1)^{(q-1)/k}\alpha_j^{-1}$.

Similarly, if we have a group-constructed set (α, α^{-1}), where $\alpha \in C_{(q-1)/k}$, and ρ is any element of $GF(q)^*$, then we may form the multiplicative set $(\alpha\rho, \alpha^{-1}\rho^{-1})$ and these two sets of hyper-reguli will normally not be isomorphic.

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