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Certain rank conditions on groups

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Abstract. This paper represents an expanded version of the content of three talks given by the author at the conference Advances in Group Theory and Applications, 2007 in Otranto, Italy.

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1 Introduction

At the conference Advances in Group Theory and Applications 2007 in Otranto, Italy I was given the opportunity to present three lectures concerned with the groups in the title of this paper. This paper represents an expanded version of the lectures. As such, the paper is somewhat informal. Some sketch proofs have been included, initially to give the audience (and now the reader) some idea of how the theory of groups of finite rank has progressed. I wanted to give a brief survey of some known results following my own preferences in the subject which hopefully would convey some of the history of groups of finite rank and which would also include work of various collaborators and myself.

Certainly many other mathematicians have done work on groups of finite rank and most of these are not mentioned in this paper. I apologize to them for any omissions and hope that I have correctly attributed results mentioned here. It is appropriate here to also point the reader to the influential texts [36] and [23] where more information can be found concerning groups of finite rank. The paper [38] is also an excellent source for anyone wishing to learn much more.

The topics of my talks were concerned, as the title implies, with the various ranks of groups. The talks were meant to be quite general with some sketch proofs, but accessible to the many students in the audience. The first talk was concerned with the various definitions and examples and in it I discussed the-

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orems of Belyaev and Šunkov in locally finite groups. Section 2 contains this material. In the second talk I discussed the general situation and included the work of Baer and Heineken and also more recent work of N. S. Černikov. This material is described in Section 3. Section 4 contains the material from the third talk and is concerned with some recent developments in the subject.

2 Locally finite groups

An important concept in group theory is the rank of a group; since the word rank is in rather common usage in mathematics some authors refer to the special rank or the Prüfer rank of a group. Various people have studied groups of finite rank and groups with related rank conditions and in this article I shall give a brief history of the subject.

1 Definition. A group G has *finite (or Prüfer or special) rank* r if every finitely generated subgroup of G can be generated by r elements and r is the least integer with this property. If no such integer r exists then we say that the group has infinite rank. We denote the rank of the group G by $r(G)$.

Mal'cev first defined the rank of a group in [25]. In this paper Mal'cev also defined the *general rank* of a group which we shall briefly discuss later. A further influential paper of Mal'cev which will be mentioned often in this paper is [26].

Of course, when discussing groups in general we should see what the situation is for abelian groups. The motivation for the following definition comes from vector spaces.

2 Definition. Let G be abelian. A subset X of G is *linearly independent* if, given distinct $x_1, \dots, x_n$ in X and $m_1, \dots, m_n \in \mathbb{Z}$, then $m_1x_1 + \dots + m_nx_n = 0$ implies $m_ix_i = 0$ for all i.

In the torsionfree case this means that $m_i = 0$ for all i.

3 Definition. Let G be abelian. The number of elements in a maximal independent subset consisting of elements of infinite order is the *0-rank* of G, $r_0(G)$. The number of elements in a maximal independent subset consisting of elements of p-power order is the *p-rank* of G, $r_p(G)$.

It is easy to see that if G abelian then two maximal linearly independent subsets of G consisting of elements of infinite order (respectively prime power order, for the same prime) have the same cardinality so that $r_0(G)$ and $r_p(G)$ are well-defined. For an abelian p-group the p-rank and the rank coincide and for a torsion-free abelian group this the 0-rank and the rank coincide. It is easy to see that for each abelian group G, $r(G) = r_0(G) + \max_p\{r_p(G)\}$. An abelian group G also has what is known as its *total rank* which is defined to be $\bar{r}(G) = r_0(G) + \sum r_p(G)$, where the sum is taken over all primes p. Clearly $r(G) \le \bar{r}(G)$, where generally the inequality is strict. We shall not discuss the total rank further.

Throughout this paper $T(G)$ will denote the torsion subgroup of G. When G is locally nilpotent $T(G)$ is the set of elements of finite order in G. In general however the set of elements of finite order does not form a subgroup and in this case $T(G)$ will denote the unique maximal normal periodic subgroup of the group G. It is very easy to see that if $A, B \triangleleft G$ are periodic then AB is also periodic so $T(G)$ is then well-defined in this case. For an abelian group G, $r_p(G) = r_p(T(G))$ and $r_0(G) = r_0(G/T(G))$. Thus to determine $r(G)$ in the abelian case it is sufficient to know about the torsionfree case and the p-group case, for p a prime. We note that Prüfer first defined the rank of an abelian group in his 1924 paper [34].

Earlier we mentioned the general rank of a group G. A group G has *general rank* R if every finite subset lies in a subgroup of G with R generators. If F is the free group of rank n then this means that the general rank is n but F has infinite special rank. Thus to say that a free group has rank n means that we are referring to the general rank. We remark that if a group has general rank R and special rank r then $R \le r$.

4 Example (General facts and examples)**.**

(1) $C_{p^\infty}, \mathbb{Q}$ are locally cyclic groups, in the sense that every finitely generated subgroup is cyclic, and hence have rank 1.

(2) $\underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{n}$ has rank n since it a vector space of dimension n over $\mathbb{Q}$ so has n linearly independent elements. Every set of $n+1$ elements is linearly dependent so the rank is exactly n.

(3) If A is torsionfree abelian of rank r then $A \cong B$ where $B \le \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{r}$. Thus the torsionfree abelian groups of finite rank are known, in some sense. However it is well-known that the subgroups of $\underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{r}$ can be rather complicated.

(4) If A is an abelian p-group then A has finite rank if and only if A has the minimal condition if and only if A is Černikov. To see this write $A = D \oplus R$, where D is divisible and R is reduced. Then D is a direct sum of finitely many Prüfer p-groups. If B is a basic subgroup of R then B is finite since it is a direct sum of cyclic groups and hence $R = B \oplus L$, for some subgroup L. However since R/B is divisible and R is reduced it follows that $L = 0$ and hence R is finite, so that A has the required structure.

(5) The class of groups of finite rank is closed under taking subgroups and homomorphic images. It is also closed under extensions and if $N \triangleleft G$ then $r(G) \le r(N) + r(G/N)$, where it is easy to see that inequality holds in general.

(6) In this way one shows that Černikov groups, polycyclic groups and soluble minimax groups are all examples of groups of finite rank.

(7) There are also very complicated groups of finite rank. The so-called Tarski monsters, infinite simple 2-generator groups with all proper subgroups cyclic of prime order constructed by Ol'shanskii [31] (and also by Rips) are of rank 2.

(8) It is very easy to see that if all finitely generated subgroups of G have rank at most r then G has rank at most r. Hence $\mathbf{L}(\mathfrak{R}_r) = \mathfrak{R}_r$, where $\mathbf{L}$ is the usual local closure operation and $\mathfrak{R}_r$ denotes the class of groups of rank at most r.

(9) On the other hand if $\mathfrak{R}$ is the class of groups of finite rank then it is not true that $\mathbf{L}(\mathfrak{R}) = \mathfrak{R}$ since there are locally finite groups that are not of finite rank, for example. The class of groups of finite rank does have the property that if all countable subgroups of a group have finite rank then G has finite rank. Thus the class $\mathfrak{R}$ is countably recognizable. For if G does not have finite rank, but every countable subgroup has finite rank then there are finitely generated subgroups H_r whose minimal number of generators is at least r. Then $\langle H_r | r \ge 1\rangle$ is countable and hence has rank s say. However H_{s+1} is not s-generator, yielding a contradiction.

