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Down a mathematical memory lane with Norm

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Abstract. NO ABSTRACT.

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1 Prolegomena

Io sono molto contento di essere qui per potere desiderare a Norm: Buon compleanno! È difficile da credere che ha quasi settanta anni! I am aware that Norm speaks English almost as well as he speaks Italian, so for the sake of several Americans at this conference (including myself!), I will continue in English.

Who is the world's most prolific author of new and interesting results on translation planes? There might be more than one candidate for this distinction, but my vote is cast for Norm Johnson. With many coauthors, but most especially Mauro Biliotti and Vik Jha, he is the author or coauthor of over 280 books and research articles, almost all related in some way to translation planes. Possibly the most impressive of these is the recent monumental *Handbook of Finite Translation Planes* [2]. Norm's energy and talent have continued to produce a flood of new results even though he has now reached an age at which proverbial wisdom (or lack of it) suggests he should be concentrating on the history of math, or perhaps just tending a rose garden. He serves as an editor of the journal *Note di Matematica*, and the rumor is that he has just submitted the manuscript of a new book on parallelisms. Wow!! Happy Birthday Norm!

Norm and I began our careers about the same time in the latter half of the 1960's. Certain central parts of our work over the years have been interrelated, although we have collaborated only once, and that was only on a major survey paper [7]. This talk is a mathematical reminiscence surveying again some of those areas where our interests overlapped. I have no new results to report, only open problems to share, so the experts may very well find nothing of interest here. However, I hope that this talk will interest some of the younger mathematicians and maybe even inspire them to contribute new results in geometry, finite or infinite.

When Norm and I started our careers not only was there no email, there was not even such a thing as a photocopy machine. Our letters were written by hand (or perhaps typed on a mechanical typewriter), and often we made copies by hand (or used carbon paper) in case our memories needed a boost when there was to be further communication. The earliest letter I have from Norm does not carry a date, but my response to it is dated Feb. 17, 1971. We were interested in getting copies of reprints and preprints from each other, but I believe the main topic of common interest was a type of generalized quadrangle I called an "amalgamation of planes." So I shall start with this.

2 Generalized Quadrangles as Amalgamations of Planes

A *generalized quadrangle of order* (s, t), (with $s, t \geq 1$), is a point-line geometry with $1+s$ points on each line, $1+t$ lines through each point, each two points on at most one line, with at least two noncollinear points and at least two nonconcurrent lines, and satisfying the main axiom: if P is a point not on a given line m, there is a unique point-line pair (Y, ℓ) for which P is on ℓ and Y is on both ℓ and m. So there are no triangles and there are the combinatorially maximum possible number of ordinary 4-gons.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a generalized quadrangle (GQ) of order (s, t), (i.e., with point set $\mathcal{P}$, line set $\mathcal{B}$, and incidence relation I). For a point P, $P^{\perp}$ denotes the set of $1 + s + st$ points collinear with P (including P itself). If P and Q are noncollinear points of $\mathcal{S}$, then $|\{P, Q\}^{\perp}| = |P^{\perp} \cap Q^{\perp}| = 1 + t$. Let $\{P, Q\}^{\perp\perp} = \{R : R \in Z^{\perp}$ for all $Z \in \{P, Q\}^{\perp}\}$. Clearly $|\{P, Q\}^{\perp\perp}| \leq 1 + t$. If equality holds we say that the pair (P, Q) is a *regular pair* of points. If P is a point for which it is true that each pair (P, Q) is regular, for all points Q not collinear with P, we say the point P is *regular*.

If $s = t$ and P is a regular point, it is easy to show that a projective plane π_P may be constructed as follows. The points of π_P are just the points of $P^{\perp}$. The lines of π_P are the $s + 1$ lines of $\mathcal{S}$ through P and the sets $\{P, Q\}^{\perp}$ as Q runs over the points of $\mathcal{S}$ not collinear in $\mathcal{S}$ with P. Incidence is the natural one.

