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Logical asides on the maximality of a subgroup

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Abstract. We point out the geometric significance of a part of the theorem regarding the maximality of the orthogonal group in the equiaffine group proved in [12].

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A. Schleiermacher and K. Strambach [12] proved a very interesting result regarding the maximaility of the group of orthogonal transformations and of that of Euclidean similarities inside certain groups of affine transformations. Although similar results have been proved earlier, this is the first time that the base field for the groups in question was not the field of real numbers, but an arbitrary Pythagorean field which admits only Archimedean orderings. They also state, as geometric significance of the result regarding the maximality of the group of Euclidean motions in the unimodular group over the reals, that there is "no geometry between the classical Euclidean and the affine geometry". The aim of this note is to point out the exact geometric meaning of the positive part of the 2-dimensional part their theorem, in the case in which the underlying field is an Archimedean ordered Euclidean field. In this case their theorem states that: (1) the group G_1 of Euclidean isometries is maximal in the group H_1 of equiaffinities (affine transformations that preserve non-directed area), and that (2) the group G_2 of Euclidean similarities is maximal in the group H_2 of affine transformations. The restriction to the 2-dimensional case is not essential but simplifies the presentation. The geometric counterpart of group-theoretic results in the spirit of the Erlanger Programm is given by Beth's theorem, as was emphasized by Büchi [1]. Let Eu denote the class of Archimedean ordered Euclidean fields. Given that Eu is not an elementary class (i. e. cannot be axiomatized in first-order logic, as all of its models can be embedded in $\mathbb{R}$, and thus cannot have models of cardinality $> 2^{\aleph_0}$, whereas, by the Löwenheim-Skolem theorem, first-order theories admitting infinite models have models of arbitrarily large cardinality), the logical interpretation one is bound to find for the above results will by necessity be one in a higher-order logic which is strong enough to express Archimedeanity. There are several options, such as weak second-order logic, logic with the Ramsey quantifier Q^2, transitive closure logic, and the infinitary logic $\mathsf{L}_{\omega_1\omega}$ (we shall use *infinitary* only in this sense throughout this paper). Of these, we shall choose the latter, given that we know that Beth's theorem holds in it (see [4], [5], [6]). $\mathsf{L}_{\omega_1\omega}$ is an extension of first-order logic in which infinite conjunctions and disjunctions of first-order sentences are allowed. Beth's theorem states that If $\mathcal{T}$ is a theory expressed in the language $\mathsf{L} := \mathsf{L}_{\omega_1\omega}$, R a k-ary relation symbol which is not in L, $\mathcal{T}'$ is a theory in $\mathsf{L} \cup \{R\}$, whose reduct to L is $\mathcal{T}$, $\mathfrak{A}$ a model of $\mathcal{T}$, $\mathfrak{A}'$ a model of $\mathcal{T}'$ extending $\mathfrak{A}$, $R_{\mathfrak{A}'}$ the interpretation of R in $\mathfrak{A}'$, then the following two statements are equivalent: (i) for every automorphism $f : \mathfrak{A} \to \mathfrak{A}$, we have $R_{\mathfrak{A}'}\ (a_1 \ldots a_k)$ holds in $\mathfrak{A}'$ if and only if $R_{\mathfrak{A}'}\ (f(a_1) \ldots f(a_k))$ holds in $\mathfrak{A}'$. (ii) there exists a formula $\varphi(x_1, \ldots, x_k)$ in L, with free variables $x_1, \ldots, x_k$ such that $R(x_1 \ldots x_k) \leftrightarrow \varphi(x_1, \ldots, x_k)$ is a theorem of $\mathcal{T}'$. Condition (i) is referred to as the *implicit definability* of the relation R, whereas (ii) is referred to as the *explicit definability* of R, the explicit definition being φ. It follows from [11, Th. 6] that equiaffine geometry over fields in Eu can be axiomatized inside $\mathsf{L} := \mathsf{L}_{\omega_1\omega}$, a language containing only one ternary relation Δ as primitive notion, with $\Delta(abc)$ standing for 'the triangle abc has area 1'. To get from equiaffine geometry to Euclidean geometry, one needs to extend the language with the quaternary relation $\equiv$, with $ab \equiv cd$ standing for the 'ab is congruent to cd', or 'the distance from a to b is equal to the distance from c to d' as well as some axioms regarding $\equiv$, such as those found in [13] or those in [14]. We don't go into details as the particular axiomatics used is irrelevant. Let R be a k-ary relation on $K \times K$, defined for all $K \in Eu$. Let $\mathcal{A}$ stand for the $\mathsf{L} \cup \{R\}$-theory of Cartesian planes over fields in Eu (i. e. the intersection over all $K \in Eu$ of the set of all $\mathsf{L} \cup \{R\}$-sentences true in the plane over K) and $\mathcal{E}$ stand for the corresponding $\mathsf{L} \cup \{\equiv, R\}$-theory. The non-existence of a group strictly between G_1 and H_1 states that any relation R, which is invariant under all isometries, but not under *all* equiaffinities, cannot be invariant under *any* equiaffinity which is not an isometry. In the explicit definition formulation, this means that: *If R is a k-ary relation, which is explicitly definable in $\mathcal{E}$ in terms of $\equiv$ and Δ, but is not explicitly definable in $\mathcal{A}$ in terms of Δ alone, then the relation $\equiv$ must be explicitly definable in $\mathcal{E}$ in terms of R and Δ.* To see why we needed Eu and not just the class $Pyth$ of all Pythagorean fields that admit only Archimedean orders, for which (1) was proved in [12], notice that a field K in $Pyth$ need not be rigid (i. e. there may be automorphisms of K different from the identity), giving rise to semi-linear mappings of the affine plane over K preserving Δ, as pointed out in [11, p. 96] (see also [15, 5.57]). For such fields the group of transformations preserving the notions of collinearity, parallelity, and unit area is strictly larger than that of linear transformations with determinant ± 1, so the groups in Theorem 1 in [12] do not have a clear geometric significance.[1] The choice of Archimedean ordered Euclidean fields for our logical interpretation was determined by the fact that they are rigid. In effect, if K is the Pythagorean closure of $\mathbb{Q}$ (the intersection of all Pythagorean fields containing $\mathbb{Q}$), then there are such non-trivial automorphisms of K. By [2], [3, 22C] and [7], K is precisely the splitting field of all irreducible polynomials in $\mathbb{Q}\,(X)$ which split over $\mathbb{R}$, and all of whose roots lie in repeated real radical extensions of $\mathbb{Q}$. The degree of such a polynomial f is a power of 2, so all real radicals appearing in any of the roots of f are square roots. Define the mapping σ on K by $\sigma(\alpha) = \alpha$ if $[\mathbb{Q}\,(\alpha) : \mathbb{Q}] = [\mathbb{Q}\,(\alpha) : \mathbb{Q}\,(\sqrt{2})]$ and by $\sigma(\alpha) = \overline{\alpha}$ if $[\mathbb{Q}\,(\alpha) : \mathbb{Q}\,] > [\mathbb{Q}\,(\alpha) : \mathbb{Q}\,(\sqrt{2})]$, where by $\overline{\alpha}$ we have denoted the number obtained from α by replacing all occurrences of $\sqrt{2}$ in it by $-\sqrt{2}$. It is easy to check that σ is an automorphism of K, and it is not the identity since $\sigma(\sqrt{2}) = -\sqrt{2}$.

Let L stand for the ternary collinearity relation, with $L(abc)$ to be read as 'a, b, c are collinear (but not necessarily distinct) points'. Let $\mathcal{A}'$ and $\mathcal{S}$ stand for the infinitary theories based on L and R, and $\equiv$ and R respectively, of Cartesian planes over fields in Eu. Notice that L is definable in terms of $\equiv$ in $\mathcal{S}$, so that we do not need to take it as an additional primitive notion for $\mathcal{S}$. The non-existence of a group strictly between G_2 and H_2 is equivalent to the following statement: *If R is a k-ary relation, which is explicitly definable in $\mathcal{S}$ in terms of $\equiv$, but is not explicitly definable in $\mathcal{A}'$ terms of L alone, then the relation $\equiv$ must be explicitly definable in $\mathcal{S}$ in terms of R and L.* Another important result of [12] states that the respective subgroups are *not* maximal if K is Pythagorean but not in $Pyth$. Again, I do not know what the exact logical equivalent of that statement is, given that for all elementary (i. e. first-order axiomatizable) classes of fields, the group of equiaffinities is strictly included in the group of all transformations preserving Δ, as shown in [11, p. 96]. I thank the referee for correcting several errors of a previous version.

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