(10) At the conference I posed the question as to whether there was an uncountable group of finite rank. We give an example, due to Obraztsov, in Section 4. Such groups will have to be quite strange since as we shall see later, for a very large class of groups it is known that a group of finite rank in this class is countable. In his paper [25] Mal'cev gave an example of an uncountable group with finite general rank.

The following problem is as yet unsolved.

- What is the structure of groups of finite rank?

We'll start to answer this question, but given the examples of Ol'shanskii we'll need to restrict attention to some "sensible" classes of groups, such as locally finite groups, soluble groups and so on. It is appropriate at this stage however to broaden the scope of the investigation. Accordingly we give some further definitions.

5 Definition.

(i) A group G has *finite abelian subgroup rank* if $r_p(A)$ is finite for all $p \geq 0$, for all abelian subgroups A.

(ii) If there exists an integer r_p, for each p, such that $r_p(A) \leq r_p$ for all abelian subgroups A then G has *bounded abelian subgroup rank*.

(iii) A group G has *finite abelian section rank* (respectively *bounded abelian section rank*) if every section of G has finite abelian subgroup rank (respectively, bounded abelian subgroup rank).

(iv) A group G has *uniformly bounded abelian subgroup rank* if $r_p \leq r$ for all $p \geq 0$.

The terminology employed here is due to Baer [1]. We shall also be interested in groups all of whose abelian subgroups have finite rank as well as other rank conditions. For the convenience of the reader a diagram of group classes is given at the end of the paper. Of course there is no obvious reason why "finite abelian subgroup rank" is closed under taking quotients since abelian subgroups in the quotient don't necessarily correspond to abelian subgroups in the original group. Indeed if F is a free group of finite (general) rank then F has bounded abelian subgroup rank since the abelian subgroups have rank 1, but since every group is an image of a free abelian group the quotients of F will not generally have finite abelian subgroup rank. A good example to also keep in mind here is the group $C_2 \times C_3 \times C_3 \times C_5 \times C_5 \times C_5 \times C_7 \times \dots$ which has bounded abelian subgroup rank but itself does not have finite rank. Furthermore if p is a prime the abelian subgroups of a free product G of elementary abelian groups of orders $p, p^2, p^3, \dots$ are either infinite cyclic or finite elementary abelian p-groups. Hence the abelian subgroups of G have finite rank, but the group G is certainly not of finite rank, nor does this group have bounded abelian subgroup rank.

Soluble groups with finite abelian section rank were called $\mathfrak{S}_0$-groups by Robinson in [35] and in their recent book [23] Lennox and Robinson call such groups FAR-groups. Studying groups with finite abelian subgroup rank is connected with the question:

- What properties does a group inherit from its abelian subgroups?

Such questions were first studied in the following papers (and in many others, rather too numerous to mention here):

(1) O. J. Schmidt [40]: A soluble group in which each abelian subgroup satisfies the minimal condition also satisfies the minimal condition.

(2) S. N. Černikov [6]: A locally finite p-group in which each abelian subgroup satisfies the minimal condition also satisfies the minimal condition.

(3) B. I. Plotkin [33] extended Schmidt's results to radical groups.

Since generalized soluble groups have numerous abelian subgroups, classes of generalized soluble groups are attractive classes in which to study the influence of the abelian subgroups. On the other hand, if we restrict attention to the class of locally finite groups then the structure of the abelian subgroups can have a profound influence on the structure of the whole group as the following theorem shows

6 Theorem. *[Hall-Kulatilaka [16], Kargapolov [20]] If all abelian subgroups of the locally finite group G are finite then G is finite.*

We mention here the well known problem.

- If G is an infinite periodic residually finite group does G have an infinite abelian subgroup?

The theory of groups of finite rank is particularly pleasing in the locally finite case and it is to this theory that we turn. There is a big trade-off in at once restricting ourselves to elements of finite order at the expense of dropping the generalized solubility hypothesis that might be desirable. However the presence of so many finite subgroups (and the power of the theory to be employed) is evident even in Theorem 6 which requires the Feit-Thompson theorem [13].

Let us suppose then that G is a locally finite group with finite abelian subgroup rank. Let P be a p-subgroup of G for some prime p and let A be an abelian subgroup of P. Then A has finite rank and hence has the minimal condition, so P is a locally nilpotent p-group with the minimal condition on abelian subgroups. By the theorem of Černikov [6] mentioned above it follows that P has the minimal condition also and hence G has min-p for all primes p. The class of groups with min-p for all primes p has been studied for some time and undoubtedly one of the highlights of infinite group theory in the past thirty years is the following theorem.

7 Theorem. *[Belyaev [4]] Let G be a locally finite group satisfying min-p for all primes p. Then G is almost locally soluble.*

This result had been conjectured for some time and can be compared to a result of Šunkov to be mentioned later. It is important to realize that these results of Šunkov and Belyaev were proved without recourse to the classification of finite simple groups. We give a "ten line proof" of Theorem 7 using the classification theorem. We observe roughly that Černikov p-groups H can be compared according to their *size*, an invariant of H which is based on the rank of the divisible part H^0 of H and also the index $|H : H^0|$. The interested reader should consult [22] for the formal definition.

Proof of Theorem 7. Suppose there is a counterexample to the theorem. The Feit-Thompson Theorem implies this counterexample contains elements of order 2. Choose a counterexample, G, of minimal 2-size. Then $G/O^{2'}(G)$ is locally soluble by the Feit-Thompson theorem, where $O^{2'}(G) = \cap\{N|N \triangleleft G$ and G/N is a $2'$-group$\}$. By employing a factor shifting argument it follows that if $O^{2'}(G)$ is almost locally soluble then so is G.

So we may assume $G = O^{2'}(G)$ and that every proper normal subgroup of G has smaller 2-size than G, and hence is almost locally soluble. Let $N = \prod\{M|M \triangleleft G, M \neq G\}$. Then N involves no infinite simple groups so N is almost locally soluble, by [22, Theorem 3.17], which tells us that in this situation $N/O_{2',2}(N)$ is finite.

Since N is a maximal normal subgroup of G it follows that G/N must be infinite simple with min-p for all primes p. Hence there is a simple counterexample to the theorem which we again call G. A result of Kegel [21] shows that CFSG implies that a simple group with min-p for even a single prime p is linear. Hence G is linear over a field of characteristic q, say and in this case the maximal q-subgroups of G must be finite. A corollary to the Brauer-Feit theorem (see [43, 9.7]) then implies that G contains an abelian normal subgroup of finite index contradicting the fact that G is infinite simple. The result follows. $\boxed{QED}$

8 Corollary. *Let G be a locally finite group with finite abelian subgroup rank. Then G is almost locally soluble.*

Of course G need not have finite rank since $C_2 \times C_3 \times C_3 \times C_5 \times C_5 \times C_5 \times \dots$ has finite abelian subgroup rank.

What can be said if our locally finite group G has the stronger property that all abelian subgroups have finite rank? This is the result of Šunkov that was mentioned earlier.

