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A class of two-dimensional translation planes admitting SL(2,5)

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Abstract. A class of translation planes of order q^2 admitting SL(2,5) is described, constructed from replaceable A_5-invariant nests of reguli in a regular spread of PG(3,q), where q is odd and $q+1 \equiv 0 \pmod 5$. The translation planes of order 29^2 in this class are determined.

Keywords: Translation planes, nest replacement, SL(2,5)

MSC 2000 classification: 51A40, 51E15

1 Introduction

Two-dimensional translation planes of order q^2 admitting SL(2,5) arise from spreads of PG(3,q) invariant under a collineation group isomorphic to A_5. Examples have been constructed by the following procedure. Consider a collineation group G, isomorphic to A_5, leaving invariant a regular spread of PG(3,q). The reguli in the regular spread are permuted by the action of G. If an orbit in this action consists of disjoint reguli, then taking the opposite reguli gives a new spread invariant under G. For example, the Prohaska plane of order q^2 arises from an orbit of length 6 consisting of disjoint reguli, in the case that $q \equiv 9$ (mod 20) [1,8]. If an orbit, or a union of orbits, in this action consists of the reguli of a replaceable nest, then replacement gives a new spread invariant under G. For example, in the case $q = 9$, there is a replaceable 12-nest orbit [4,5] and, in the case $q = 19$, there is a replaceable 12-nest orbit [6] and also three replaceable 24-nests, each a union of orbits of length 12 [7].

The purpose of this paper is to investigate when other examples arise in this way. It is well known that a regular spread of PG(3,q) gives a model for the Miquelian inversive plane of order q, with the lines of the spread as points and the reguli in the spread as circles. Another model for the Miquelian inversive plane of order q is the projective line GF(q^2)$\cup\{\infty\}$, where the points of the plane are the points of the projective line and the circles are the projective sublines of order q. We assume that q is odd and that 5 divides $q+1$. The model we use for the action of a group isomorphic to A_5 on the projective line GF(q^2) $\cup \{\infty\}$ is as follows. Let θ be a primitive element of GF(q^2). Then $a = \theta^{(q^2-1)/5}$ has order

5 and also norm 1. Let $b = (a - \overline{a})^{-1}$ and choose c such that $b\overline{b} + c\overline{c} = 1$ (where $\overline{x}$ denotes the conjugate of x). The matrices $\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}$, $\begin{pmatrix} b & c \\ -\overline{c} & \overline{b} \end{pmatrix}$ generate a group isomorphic to SL(2,5) [1]. The induced group H of permutations of the projective line is generated by $\alpha : x \mapsto ax$, $\beta : x \mapsto \frac{bx+c}{-\overline{c}x+\overline{b}}$ and is isomorphic to A_5 [3, Satz 8.13]. Consider the orbits of H on the projective line $\mathrm{GF}(q^2) \cup \{\infty\}$. We specify each orbit by the smallest positive exponent k such that θ^k is in the orbit and call k the *exponent* of the orbit. Define a function f as follows: $f(i)$ is the exponent of the orbit of θ^i, for $1 \leq i \leq q^2 - 1$. Since multiplication by a is an element of order 5 in H, the function $f(i)$ is periodic and is determined by the values $f(i)$ for $i = 1, \frac{q^2-1}{5}$. These values can be listed in a table with $q - 1$ rows and $\frac{q+1}{5}$ columns, called the *orbit table*. The first column contains the first $q - 1$ values (in order), the second the next $q - 1$ values, and so on. For example, the orbit tables for $q = 19, 29$ (based on a primitive element θ satisfying $\theta^2 + \theta + 2 = 0$ for $q = 19$ and $\theta^2 + \theta + 3 = 0$ for $q = 29$) are given in Table 1.