The point-line dual of a GQ of order (s, t) is a GQ of order (t, s). So a definition or theorem stated for all GQ of order (s, t) has an immediate dual definition or theorem for all GQ of order (t, s). So we immediately have the concept of regular line (or pair of lines) of a GQ. Suppose that $\mathcal{S}$ is a GQ of order s and that the point P and the line ℓ are both regular and they are incident. So there are projective planes π_P and π_ℓ that have to "fit together" in a special way. We say that $\mathcal{S}$ is an *amalgamation* of the planes π_P and π_ℓ. It turns out that every known GQ of order s with s even (so $s = 2^e$ for some positive integer e) is an amalgamation of planes, and in every known case the planes are Desarguesian. (Verification of this rather elementary result was the main content of that early Johnson-Payne correspondence. We hoped for a construction using non-Desarguesian planes, but that still eludes us.) Both Ted Ostrom (Norm's Doctor-Father) and I claimed in the 1970's to have shown that no amalgamation of odd-order planes could exist if we made certain minimal and natural additional hypotheses on the planes. Ted never published such a result as far as I know, and the strongest result in that direction that I published was that the amalgamation of two Desarguesian planes must have even order. The following two theorems appeared in [13].

Let (α, β) be a pair of permutations of the elements of the Galois field $GF(2^e)$ fixing 0 and 1. The pair is called *admissible* provided the following holds:

$$0 = \sum_{i=1}^{3} u_i(z_{i+1} - z_{i-1}) \implies 0 \neq \sum_{i=1}^{3} u_i^{\alpha}(z_{i+1}^{\beta} - z_{i-1}^{\beta}). \tag{1}$$

Here $u_1,\ u_2,\ u_3$ are distinct elements of $GF(2^e)$, $z_1,\ z_2,\ z_3$ are distinct elements of $GF(2^e)$, and subscripts are taken modulo 3.

1 Theorem. *(S. E. Payne [13]) A GQ $\mathcal{S}$ of order $s = t = 2^e$ is the amalgamation of two Desarguesian planes if and only if there is an admissible pair (α, β) of permutations of the elements of $GF(2^e)$ such that the GQ can be coordinatized as follows. The points of $\mathcal{S}$ are of the form (∞), (a), (m, v), (a, b, c). The lines of $\mathcal{S}$ are of the form $[\infty]$, $[m]$, $[a, b]$, $[m, v, w]$, where a, b, c, m, v, w are elements of $GF(2^e)$, and incidence is as follows:*

(∞) is incident with $[\infty]$ and with $[m]$, for all $m \in GF(2^e)$.

(a) is incident with $[\infty]$ and with $[a, b]$ for all $a, b \in GF(2^e)$.

(m, v) is incident with $[m]$ and with $[m, v, w]$ for all $m, v, w \in GF(2^e)$.

(a, b, c) is incident with $[a, b]$ and with $[m, v, w]$ provided $b = am + w$ and $c = a^{\alpha}m^{\beta} + v$, for all $a, b, c, m, v, w \in GF(2^e)$.

2 Theorem. *(S. E. Payne [13]) If α and β are both additive maps, then $\alpha^{-1}\beta$ is an automorphism of $GF(2^e)$ of maximal order e.*

I tried to prove that if both α and β are multiplicative, then at least one of them is also additive, and hence an automorphism of the field. If α, say, is the automorphism, then WLOG we may assume that $\alpha = id$, and β is an O-permutation (giving an oval). While I was visiting Tim Penttila in Perth, Australia, in 2004, Tim checked this on a computer for several values of e always with this same result. I believe that some clever researcher could prove this.

3 Problem. Is it possible to have a generalized quadrangle that is an amalgamation of planes for which either

(i) the planes have odd order, or

(i) at least one of the planes is non-Desarguesian?