9 Theorem. *[Šunkov [41]] Let G be a locally finite group and suppose that all abelian subgroups of G have finite rank. Then G has finite rank and is almost locally soluble.*

Proof. G has min-p for all primes p so G is almost locally soluble. Thus the result follows if we know it for locally soluble groups. $\boxed{QED}$

The locally soluble case is the following result.

10 Theorem. *[Gorčakov [15]] If G is a periodic locally soluble group all of whose abelian subgroups have finite rank then G has finite rank*

Gorčakov's theorem itself depends upon

11 Theorem. *[Kargapolov [18]] Let G be a periodic locally soluble group of finite rank. Then $G/\rho(G)$ is abelian-by-finite with finite Sylow p-subgroups for all primes p.*

The theory of linear groups underpins much of this work. For example to prove Theorem 11 let V be a chief factor of G so that V is a finite elementary abelian p-group for some prime p. Thus V is a vector space over $\mathbb{Z}_p$, the field with p elements. Then $\bar{G} = G/C_G(V)$ is a finite soluble irreducible group of linear transformations of V of degree at most $n = r(G)$. A Theorem of Mal'cev (see Theorem 12 below) shows that $\bar{G}$ has a normal abelian subgroup of index $f(n)$. Then $\bar{G}^{f(n)}$ is abelian and hence $(G^{f(n)})' \leq \rho(G)$, the Hirsch-Plotkin radical of G. Thus $G/\rho(G)$ is abelian-by-finite. Since G has min-p for all primes p, all radicable subgroups lie in $\rho(G)$ and hence the maximal p-subgroups of $G/\rho(G)$ are finite, using results concerning groups with min-p for all p.

The theorems of Mal'cev referred to above come from his influential paper of 1951.

12 Theorem. *[Mal'cev [26]]*

(i) Let G be a soluble subgroup of $GL(n, F)$, for some algebraically closed field F. Then there exists an integer valued function $d(n)$ such that G has a normal subgroup of index dividing $d(n)$ which is conjugate in $GL(n, F)$ to a group of triangular matrices.

(ii) If G is an irreducible soluble group of linear transformations of a vector space of finite dimension n over a field then there is an integer valued function $e(n)$ such that G has an abelian normal subgroup of index dividing $e(n)$.

It is quite pleasing how this result of Kargapolov's was being obtained at about the same time that the structure, at least of locally soluble groups with min-p for all primes p, was being obtained.

3 Non-torsion groups

In the previous section we saw how locally finite groups behave when the various rank conditions in which we are interested are placed on them. We now remove the restriction of having all elements of finite order. The goal in this section is to consider two major results, one due to Baer and Heineken [3], the other due to N. S. Černikov [5].

First we note that if G is a torsionfree nilpotent group of rank r then G has nilpotency class bounded by a function of r only. The easiest way to see this is as follows: If N is a maximal normal abelian subgroup of G then $N = C_G(N) \leq$

$Z_r(G)$, by a result due to Čarin. Since N is a subgroup of a direct sum of r copies of the rationals it follows that G/N can be embedded in the group of $r \times r$ unitriangular matrices over $\mathbb{Q}$ and hence G/N has nilpotency class at most $r-1$, whence G has class at most $2r-1$. This observation is very useful in some of the results that follow. An important initial result which is needed for the Baer-Heineken theorem is the following result.

13 Theorem. *[Mal'cev [26]] Let G be a locally nilpotent group. The abelian subgroups of G have finite* 0*-rank if and only if $G/T(G)$ is a torsionfree nilpotent group of finite rank.*

Periodic locally nilpotent groups with finite abelian subgroup rank are easily seen to be hypercentral groups with Černikov p-components and such a group G has finite rank if and only if each abelian subgroup has finite rank if and only if the p-components have bounded rank. Notice however that such a group G need not be nilpotent as the example of the locally dihedral 2-group shows.

A very deep theorem of Kargapolov [19] asserts that a soluble group whose abelian subgroups have finite rank itself has finite rank. The results we now describe are more general. Historically however the results in the soluble and locally nilpotent cases were obtained first.

14 Definition.

(i) A group G is *radical* if it has an ascending normal series each factor of which is locally nilpotent.

(ii) The *upper Hirsch-Plotkin series* $\{\rho_\alpha(G)\}$ of the group G is defined by

$$\rho_0(G) = 1 \tag{1}$$
$$\rho_1(G) = \text{Hirsch-Plotkin Radical of } G \tag{2}$$
$$\rho_{\alpha+1}(G)/\rho_\alpha(G) = \rho(G/\rho_\alpha(G)) \text{ for ordinals } \alpha \tag{3}$$
$$\rho_\gamma(G) = \bigcup_{\beta<\gamma} \rho_\beta(G) \text{ for limit ordinals } \gamma. \tag{4}$$

It is a well-known theorem of Hirsch [17] and also Plotkin [32] that the product of the normal locally nilpotent subgroups of a group is also locally nilpotent, the Hirsch-Plotkin radical of G. The class of radical groups, then, allows us to talk about locally nilpotent groups and also soluble groups concurrently. It is clear that a group G is radical if and only if its upper Hirsch-Plotkin series terminates in G.

The most far reaching results obtained concerning groups with finite abelian subgroup rank lie in the following theorem, published in 1972. As usual $T(G)$ is the unique maximal normal torsion subgroup of G.

15 Theorem. *[Baer-Heineken [3]] Let G be a radical group. Suppose that $T(G)$ has finite abelian subgroup rank and that the torsionfree subgroups of G have finite rank. Then*

(i) G is countable

(ii) G has bounded abelian section rank

(iii) G has an ascending series of characteristic subgroups with abelian factors. In particular G is hyperabelian.

(iv) $G^{\mathfrak{F}}$, the finite residual, is nilpotent

(v) $T(G)$ is locally soluble

(vi) $G/T(G)$ is soluble of finite rank and its abelian factors have finite torsion subgroups.

As we remarked previously there are uncountable groups of finite rank. However in general even a locally soluble group with min-p for all primes p can be uncountable, as exhibited by Baer [2]. It is interesting that even though we only hypothesize knowledge concerning the *subgroups* of G we actually get information concerning the *sections* of G. This works essentially because if A/N is an abelian p-subgroup of the subgroup $T(G)/N$ then there is a maximal p-subgroup P of $T(G)$ such that $A \leq PN$. Thus A/N has rank at most the rank of P. The fact that $T(G)$ is locally soluble is a consequence of Belyaev's theorem, Theorem 7, but this fact can be proved independently of that. Also $G^{\mathfrak{F}}$ is the subgroup generated by all the radicable subgroups of G. During my talk I asserted that G is also locally soluble, but it was pointed out to me that this is not so clear. Thus although G is hyperabelian, there is the question:

- Is a group G satisfying the hypotheses of Theorem 15 locally soluble?

Much more can be said here and we list some of the other major points.

- $T(G)$ has min-p for all primes p.
- $T(G)$ has a radicable part (Kargapolov 1961).
- $T(G)$ has conjugate maximal p-subgroups for all p (Šunkov [42]).
- $T(G)$ is residually Černikov.
- $H/T(G)$, the Hirsch-Plotkin radical of $G/T(G)$, is nilpotent by Theorem 13.