The orbit lengths for the action of A_5 on the points of the projective line are easily read off from the orbit tables. Since the tables record only the first $(q^2 - 1)/5$ values of the periodic function $f(i)$, the orbit length is obtained by multiplying the number of occurences of each exponent by 5 (adding 2 for the length of the orbit containing $0, \infty$). For $q = 19$, the orbit lengths are $[12, 20, 30, 60^5]$ and for $q = 29$ they are $[12, 20, 30, 60^{13}]$. The short orbits (of length 12, 20 30) in each case are those with exponents as follows.

	length 12	length 20	length 30
$q = 19$:	14	9	2
$q = 29$:	92	32	14

We now consider the action of H on the circles of the projective line, viewed as a Miquelian inversive plane. For each $\delta \in \mathrm{GF}(q)^*$, let C_δ denote the circle consisting of all the elements in $\mathrm{GF}(q^2)$ of norm δ (the norm is the norm into $\mathrm{GF}(q)$ and is the map: $x \mapsto x\overline{x} = x^{q+1}$). The orbits in this action of length dividing 12 are precisely the orbits containing some C_δ (since these are the only circles invariant under multiplication by the element a of order 5 and norm 1). The cyclic subgroup of $\mathrm{GF}(q^2)^*$ generated by θ^{q-1} consists of all the elements of norm 1 and the circles C_δ are just the cosets of this subgroup. The element θ^{q+1} is a primitive element of $\mathrm{GF}(q)$. Thus, the rows in the orbit table described above correspond exactly to the circles C_δ, the jth row corresponding to C_δ with $\delta = (\theta^{q+1})^j$. The rows give the *orbit profile* of each circle C_δ, that is the distribution of its points amongst the point orbits of H. The circles C_δ and $C_{\delta^{-1}}$ are in the same H-orbit of circles and have the same orbit profile. The orbit of C_δ has length 12, if $\delta \neq \pm 1$. The circle C_{-1} is invariant under H, while the circle C_1 is in an orbit of length 6. The latter is the orbit giving rise to the Prohaska planes in the case $q \equiv 1 \pmod 4$. The orbit of C_δ, $\delta \neq \pm 1$, is a 12-nest if and only if the corresponding row of the orbit table involves only exponents of orbits of length 60, with each exponent occuring twice. In this case, the circle C_δ has 10 points in each of the occuring 60-orbits and its H-orbit consists of 12 circles covering each point of these 60-orbits twice. In contrast, an orbit profile consisting only of orbits of length 60, with no exponent repeated, corresponds to an orbit of 12 pairwise disjoint circles since the 12 circles cover $12(q+1)$ points, the maximum possible.

1	4	12	1
2	3	1	3
3	3	4	12
4	1	12	14
5	1	1	5
3	4	5	12
4	12	5	5
4	3	12	2
9	9	9	9
2	12	3	4
5	5	12	4
12	5	4	3
5	1	1	5
14	12	1	4
12	4	3	3
3	1	3	2
1	12	4	1
5	2	5	2

1	3	5	28	31	4
2	3	7	32	49	31
3	31	49	28	5	7
4	32	1	9	6	9
5	8	8	28	3	31
6	4	13	1	6	8
7	2	1	4	2	49
8	5	1	92	4	28
9	9	13	7	49	13
8	9	8	49	2	7
7	31	3	49	6	6
2	5	13	28	2	9
13	1	3	31	4	13
14	14	14	14	14	14
13	13	4	31	3	1
2	9	2	28	13	5
7	6	6	49	3	31
8	7	2	49	8	9
9	13	49	7	13	9
8	28	4	92	1	5
7	49	2	4	1	2
6	8	6	1	13	4
5	31	3	28	8	8
4	9	6	9	1	32
3	7	5	28	49	31
2	31	49	32	7	3
1	4	31	28	5	3
28	6	5	5	6	28

Table 1

The orbit table for $q = 19$ shows that there is a unique 12-nest orbit (corresponding to the 5th row) and a unique orbit of 12 disjoint circles (corresponding to the 6th row). Each of these orbits gives rise to a translation plane of order 19^2 admitting SL(2,5) [6]. Further such translation planes were constructed in [7] by replacement of certain 24-nests, obtained by taking the union of any two of the orbits corresponding to rows 1,3,7 of the orbit table. The proof that the nests were replaceable depended crucially on properties of the orbit table.

2 Main result

We describe a class of translation planes of order q^2 admitting SL(2,5), constructed from replaceable A_5-invariant nests of reguli in a regular spread of PG(3,q). We assume that q is odd and that $q + 1 \equiv 0 \pmod 5$. Our main theorem is the following.