This problem suggests another one that is of great interest to me. If q is a power of a prime, there is a GQ (denoted $W(q)$) of order q obtained by taking as points all the points of $PG(3, q)$ and as lines those lines which are self-conjugate with respect to a symplectic polarity. The point-line dual of $W(q)$ is isomorphic to the point-line geometry obtained from a parabolic quadric $Q(4, q)$ in $PG(4, q)$. When q is odd, $W(q)$ is not self-dual. However, up to duality this is the only known GQ of odd order. If $\mathcal{S}$ is a GQ of order s with two regular points P and Q that are not collinear, then J. A. Thas [20] proved fairly recently that the projective plane π_P is isomorphic to the dual of π_Q. Moreover, if one of these planes is Desarguesian (so the other is also), then $\mathcal{S}$ must be isomorphic to $W(q)$. This GQ and its point-line dual are said to be classical. J. A. Thas and I have worked on the problem of constructing a GQ of odd order having two noncollinear, regular points at which the planes are known non-Desarguesian planes. We will have a contribution to make in this area, but so far no actual construction has been found. The general problem still defies us.

4 Problem. Is there a non-classical GQ of odd order s?

Some computer searches seemed to indicate that certain small self-dual planes cannot be made to work, but much more theoretical work needs to be done before too much more computer time is devoted to this problem.

Accordng to my records all Johnson-Payne correspondence until the mid-1980's dealt with amalgamations of planes.

3 Generalized Quadrangles as Kantor's Coset Geometries & Flock GQ

At a conference in Han-Sur-Lesse, Belgium, held in 1979, W. M. Kantor [8] surprised us with a construction of GQ with parameters (q^2, q) for each prime power q with $q \equiv 2 \pmod 3$. His construction began with a family of classical generalized hexagons, but his description of them also utilized a group coset construction that has been used for all new GQ discovered since then. The appearance of [8] inspired the development in Payne [14] of a more specific recipe for constructing GQ. Then in Kantor [9] the conditions in [14] for q odd were shown to be equivalent to what today we call a q-clan. This in turn inspired an analogous interpretation in Payne [15] for $q = 2^e$, along with the discovery of a new infinite family of GQ with parameters (q^2, q) where $q = 2^e$ and e is odd. About the time that the concept of q-clan had become fixed, and the connection with GQ with parameters (q^2, q) was understood at least algebraically, J. A. Thas showed that a q-clan was equivalent to a flock of a conical cone, i.e., a partition of the cone minus its vertex into conics. At that point he was able to add one new family of GQ to our list of known ones, because he knew of a flock constructed earlier by C. Fisher. This event probably took place in 1986, because in the late winter of 1986 H. Gevaert from Ghent University visited Norm Johnson at the University of Iowa and gave a talk on the subject showing that q-clans were equivalent to flocks of quadratic cones (a recent result of J. A. Thas). Years before, J. A. Thas and independently M. Walker had shown that a flock of a quadric in $PG(3, q)$ coexisted with a spread of $PG(3, q)$. Apparently Johnson already knew this. When he heard the talk by Gevaert he quickly realized that those spreads that coexisted with flocks of conical cones were precisely those that consisted of q reguli that pariwise intersected in the same line. This led almost immediately to the discovery of two new infinite families of flocks, and hence two new infinite families of GQ. Norm sent a brief hand-written letter (to someone? Maybe to H. Gevaert?) describing these two new families as q-clans. (I do not recall how I happened to possess it for a short time.) I suffered greatly from not being able to read Norm's handwriting and hence could not immediately check out the new GQ for myself. Fortunately, that summer there was a wonderful conference in Italy at Paso della Mendola and Vik Jha was there. The day we all arrived, a small group of people formed around me trying to figure out Norm's letter. Vik appeared quite confident that he would be able to read the letter. When Vik looked at it he said we should first guess what Norm would most likely want to be saying. He made a guess and we reread the letter and tried out the resulting computations and they worked! This gave new examples of GQ with parameters (q^2, q) for $q = 5^r$ and for $q = 3^r$ for r large enough. (According to correspondence from Norm, throughout 1986 he was busy trying out other possible parameters, but apparently without success.)

This was very exciting! New q-clans meant new GQ, new conical flocks, new spreads of $PG(3, q)$, and hence new translation planes. Because of ideas introduced later by G. Ebert, there were also new "regular hyperbolic fibrations with constant back" along with even more planes. For more on this subject see [1].