- G/H is abelian-by-finite.
- $H/T(G)$ is the join of the radicable subgroups of $G/T(G)$.

Some issues that have to be addressed in the proof (which uses a massive amount of established machinery) include

- We need to show that if G is a radical group with $T = T(G)$ of finite abelian subgroup rank and if A/T is free abelian of countable rank then G has a free abelian subgroup of the same rank. Thus we have to show that $G/T(G)$ is well-behaved, from our point of view.
- To do this let $\langle x_1T, x_2T, \dots\rangle$ be free abelian. Then some work is required to show that $X = \langle x_i, x_j\rangle$ is polycyclic, for each i, j, and hence that X' is finite. Then $X/C_X(X')$ is finite and a commutator argument can be used to show that there are $m_i \in \mathbb{Z}$ such that $\langle x_1^{m_1}, x_2^{m_2} \dots\rangle$ is free abelian.
- Finite abelian subgroup rank sometimes passes to homomorphic images: In general if $A \triangleleft G$ and A abelian then G/A has finite abelian subgroup rank also. This follows with some effort from the previous item.
- $G/T(G)$ has torsion subgroups of finite bounded order essentially because this is true of the torsion subgroups of $GL(n, \mathbb{Q})$, a result due to Schur in 1911 (see [43] for details).

We can now sketch the rest of the proof of Theorem 15. The upper central series of $H/T(G)$ is refined to one in which the factors are "rationally irreducible"; here if $L \triangleleft K$ are two terms of the refined series then K/L is rationally irreducible if whenever M is a G-invariant subgroup such that $L \le M \le K$ we have K/M periodic. Without loss of generality we may assume $T = T(G) = 1$ and if

$$1 = T(G) = H_0 \le H_1 \le H_2 \le \cdots \le H_k = H$$

is a central series whose factors are rationally irreducible then, letting $W_i/C_G(H)$ be the group of automorphisms induced by G that act trivially on H_{i+1}/H_i and $W = \cap_{i\ge 1} W_i$, we see that $W/C_G(H)$ is the stability group of the series so is nilpotent, and W is nilpotent since the series is a central series. Also G/W_i is the group of automorphisms induced by G acting on H_{i+1}/H_i, which is abelian of rank at most r, so that there is an embedding of G/W_i into $GL(r, \mathbb{Q})$. Since a radical linear group of degree n is soluble of derived length bounded by a function of n, by a theorem of Zassenhaus (see [43, Corollary 3.8], for example), it follows from Theorem 12 that G/W_i is abelian-by-finite and hence G/W is also abelian-by-finite. Since it is possible to show that $W = H$ the result follows.

The following quantitative version of the theorem also holds:

16 Theorem. *[Baer-Heineken [3]] Let G be a radical group with finite abelian subgroup rank and let r_p be the maximal p-rank of an abelian subgroup, for $p \geq 0$. Let T be the maximal normal torsion subgroup of G. Then*

(i) Every p-section of T has rank at most $\frac{1}{2}r_p(5r_p + 1)$

(ii) There exist integers, $l(r_0), m(r_0), n(r_0)$ depending only on r_0 such that G/T has a series of characteristic subgroups of length at most $l(r_0)$ whose factors are either finite of order at most $m(r_0)$ or torsion-free abelian of rank at most $n(r_0)$

(iii) G/T is soluble of derived length bounded by a function of r_0

(iv) The p-rank of an abelian section of G is bounded by a function of r_0 and r_p.

This depends heavily upon the following facts.

- A theorem due to Kargapolov [19], Hall (see [39]), Gorčakov [15] and Baer and Heineken [3] asserts that if A is an abelian p-group of finite rank n then the p-component of $\mathrm{Aut}\, A$ is finite of rank at most $\frac{1}{2}n(5n-1)$. Thus if P is a p-subgroup it has a maximal abelian normal subgroup A and then $A = C_P(A)$. Hence P/A embeds in the automorphism group of A so P has rank at most $\frac{1}{2}r_p(5r_p - 1) + r_p$.

- If $A/T(G)$ is a maximal normal abelian subgroup of the nilpotent group $H/T(G)$ then $A/T(G)$ has rank at most r_0 and H/A has class at most $r_0 - 1$, as a group of rational unitriangular $r_0 \times r_0$ matrices, and rank at most $1 + 2 + \cdots + (r_0 - 1) = \frac{1}{2}r_0(r_0 - 1)$.

To complete our discussion of the Baer-Heineken theorem we note the corresponding result when the abelian subgroups are all of finite rank.

17 Theorem. *[Baer-Heineken [3]] Let G be a radical group with maximal normal torsion subgroup T and let H be the Hirsch-Plotkin radical of G. Suppose all the abelian subgroups of T have finite rank and that the torsion-free abelian subgroups of G have finite rank. Then G has finite rank and $G/H \cap T$ is soluble.*

The following generalization has also been proved.

18 Theorem. *[Robinson [38]] Let G be a generalized radical group which has finite abelian subgroup rank. Then G is radical-by-finite*

We shall say more concerning generalized radical groups in Section 4 and refrain from giving the definition now. Also in Section 4 we shall see some examples which show the limitations here, but now we change our emphasis to groups of finite rank in general. The results mentioned earlier tell us rather a lot about groups of finite rank when the groups are radical groups. In general locally soluble groups are not radical, so this is the next type of group to consider. First we have a structure theorem which follows with a small amount of work from Theorem 13 and the result of Zassenhaus mentioned above.

19 Theorem. *Let G be a locally soluble group with finite rank r. Then there is a non-negative integer $n = n(r)$ such that $G^{(n)}$ is a periodic hypercentral group with Černikov p-components.*

A natural question to ask here of course is whether locally soluble groups of finite rank are soluble. This is false even for locally nilpotent groups as the following example shows.

20 Example. [Kegel] A periodic locally soluble and hypercentral group with finite rank need not be soluble.

To show this we work as follows, where p is prime.

- Let H_r be the group of all 2×2 matrices X with coefficients in $\mathbb{Z}_{p^r}$, the ring of integers modulo p^r, and such that $X \equiv I \mod p$.
- F, generated by the integer matrices $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$, is a free group.
- If $X \in F$ then the map $X \longmapsto X \mod p^r$ is a homomorphism of F into H_r with kernel K_r consisting of all matrices X in F such that $X \equiv I \mod p^r$.
- There is no bound on the derived lengths of the H_r as r varies.
- So for each prime p there is a finite p-group $H(p)$ whose derived length increases with p and the group $H(p)$ can be thought of as a group of automorphisms of an abelian p-group of rank 2.
- Then $H(p)$ has rank at most 9 by the result mentioned after Theorem 16 so G, the direct product of the groups $H(p)$ as p varies, is a periodic hypercentral group of rank at most 9 which is not soluble.

We have already noted that groups of finite rank could be too hard to classify in general because of the Tarski monsters. In order to exclude such infinite simple groups we make the following definition.

21 Definition. A group G is locally graded if every nontrivial finitely generated subgroup of G has a nontrivial finite image.

The following classes of groups are all subclasses of the class of locally graded groups.

- locally finite groups.
- locally soluble groups.
- residually finite groups.
- locally (soluble by finite) groups, locally residually finite groups, radical groups.