1 Theorem. *Amongst the first $(q-3)/2$ rows of the orbit table, consider those involving only exponents of orbits of length 60. Suppose that we can find k of these rows with the property that each exponent occurs precisely twice in the rows or not at all. Then, these k rows correspond to an A_5-invariant $12k$-nest of reguli in a regular spread of PG(3,q). If, in each of the k rows, we can assign a parity + or − to each exponent such that the parities alternate along each row and such that each exponent is assigned + in one of its occurences and − in the other, then the nest is replaceable and replacement gives rise to a translation plane of order q^2 admitting SL(2,5).*

Proof. Each of the k rows corresponds to an orbit of length 12 in the action of H on the circles of the Miquelian plane. Since each exponent occurs precisely twice or not at all, the $12k$ circles form a nest (as there are $k(q+1)/10$ point orbits covered). We translate to the context of PG(3,q). The group H corresponds to a collineation group K, isomorphic to A_5, leaving invariant a regular spread. The lines and reguli of the regular spread correspond to the points and circles of the Miquelian plane. There is a collineation σ of order $q+1$ which fixes each line of the regular spread, acting transitively on its points. The collineation σ centralises K. In addition, there is another collineation τ of order $q+1$ leaving the regular spread invariant, which fixes two of its lines and permutes the remaining lines in $q-1$ cycles of length $q+1$, which are reguli of the regular spread. These facts are proved in [2]. There is no loss of generality in assuming that the fixed lines of this collineation correspond to $0, \infty$, that the $q-1$ reguli correspond to the circles C_δ, that τ corresponds to multiplication by θ^{q-1} and that $\tau^{(q+1)/5} \in K$. Let A_δ be the regulus corresponding to C_δ. An additional fact, proved in [6], is crucial to our argument. Consider the regulus of the regular spread fixed by K. The external points to the corresponding hyperbolic quadric can be partitioned into two subsets Δ_1, Δ_2, which are the orbits of either PSL(2,q)-subgroup fixing the regulus, or its opposite regulus, line-wise. The collineations σ and τ each interchange Δ_1 and Δ_2. The subgroup $K \times \langle \sigma^2 \rangle$ leaves each of Δ_1, Δ_2 invariant.

Let ℓ be a line of A_δ^{opp}, where A_δ corresponds to one of the k rows defining the nest. We claim that the $(K \times \langle \sigma^2 \rangle)$-orbit of ℓ is a partial spread. Since exponents occur at most twice in the row, the regulus A_δ contains either 5 or 10 lines from certain K-orbits of lines of length 60. The line ℓ meets each line of A_δ. The points on the lines of any orbit of length 60 form a single $(K \times \langle \sigma \rangle)$-orbit of points, which splits into two $(K \times \langle \sigma^2 \rangle)$-orbits, one in Δ_1 and one in Δ_2. The $q+1$ points of ℓ are permuted transitively by τ and the action of τ on the points of ℓ corresponds to the action of τ on the lines of A_δ. The action of τ on ℓ gives rise to two half-lines, the orbits of $\langle \tau^2 \rangle$, one in Δ_1 and one in Δ_2. The orbit of ℓ under $K \times \langle \sigma^2 \rangle$ consists of $6(q+1)$ lines (the K-orbit of ℓ has length 12). Since repeated exponents in the row have opposite parities assigned, these form a partial spread since, if ℓ meets 10 lines of any orbit of length 60, the intersection points are not all in the same $(K \times \langle \sigma^2 \rangle)$-orbit. This proves the claim. The $\langle \sigma \rangle$-orbit of ℓ consists of the lines of A_δ^{opp} and the $\langle \sigma^2 \rangle$-orbit of ℓ is a $\frac{1}{2}$-regulus of A_δ^{opp}. The complementary $\frac{1}{2}$-regulus (corresponding to a choice of ℓ in the other orbit of $\langle \sigma^2 \rangle$ on the lines of A_δ^{opp}) gives rise to a different partial spread. If we now consider a second orbit of reguli involved in the nest, the parity assignment ensures that it is possible to choose one of the two associated partial spreads in such a way that the $(K \times \langle \sigma^2 \rangle)$-orbits of points covered by its lines are different from those covered by the lines of the first partial spread. Thus, the union of the two partial spreads obtained from the two orbits of reguli is itself a partial spread. In fact, the parity assignment ensures that there is a compatible choice of partial spreads across all k of the orbits. The union of these k partial spreads is then a partial spread of lines covering the same set of points as the lines of the nest. Thus, the nest is replaceable. Replacement gives a new spread of PG(3,q) invariant under K and hence a translation plane of order q^2 admitting SL(2,5). $\boxed{QED}$