The conference in Italy was really special for GQ. Even though Norm could not be at Paso della Mendola at that time because of a trip to Brazil, his influence was very real to me.

In March of 1987 I received a letter from Norm with a proof that every semifield flock of a conical cone of even order is classical. In Norm's notation for q-clans, suppose that the functions f and g that give a q-clan in the form

$$A_t = \begin{pmatrix} t & g(t) \\ 0 & -f(t) \end{pmatrix}, \quad t \in GF(2^e)$$

are both additive. Then the flock is linear, the GQ is classical, the spread is Desarguesian, etc. The proof is truly technical, especially if the earlier computations of Cohen and Ganley, which are needed to complete the argument, are all included. In response to my pleading, in March of 1990 Norm was gracious enough to send me a complete proof (thirteen pages of his handwriting!) with all the details! Norm's original proof appeared in [4]. Many years later Tim Penttila showed me a proof that depends only on my result (see [10]) characterizing translation ovals, but is otherwise quite simple. This appears in [3] as the fairly short proof of Theorem 9.3.1.

Jef Thas and I wrote a paper [16] in which we showed (among several other things) that a conical partial flock of deficiency 1 could always be completed to a full flock. The proof for q odd was quite involved, and that for q even, while simple enough, was quite different. We sent an early preprint to Norm and he really ran away with this result. One of my letters from him is accompanied by a preprint of the paper [5] that appeared in the proceedings of an AMS regional conference held in Lincoln, Nebraska in November of 1987. Norm was able to conclude several results about translation planes from this extension result on conical flocks. In the meantime L. Storme and J. A. Thas [17] greatly extended this result (for q even), but I still occasionally get something from Norm illustrating once again how fruitful this result has been in his hands.

I subscribe to the philosophy that each talk of this nature should contain one proof. The proof I have chosen for this talk is a short, sweet proof of the Payne-Thas theorem that a partial conical flock of deficiency 1 can be completed to a flock. This proof was shown to me by Jef Thas (probably in 2004) and was attributed by him to Peter Sziklai. (In the meantime the paper [18] has appeared.)

Proof. Let K be the cone in $PG(3,q)$ with vertex $V = (0,0,0,1)$ given as follows:

$$K = \{(x_0, x_1, x_2, x_3) \in PG(3,q) : x_0x_2 = x_1^2\}.$$

Here q is any prime power. A plane π not containing the vertex V has unique coordinates of the form $\pi = [x, y, z, 1]$. A (partial) flock of K may be given as a set of (at most) q planes whose intersections with K form the actual (partial) flock. With this understanding, let $\mathcal{F} = \{\pi_i = [a_i, b_i, c_i, 1] : 2 \le i \le q\}$ be a partial flock of K with deficiency 1.

For fixed $t \in F_q$ the map $i \mapsto f_i(t) = a_i + b_i t + c_i t^2$, $2 \leq i \leq q$, is an injection into F_q, and the point of π_i on the generator through the point $(1, t, t^2, 0)$ is $(1, t, t^2, -f_i(t))$. Similarly, the point of π_i on the generator through $(0, 0, 1, 0)$ is $(0, 0, 1, -c_i)$. Also, when $q > 2$ the sum of the elements of F_q is equal to 0. Hence the missing point on the generator $\langle (0, 0, 0, 1), (1, t, t^2, 0)$ (i.e., not on any of the π_i) is

$$\left(1, t, t^2, \sum_{i=2}^{q} (a_i + b_i t + c_i t^2)\right).$$

Similarly, the missing point on the generator $\langle (0, 0, 0, 1), (0, 0, 1, 0) \rangle$ is

$$(0, 0, 1, \sum_{i=2}^{q} c_i).$$

All these points lie on the plane $[-\sum a_i, -\sum b_i, -\sum c_i, 1]$. Hence $\mathcal{F}$ extends to a flock. $\boxed{QED}$

4 The Johnson-Payne Survey

In 1995 Norm invited me to prepare a survey on topics related to flock GQ to be given at the conference he was planning for the following year to celebrate the 80th birthday of his Doctor Father Ted Ostrom. His vision of my survey included material that I was not comfortable surveying, especially stuff related to partial flocks of cones coordinatized by infinite fields and material related to translation planes. I responded by saying that I would be happy to write a survey on the part with which I was comfortable if he would be my co-author and fill out the survey to fit his vision. He agreed, and I think the result, titled "Flocks of Laguerre Planes and Associated Geometries," was quite satisfying (cf. [7]).