In 1990 a paper of N. S. Černikov [5] appeared in which a very large class of locally graded groups was defined. We next define Černikov's class. First we need some notation.

Let $\mathbf{L}, \mathbf{R}, \acute{\mathbf{P}}, \grave{\mathbf{P}}$ be the usual closure operations as defined in [36]. Thus if $\mathfrak{Y}$ is a class of groups then

- $G \in \mathbf{L}\mathfrak{Y}$ if every finite subset of G is a subset of a $\mathfrak{Y}$-group.
- $G \in \mathbf{R}\mathfrak{Y}$ if for each $1 \neq x \in G$ there is a normal subgroup N_x of G such that $x \notin N_x$ and $G/N_x \in \mathfrak{Y}$.
- $G \in \acute{\mathbf{P}}\mathfrak{Y}$ if G has an ascending series each of whose factors is a $\mathfrak{Y}$-group.
- $G \in \grave{\mathbf{P}}\mathfrak{Y}$ if G has a descending series each of whose factors is a $\mathfrak{Y}$-group.

We let Λ denote the set of closure operations $\{\mathbf{L}, \mathbf{R}, \acute{\mathbf{P}}, \grave{\mathbf{P}}\}$ and let $\mathfrak{Y}$ denote the class of periodic locally graded groups. Thus $\mathfrak{Y}$ contains all locally finite groups but is somewhat larger. We define the class $\mathfrak{X}$ to be the Λ-closure of the class of periodic locally graded groups; the class $\mathfrak{X}$ is Černikov's class. More concretely, let $\mathfrak{Y}_0 = \mathfrak{Y}$. For each ordinal α let $\mathfrak{Y}_{\alpha+1} = \mathbf{L}\mathfrak{Y}_\alpha \cup \mathbf{R}\mathfrak{Y}_\alpha \cup \acute{\mathbf{P}}\mathfrak{Y}_\alpha \cup \grave{\mathbf{P}}\mathfrak{Y}_\alpha$ and $\mathfrak{Y}_\beta = \cup_{\alpha<\beta}\mathfrak{Y}_\alpha$ for limit ordinals β. Then $\mathfrak{X} = \cup_\alpha \mathfrak{Y}_\alpha$.

It is not too hard to see that

22 Lemma. *$\mathfrak{X}$ is a subclass of the class of locally graded groups*

It is hard to imagine how the class of locally graded groups can be larger than $\mathfrak{X}$ but the solution to the following questions still appear to be unknown.

- If G is locally graded then is $G \in \mathfrak{X}$?
- What is the structure of locally graded groups of finite rank?

In his paper [5], Černikov gave a broad generalization of Theorem 9. Certainly this is the most general theorem known concerning groups of finite rank.

23 Theorem. *[N. S. Černikov [5]] Let G be an $\mathfrak{X}$-group of finite rank. Then G is almost locally soluble and almost hyperabelian.*

As a special case we have

24 Theorem. *[Lubotzky-Mann [24]] Let G be a residually finite group of finite rank. Then G is almost locally soluble.*

The Lubotzky-Mann theorem was proved using pro-p methods, but Černikov's theorem is independent of such work. We shall not go into the proof too much other than to say that the proof basically goes by transfinite induction. Limit ordinals are easy to handle: if α is a limit ordinal and it is known that all groups in $\mathfrak{Y}_\beta$, $(\beta < \alpha)$ that are of finite rank are almost locally soluble then if $G \in \mathfrak{Y}_\alpha$ has finite rank we have $G \in \mathfrak{Y}_\beta$ for some $\beta < \alpha$ and hence G is almost locally soluble. So it suffices to consider the case when $\alpha - 1$ exists and G is of finite rank and in one of the classes $\mathbf{L}\mathfrak{Y}_{\alpha-1}$, $\mathbf{R}\mathfrak{Y}_{\alpha-1}$, $\acute{\mathbf{P}}\mathfrak{Y}_{\alpha-1}$, or $\grave{\mathbf{P}}\mathfrak{Y}_{\alpha-1}$. Each case is then considered in turn and real work is required to establish the result. Notice however that an $\mathfrak{X}$-group of finite rank is countable.

Finally in this section we note that Platonov has proved that a linear group (a subgroup of $GL(n, F)$, for some field F) of finite rank is soluble-by-finite (see [43] for example).

4 Some later results

In this section we see how to generalize Theorems 15 and 23. One way to think of radical groups is that they can be used to simultaneously handle soluble and also locally nilpotent groups. The next class of generalized soluble groups with finite abelian subgroup rank that we might consider is the class of locally soluble groups satisfying this condition. Unfortunately however the anticipated theorem is no longer true as the following theorem of Merzljakov shows!

25 Theorem. *[Merzljakov [27–29]]*

(i) There exists a locally polycyclic group G with finite abelian section rank which does not have bounded abelian section rank. The group G has infinite rank.

(ii) A locally soluble group with all abelian subgroups of bounded rank at most r is of finite rank at most $f(r)$ for some function f.

As we see, these examples of Merzljakov appeared quite early in the proceedings. The original paper, appearing in 1964, asserted that a locally soluble group with all abelian subgroups of finite rank need not have finite rank. Subsequently a mistake was found and this was corrected in the 1969 paper. In the 1984 paper it was shown that the construction could be used to obtain groups with the stated property in Theorem 25(i). We shall give very brief details of this example here.

26 Example (Merzljakov's example).

- The group G is torsionfree with all abelian subgroups of finite but unbounded ranks.
- To begin the construction let H be a group that contains a free abelian normal subgroup $A = \langle a_1, \dots, a_s \rangle$ of rank $s < \infty$ such that $|H : A| = m$ is finite.
- Let $\kappa = (r_1, \dots, r_s)$ be an s-tuple of relatively prime integers and let $k = r_1 r_2 \dots r_s$. Define $\phi(k) = n$, where ϕ is Euler's function.
- Let $A^\kappa = \langle a_1^{r_1}, \dots, a_s^{r_s} \rangle$ and $A^k = \{a^k \mid a \in A\}$.
- Then $A^k \leq A^\kappa$. If $\xi_1, \dots \xi_n$ are the primitive n th roots of unity then let $\Phi_k(x) = \prod_{i=1}^n (x - \xi_i) = x^n - c_1 x^{n-1} - \dots - c_n$ be the cyclotomic polynomial of degree n. Let

$$\zeta_k = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots 1 & 0 \end{pmatrix}$$

  be the companion matrix of Φ_k. Then the characteristic polynomial of ζ_k is Φ_k and ζ_k is cyclic of order k.
- Since A/A^κ is cyclic of order k there is a representation $\sigma : A \longrightarrow A/A^\kappa \longrightarrow GL(n, \mathbb{Z})$.
- The induced representation $H \longrightarrow GL(mn, \mathbb{Z})$ can be formed. Let B be the induced module, which is free abelian of rank mn.
- Form $H_1 = B \rtimes H$, $A_1 = A^k B$, which is free abelian of rank smn and $|H_1 : A_1| = k^s m$. Note that $H = H_0 \leq H_1$.
- We now iterate this procedure to form $G = \bigcup_{i \geq 0} H_i$.

This group G is the group that is required. We shall not go into the remaining details of the proof, but certainly G is locally (free abelian-by-finite).