3 The case $q = 29$

We illustrate the main theorem in the case $q = 29$. The orbit table is listed in Table 1. We restrict to rows not involving the exponents 14,32,92 (corresponding to orbits of lengths 30,20,12 respectively) and also restrict to the first 13 rows (since C_δ and $C_{\delta^{-1}}$ are in the same orbit). Row 28 corresponds to the orbit of circle C_1 of length 6, giving the Prohaska plane.

Rows 1 and 3 have distinct exponents so each corresponds to an orbit of length 12 consisting of disjoint circles or, in the context of PG(3,29), to an orbit of 12 disjoint reguli in a regular spread. For each orbit, the spread obtained by replacing the reguli by their opposites gives a translation plane of order 29^2 admitting SL(2,5).

Table 2 lists the exponents in the rows under consideration. Each exponent occurs at most twice in these rows.

	repeated exponents	single exponents
row 1		1, 3, 4, 5, 28, 31
row 3		3, 5, 7, 28, 31, 49
row 5	8	3, 5, 28, 31
row 6	6	1, 4, 8, 13
row 7	2	1, 4, 7, 49
row 9	9, 13	7, 49
row 10	8	2, 7, 9, 49
row 11	6	3, 7, 31, 49
row 12	2	5, 9, 13, 28
row 13	13	1, 3, 4, 31

Table 2

By examining the table, we see that:
rows 1,3,7 give rise to a 36-nest (1,2,3,4,5,7,28,31,49 occur twice);
rows 3,5,9 give rise to a 36-nest (3,5,7,8,9,13,28,31,49 occur twice);
rows 7,11,13 give rise to a 36-nest (1,2,3,4,6,7,13,31,49 occur twice);
rows 1,5,7,9 give rise to a 48-nest (1,2,3,4,5,7,8,9,13,28,31,49 occur twice).

The rows involved in the nests all satisfy the condition that any repetitions occur in different parity column positions. This is not true of rows 6, 10 and 12. We now test the complete parity assignment in each case.

Consider the nest given by rows 1,3,7. We can assign parities to row 1 as follows:

1	3	5	28	31	4
+	−	+	−	+	−

In row 3, we must initially assign + to exponent 3 (since its parity in row 1 is −), which gives the assignment

3	31	49	28	5	7
+	−	+	−	+	−

which leads to 5 (and 28) being assigned the same parity twice. Thus, there is no parity assignment as in the main theorem.

Consider now the 36-nest given by rows 3,5,9. We can assign parities to row 3 as follows:

3	31	49	28	5	7
+	−	+	−	+	−

This forces the following assignment in row 5

5	8	8	28	3	31
−	+	−	+	−	+

and the following assignment in row 9

9	9	13	7	49	13
−	+	−	+	−	+

which satisfies the hypotheses of the main theorem. Hence, this 36-nest is replaceable.

Consider now the 36-nest given by rows 7,11,13. We can assign parities to row 7 as follows:

7	2	1	4	2	49
+	−	+	−	+	−

This forces the following assignment in row 11

7	31	3	49	6	6
−	+	−	+	−	+

and the following assignment in row 13

13	1	3	31	4	13
+	−	+	−	+	−

which satisfies the hypotheses of the main theorem. Hence, this 36-nest is replaceable.

Finally, consider the 48-nest given by rows 1,5,7,9. The assignment for row 1

1	3	5	28	31	4
+	−	+	−	+	−

forces the following assignment for row 5

5	8	8	28	3	31
−	+	−	+	−	+

which leads to 3 (and 31) being assigned the same parity twice. Thus, there is no parity assignment as in the main theorem.

Our analysis has yielded the following theorem.

2 Theorem. *There are two translation planes of order* 29^2 *admitting SL(2,5) obtained by replacing the disjoint reguli of a* K*-orbit of length 12 by their opposites. In addition, there are two translation planes of order* 29^2 *admitting SL(2,5) in the class described in Theorem 1, arising from replaceable 36-nests.*

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