5 Norm never stops!

Just recently I discovered the followng article by Norm: Homology groups of translation planes and flocks of quadratic cones, II; j-planes. This appeared only last year in *Note di Matematica* and obviously is a paper in which I am forced to be interested. It really emphasizes the connection between (partial) conical flocks and various other topics connected with translation planes. (I think the community should know that Norm is older than I am. Norm, I hope that when I reach your age I will still be working on new results in mathematics.)

NormFest is over, but our memories will last much longer and we will continue to be inspired, encouraged, enthused by Norm and his work. I thank the organizers of this conference for bringing together such a diverse and productive group of mathematicians to honor the career of one of the truly influential mathematicians of our era. Happy Birthday, Norm!

References

[1] R. D. Baker, G. L. Ebert, T. Penttila: *Hyperbolic fibrations and q-clans*, Des. Codes Cryptogr., **34** (2005), 295–305.

[2] M. Biliotti, V. Jha, N. L. Johnson: Handbook of Finite Translation Planes, Chapman and Hall/CRC, 2007.

[3] I. Cardinali, S. E. Payne: q-Clan Geometries in Characteristic 2, Frontiers in Mathematics, Birkhäuser, 2007.

[4] N. L. Johnson: *Semifield flocks of quadratic cones*, Simon Stevin, **61** (1987), 313–326.

[5] N. L. Johnson: *Flocks and partial flocks of quadric sets*, Contemporary Mathematics, **111** (1990), 111–116.

[6] N. L. Johnson: *Homology groups of translation planes and flocks of quadratic cones, II; j-planes*, Note di Matematica **28** (2008), 77–93.

[7] N. L. Johnson, S. E. Payne:*Flocks of Laguerre Planes and Associated Geometries*, Mostly Finite Geometries, Ed. N. L. Johnson, Marcel Dekker, (l997), 51–122.

[8] W. M. Kantor: *Generalized quadrangles associated with $G_2(q)$*, Jour. Comb. Theory (A), **29** (1980), 212–219.

[9] W. M. Kantor: *Some generalized quadrangles with parameters q^2, q*, Math. Zeit., **192** (1986), 45–50.

[10] S. E. Payne: *A complete determination of translation ovoids in finite desarguesian planes*, Rend. Accad. Naz. Lincei, **LI** (1971), 226–229.

[11] S. E. Payne: *Generalized quadrangles as amalgamations of projective planes*, Journal of Algebra, **22** (1972), 102–136.

[12] S. E. Payne: *Finite generalized quadrangles: a survey*, Proceedings of an International Conference on Projective Planes, Washington State University Press, 219–261.

[13] S. E. Payne: *Generalized quadrangles with symmetry, II*, Simon Stevin, **50** (1977), 209–245.

[14] S. E. Payne: *Generalized quadrangles as group coset geometries*, Congr. Num., **29** (1980), 717–734.

[15] S. E. Payne: *A new infinite family of generalized quadrangles*, Congr. Num., **49** (1985), 115–128.

[16] S. E. Payne, J. A. Thas: *Conical flocks, partial flocks, derivation and generalized quadrangles*, Geom. Dedicata, **38** (1991), 229–243.

[17] L. Storme, J. A. Thas: *k-Arcs and partial flocks*, Linear Algebra Appl., **226** (1995), 33–45.

[18] P. Sziklai: *Partial flocks of the quadratic cone*, Jour. Comb. Theory (A), **113** (2006), 698–702.

[19] J. A. Thas: *Generalized quadrangles and flocks of cones*, Europ. Jour. Comb., **8** (1987), 441–452.

[20] J. A. Thas: *A characterization of the generalized quadrangle $W(q)$ and applications*, submitted.