The result of Theorem 23 was partially generalized in 1996 and in the same paper and a subsequent one Theorem 25(ii) was generalized. This is the content of the next theorem.

27 Theorem. *[Dixon, Evans, Smith [7, 9]]*

(i) *Let G be locally soluble-by-finite and suppose that all locally soluble subgroups of G have finite rank. Then G has finite rank and is almost locally soluble.*

(ii) *Let G be locally soluble-by-finite with all abelian torsion groups of finite rank and all torsionfree abelian subgroups of bounded rank. Then G has finite rank and is almost locally soluble.*

(iii) *Let G be a locally soluble-by-finite group with finite abelian subgroup rank and suppose that G has bounded torsionfree abelian subgroup rank. Then G is almost locally soluble.*

The original goal here was to somehow meld together the results of Šunkov and Merzljakov and in fact, at the time the paper was written, we were unaware of Černikov's theorem. The paper [7] depended heavily upon the classification of finite simple groups. We show in [10] that the classification theorem is not necessary for the proof. We here sketch some of the important points.

- Since the class of groups of finite rank is countably recognizable we may assume that we have a countable counterexample G to the theorem.
- Then $G = \cup_{i\in\omega} H_i$, where H_i is a finitely generated soluble-by-finite group.
- Let R_i denote the soluble radical of H_i and let $R = \langle R_i | i \in \omega\rangle$. Then R is locally soluble of rank r, say, by hypothesis.
- It turns out that the indices $|H_i : R_i|$ must be unbounded.
- $F_i = H_r/R_i$ is semisimple for each i and F_i is a section of F_{i+1}.
- We may assume that the socle of F_i has rank n_i and that $n_i < n_{i+1}$ for each i.
- It can be shown that then H_i contains a normal locally finite subgroup A_i of rank $n_i - kd$ where d is chosen such that $R^{(d)}$ is periodic and k is the order of the largest finite subgroup of $GL(r, \mathbb{Q})$.
- Then $A = \prod_i A_i$ is locally finite and its abelian subgroups have finite rank.
- Šunkov's theorem implies that A has finite rank, which is a contradiction that proves Theorem 27.

This result actually applies to locally radical-by-finite groups. For example, if H is a finitely generated radical-by-finite group with all abelian subgroups of finite rank and if $K \lhd H$ with H/K finite and K radical then by the Baer-Heineken theorem K has finite rank. By Černikov's theorem K is soluble so H is soluble-by-finite.

Černikov's theorem shows that a periodic locally graded group with finite rank is almost locally soluble and hence is locally finite. This suggests the following questions.

- Let G be a periodic locally graded group with all locally soluble (respectively abelian) subgroups of finite rank. Is G of finite rank? Is G locally finite?
- In particular is a periodic residually finite group with all abelian subgroups of finite rank necessarily finite?
- Are there infinite residually finite groups with all subgroups either finite or of finite index?

Of course a free group has all abelian subgroups of (bounded!) finite rank and does not have finite rank, hence the restriction to periodic groups here.

What of groups all of whose proper subgroups have finite rank?

28 Theorem. *Let $G \in \mathfrak{X}$ and suppose that all proper subgroups of G have finite rank. Then G is almost locally soluble and of finite rank.*

Proof. Let F be a nontrivial finitely generated subgroup of G. Since G is locally graded F contains a proper normal subgroup N of finite index. Since F has finite rank, Černikov's theorem implies that F is soluble-by-finite, so G is locally soluble-by-finite. Hence by Theorem 27 G has finite rank or is locally soluble. In the former case G is almost locally soluble by Černikov's theorem. In the latter case let $J = \prod\{N \lhd G | N \neq G\}$. Since a simple locally soluble group is finite, G/J is finite so if $J \neq G$ then G will be of finite rank. If $G = J$ and if $N \lhd G, N \neq G$ then N is of finite rank and has a characteristic ascending series with abelian factors so G is hyperabelian. By the Baer-Heineken theorem G has finite rank. $\boxed{QED}$

Lest anyone believes that if all proper subgroups have finite rank then the group should have finite rank we quote the following theorem of Obraztsov [30] (which I incorrectly attributed to A. Yu. Ol'shanskii during my talk). There are some very exotic examples of such groups. The group is constructed using the "Ol'shanskii method". It is this result which enables us to construct uncountable groups of finite rank.

29 Theorem. *Let $\{G_i\}_{i\in I}$ be a finite or countable set of non-trivial finite or countably infinite groups without involutions. Suppose $|I| \geq 2$ and n is a sufficiently large odd number. Suppose $G_i \cap G_j = 1$ for $i \neq j$. Then there is a countable simple group G containing a copy of G_i for all i with the following properties:*

(i) If $x, y \in G$ and $x \in G_i \setminus \{1\}, y \notin G_i$, for some i, then G is generated by x and y.

(ii) Every proper subgroup of G is either a cyclic group of order dividing n or is contained in some subgroup conjugate to some G_i.

This theorem has many important consequences as follows.

- For each $i \in \mathbb{N}$ let G_i be a countable group without involutions that has rank r_i and suppose that $r_{i+1} > r_i$ for $i \geq 1$. Then the group G which can be constructed in this case is 2-generator but clearly does not have finite rank. However every proper subgroup of G does have finite rank.

- Suppose G_i is free abelian of infinite rank for $i \in \mathbb{N}$. Then the group G constructed using Obraztsov's construction has all proper non-abelian subgroups of finite rank, precisely because it has no proper non-abelian subgroups, although the group itself has infinite rank.

- Let G_i be an abelian group of rank i without involutions. Form the Obraztsov group G constructed from the groups G_i. Then G is a 2-generator group of infinite rank with all proper subgroups abelian.

- To construct an uncountable group of finite rank let H_α, for $\alpha < \beta$, be groups of rank at most 2, where β is a countable ordinal and the H_α satisfy the hypotheses of Theorem 29. Let $H_{\alpha+1}$ be the Obraztsov group obtained. Then clearly $H_{\alpha+1}$ has rank 2. For limit ordinals γ define $H_\gamma = \bigcup_{\alpha<\gamma} H_\alpha$ so that H_γ has rank 2 if each of the H_α do. If ρ is the first uncountable ordinal and for $\alpha < \rho$ we have constructed H_α as above then $H_\rho = \bigcup_{\alpha<\rho} H_\alpha$ has rank 2 and is uncountable. Note also that H_ρ will also be simple.

On the other hand it is very easy to see that if G is a group with all proper subgroups of rank at most r then G has rank at most $r + 1$.

We pursue this idea of looking at the proper subgroups of a group. We have focused on the abelian subgroups of a group. What if the non-abelian ones are well-behaved? Of course one would naturally expect a group to have lots of non-locally soluble subgroups if it is not locally soluble, so what if we ensure that the non-locally soluble subgroups are well-behaved. We can be quite explicit as to the kind of groups we get, at least in Černikov's class $\mathfrak{X}$.

30 Theorem. *[Dixon, Evans, Smith [8]] Let $G \in \mathfrak{X}$ and suppose that every proper subgroup is either locally soluble or of finite rank. Then either*

(i) G is locally soluble, or

(ii) G has finite rank, or

(iii) G is isomorphic to one of $SL(2,\mathbb{F}), PSL(2,\mathbb{F})$ or $Sz(\mathbb{F})$ for some infinite locally finite field $\mathbb{F}$ in which every proper subfield is finite.

In particular, if $G \in \mathfrak{X}$ and all non-abelian subgroups have finite rank then G is either abelian or of finite rank. To make further progress here we definitely need more information concerning locally graded groups.

We now change topic again slightly and look at the Hirsch-Zaicev rank of a group. The motivation here comes from polycyclic groups where it is well-known that the number of infinite cyclic factors is an invariant of the group called the Hirsch length.

31 Definition.

(i) A group G has *finite Hirsch-Zaicev rank* $r_{hz}(G) = r$ if G has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly r.

(ii) G has *finite section* 0*-rank* $r_0(G) = r$ if, for every torsion-free abelian section, U/V of G, $r_0(U/V) \leq r$ and *there is* an abelian torsion-free section A/B such that $r_0(A/B) = r$.

(iii) G is *generalized radical* if G has an ascending series whose factors are locally nilpotent or finite.

The examples of Merzljakov have finite section 0-rank but infinite Hirsch-Zaicev rank. Also periodic generalized radical groups are locally finite.

For abelian groups the section 0-rank and the Hirsch-Zaicev rank are the same and this implies that a soluble group has finite Hirsch-Zaicev rank if and only if it has finite section 0-rank. The natural setting for groups of finite Hirsch-Zaicev rank is the class of generalized radical groups. The following theorem is based very closely on [14].

32 Theorem. *[Dixon, Kurdachenko, Polyakov [12]] Let G be a locally generalized radical group of Hirsch-Zaicev rank r. Then G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that* *(i) T is locally finite and G/T is soluble-by-finite of finite rank,*

(ii) L/T is a torsion-free nilpotent group,

(iii) K/L is a finitely generated torsion-free abelian group,

(iv) G/K is finite and S/T is the soluble radical of G/T.

There are functions $f_1, f_2, f_3 : \mathbb{N} \longrightarrow \mathbb{N}$ such that $|G/K| \leq f_1(r)$, $d(S/T) \leq f_2(r)$ and $r(G/T) \leq f_3(r)$.

Interestingly, there is a local version of this result. So in fact we may take G to be locally (generalized radical with finite abelian subgroup rank).

33 Theorem. *[[12]] Let G be a group and suppose that every finitely generated subgroup of G is a generalized radical group of finite abelian subgroup rank. If there is a positive integer r such that $r_{hz}(A) \leq r$ for each abelian subgroup A, then*

(i) G has finite Hirsch-Zaicev rank at most $f_4(r)$

(ii) $G/T(G)$ is soluble-by-finite of finite rank at most $f_3(f_4(r))$.

Note that if we remove the bound here not only does the result not hold because of the example of Merzljakov, but because of the more critical situation that there are infinite simple examples, as exhibited in [11].

In his paper [37] Derek Robinson showed, using cohomological methods that a finitely generated soluble group with finite abelian subgroup rank is minimax. This can now be generalized as follows.

34 Theorem. *[[12]] Let G be a finitely generated generalized radical group. If G has finite abelian subgroup rank, then G is minimax and soluble-by-finite.*

There is also a further generalization of Theorem 25(ii) which runs as follows.

35 Theorem. *[[12]] Let G be a locally generalized radical group and suppose that there is an integer r such that $r(A) \leq r$ for all abelian subgroups A. Then G has finite rank at most $f_3(f_4(r)) + r(5r+1)/2 + 1$.*

Finally we give a result which holds for just a single prime p or 0. Here, if p is prime then, a group G has *finite section p-rank $r_p(G) = r$* if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section K/L such that $K/L = p^r$.

36 Theorem. *[[12]] Let p be a prime or 0. Let G be a locally generalized radical group. If every finitely generated subgroup of G has finite section p-rank at most r, then G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that*

(i) T is locally finite and G/T is soluble-by-finite of finite rank,

(ii) L/T *is a torsion-free nilpotent group,*

(iii) K/L *is a finitely generated torsion-free abelian group,*

(iv) G/K *is finite and* S/T *is the soluble radical of* G/T.

Furthermore, when p *is prime, the Sylow* p*-subgroups of* T *are Černikov of finite rank at most* $f_5(r)$. G *has finite Hirsch-Zaicev rank at most* $r(r+3)/2$ *and* $|G/K| \leq f_6(r)$ *and* $d(S/T) \leq f_7(r)$.

On the next page we give a diagram of group classes which hopefully helps delineate the classes of groups that we have been discussing.

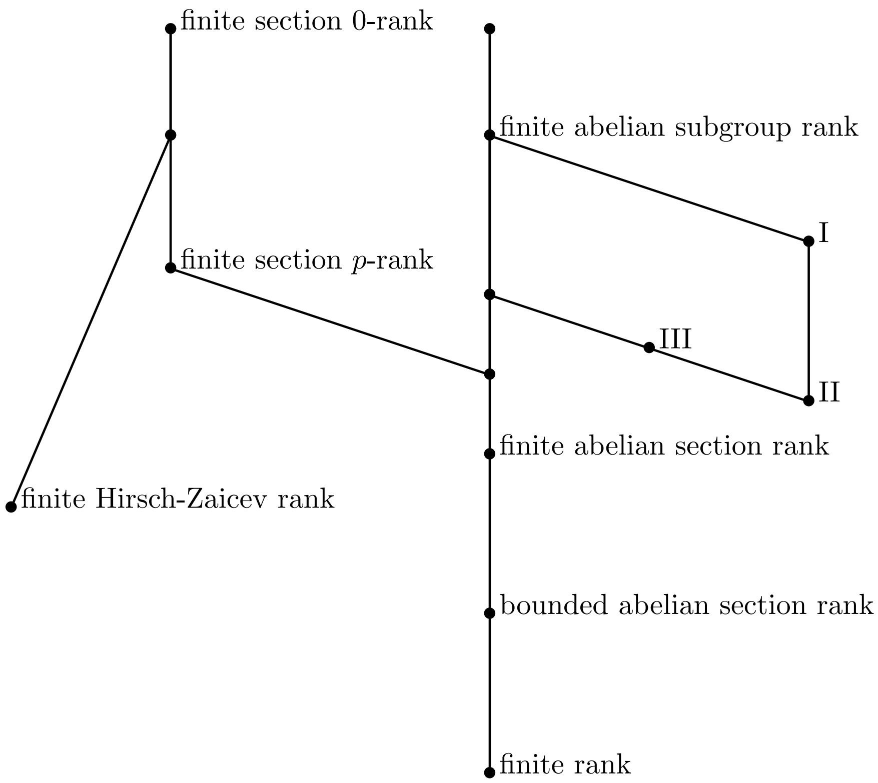

Diagram of Group Classes

Here I represents the class of groups with all abelian subgroups of finite rank, II represents the class of groups with all abelian subgroups of bounded rank and III represents the class of groups with bounded abelian subgroup rank. Examples to show many of the strict containments here have been mentioned throughout the paper.

References

[1] R. Baer: *Local and global hypercentrality and supersolubility*, Nederl.Akad. Wetensch. Proc., Ser. A **69** (1966), 93–126.

[2] R. Baer: *Lokal endlich-auflösbare Gruppen mit enlichen Sylowuntergruppen*, J. Reine Angew. Math. **239/240** (1970), 109–144.

[3] R. Baer and H. Heineken: *Radical groups of finite abelian subgroup rank*, Illinois J. Math. **16** (1972), 533–580.

[4] V. V. Belyaev: *Locally finite groups with Černikov Sylow p-subgroups*, Algebra i Logika **20** (1981), 605–619 (Russian), English transl. in Algebra and Logic, **20** (1981) 393–402.

[5] N. S. Černikov: *A theorem on groups of finite special rank*, Ukrain. Mat. Zh. **42** (1990), 962–970 (Russian), English transl. in Ukrainian Math. J. **42** (1990), 855–861.

[6] S. N. Černikov: *On the theory of special p-groups*, Doklady Akad. Nauk SSSR (N.S.) **63** (1948), 11–14.

[7] M. R. Dixon, M. J. Evans, and H. Smith: *Locally (soluble-by-finite) groups of finite rank*, J. Algebra **182** (1996), 756–769.

[8] M. R. Dixon, M. J. Evans, and H. Smith: *Locally (soluble-by-finite) groups with all proper insoluble subgroups of finite rank*, Arch. Math. (Basel) **68** (1997), 100–109.

[9] M. R. Dixon, M. J. Evans, and H. Smith: *On groups with rank restrictions on subgroups*, Groups St. Andrews 1997 in Bath, I, London Math. Soc. Lecture Note Ser., vol. 260, Cambridge Univ. Press, Cambridge, (1999), 237–247.

[10] M. R. Dixon, M. J. Evans, and H. Smith: *Locally (soluble-by-finite) groups with various rank restrictions on subgroups of infinite rank*, Glasgow Math. J. **47** (2005), 309–317.

[11] M. R. Dixon, M. J. Evans, and H. Smith: *Embedding groups in locally (soluble-by-finite) simple groups*, J. Group Theory **9** (2006), 383–395.

[12] M. R. Dixon, L. A. Kurdachenko, and N. V. Polyakov: *Locally generalized radical groups satisfying certain rank conditions*, to appear, Ricerche di Matematica.

[13] W. Feit and J. G. Thompson: *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029.

[14] S. Franciosi, F. De Giovanni, and L. A. Kurdachenko: *The Schur property and groups with uniform conjugacy classes*, J. Algebra **174** (1995), 823–847.

[15] Ju. M. Gorčakov: *On the existence of Abelian subgroups of infinite rank in locally solvable groups*, Dokl. Akad. Nauk SSSR **156** (1964), 17–20 (Russian), English transl. in Soviet Math. Dokl., **5** (1964), 591–594.

[16] P. Hall and C. R. Kulatilaka: *A property of locally finite groups*, J. London Math. Soc. **39** (1964), 235–239.

[17] K. A. Hirsch: *Über lokal-nilpotente Gruppen*, Math. Z. **63** (1955), 290–294.

[18] M. I. Kargapolov: *Some problems in the theory of nilpotent and solvable groups*, Dokl. Akad. Nauk SSSR **127** (1959), 1164–1166.

[19] M. I. Kargapolov: *On solvable groups of finite rank*, Algebra i Logika Sem. **1** no.5 (1962), 37–44.

[20] M. I. Kargapolov: *On a problem of O. Ju. Šmidt*, Sibirsk. Mat. Ž. **4** (1963), 232–235.

[21] O. H. Kegel: *Uber einfache, lokal endliche Gruppen*, Math. Zeit. **95** (1967), 169–195.

[22] O. H. Kegel and B. A. F. Wehrfritz: *Locally Finite Groups*, North-Holland Mathematical Library, North-Holland, Amsterdam, London, Volume 3 (1973).

[23] J. C. Lennox and D. J. S. Robinson: *The Theory of Infinite Soluble Groups*, Oxford Mathematical Monographs, Oxford University Press, Oxford, (2004).

[24] A. Lubotzky and A. Mann: *Residually finite groups of finite rank*, Math. Proc. Camb. Phil. Soc. **106** (1989), 385–388.

[25] A. I. Mal'cev: *On groups of finite rank*, Mat. Sbornik **22** (1948), 351–352.

[26] A. I. Mal'cev: *On certain classes of infinite soluble groups*, Mat. Sbornik **28** (1951), 367–388 (Russian), English transl. Amer. Math. Soc. Translations.,**2** (1956), 1–21.

[27] Ju I. Merzljakov: *Locally solvable groups of finite rank*, Algebra i Logika Sem. **3** no.2 (1964), 5–16.

[28] Ju I. Merzljakov: *Locally solvable groups of finite rank. II*, Algebra i Logika **8** (1969), 686–690.

[29] Ju I. Merzljakov: *On the theory of locally polycyclic groups*, J. London Math. Soc. (2) **30** no.1 (1984), 67–72.

[30] V. N. Obraztsov: *An embedding theorem for groups and its corollaries*, Mat. Sb. **180** no.4 (1989), 529–541, 560 (Russian), English transl. in Math. USSR-Sb.,**66** (1990), 541–553.

[31] A. Yu Ol'shanskii: *Groups of bounded exponent with subgroups of prime order*, Algebra i Logika **21** (1982), 553–618 (Russian), English transl. in Algebra and Logic, **21** (1982), 369–418.

[32] B. I. Plotkin: *On some criteria of locally nilpotent groups*, Uspehi Mat. Nauk (N.S.) **9** no.3 (1954), (61), 181–186 (Russian), English transl. Amer. Math. Soc. Translations.,**17** (1961), 1–7.

[33] B. I. Plotkin: *On groups with finiteness conditions for abelian subgroups*, Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 648–651.

[34] H. Prüfer: *Theorie der Abelschen Gruppen*, Math. Z. **20** (1924), 165–187.

[35] D. J. S. Robinson: *Residual properties of some classes of infinite soluble groups*, Proc. London Math. Soc. (3) **18** (1968), 495–520.

[36] D. J. S. Robinson: *Finiteness Conditions and Generalized Soluble Groups vols. 1 and 2*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, Heidelberg, New York, (1972), Band 62 and 63.

[37] D. J. S. Robinson: *On the cohomology of soluble groups of finite rank*, J. Pure and Applied Algebra **6** (1975), 155–164.

[38] D. J. S. Robinson: *A new treatment of soluble groups with a finiteness condition on their abelian subgroups*, Bull. London Math. Soc. **8** (1976), 113–129.

[39] J. E. Roseblade: *On groups in which every subgroup is subnormal*, J. Algebra **2** (1965), 402–412.

[40] O. J. Schmidt: *Infinite soluble groups*, Mat. Sb. **17(59)** (1945), 145–162.

[41] V. P. Šunkov: *On locally finite groups of finite rank*, Algebra i Logika **10** (1971), 199–225 (Russian), English transl. in Algebra and Logic,**10** (1971), 127–142.

[42] V. P. Šunkov: *On the conjugacy of the Sylow p-subgroups in SF-groups*, Algebra i Logika **10** (1971), 587–598 (Russian), English transl. in Algebra and Logic, **10** (1971), 363–368.

[43] B. A. F. Wehrfritz: *Infinite Linear Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, New York, Heidelberg, Berlin, (1973), Band 76.