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Symmetric spread sets

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Abstract. Some new results on symplectic translation planes are given using their representation by spread sets of symmetric matrices. We provide a general construction of symplectic planes of even order and then consider the special case of planes of order q^2 with kernel containing $\mathrm{GF}(q)$, stressing the role of Brown's theorem on ovoids containing a conic section. In particular we provide a criterion for a symplectic plane of even order q^2 with kernel containing $\mathrm{GF}(q)$ to be desarguesian. As a consequence we prove that a symplectic plane of even order q^2 with kernel containing $\mathrm{GF}(q)$ and admitting an affine homology of order $q-1$ or a Baer involution fixing a totally isotropic 2-subspace is desarguesian. Finally a short proof that symplectic semifield planes of even order q^2 with kernel containing $\mathrm{GF}(q)$ are desarguesian is given.

Keywords: translation plane, symplectic spread, line-oval, affine homology, Baer involution

MSC 2000 classification: primary 51E23, secondary 51A40

1 Introduction

Let K be a field and V a vector space of dimension $2n$ over K. A *spread* of V is a set Σ of subspaces of V, each of dimension n, partitioning the set of non-zero vectors of V. For the theory of spreads and their associated translation planes we refer to [2] or [21, Chapter I].

In this paper we are mainly interested in the case where K is a finite field with q elements and V is equipped with a *symplectic form* β, that is a non-degenerate alternating bilinear form. The pair (V, β) is called a *symplectic space.* We refer to [30] for symplectic and orthogonal geometries . We consider spreads consisting of *totally isotropic subspaces* (with respect to β). Such a spread, as well the corresponding translation plane, is called *symplectic* (see also [15], [16], [24]). If we fix suitable coordinates (*symplectic coordinates*), to the symplectic spread Σ we associate a set $\mathcal{M}$ of q^n symmetric $n \times n$ matrices with entries in K, such that $\mathcal{M}$ contains the zero matrix and the difference of any two distinct matrices is non-singular. Such a set of matrices will be called a *symmetric spread set.*

In section 2 we will briefly recall some of the main properties of symplectic planes, emphasizing the difference between the cases q *even* and q *odd.* In the even case any symplectic plane admits a very special line-oval, that we called *completely regular* (see [24] and [26]). A very short construction of such line-ovals is given in Theorem 11. The existence of completely line-ovals allows us to provide a general construction of all possible symplectic translation planes of even order. This construction comprises those given by Kantor [16] and Kantor and Williams ([17], [18]). We remark that Kantor's construction applies only when the plane has *odd* dimension over its kernel.

The last topic we consider is the very special case of symplectic translation planes of even order q^2, having kernel containing a field of (even) order q. These planes are equivalent objects of ovoids of $\mathrm{PG}(3,q)$ (see [31]). Only two families of them are known, namely the elliptic quadrics and the Tits ovoids. It is largely conjectured that these are the only ones ([27], [29]). The corresponding symplectic planes are the desarguesian ones and the Lüneburg planes [21, Chapter IV]. We will illustrate the important role played by Brown's theorem [4] on ovoids containing a conic section in deriving some consequences for the corresponding symplectic planes. The main result we prove is Theorem 27, which provides a criterion for a symplectic plane of even order q^2 with kernel containing $\mathrm{GF}(q)$ to be desarguesian. As a consequence we derive that every symplectic plane of order q^2 (with kernel containing $K = \mathrm{GF}(q)$) admitting an affine homology of order $q-1$ or a Baer involution, whose set of fixed points is a totally isotropic 2−subspace, is desarguesian. Finally a short proof, based on Brown's theorem, that any symplectic semifield plane of even order q^2 with kernel containing $K = \mathrm{GF}(q)$ is desarguesian will be given. In view of such an important role played by Brown's theorem we state the following

Problem. Prove that Theorem 27 implies Brown's theorem.

2 A general construction of symplectic planes

Let Π be a finite projective plane of order q. We consider lines as set of points, so that the incidence relation coincides with set-theoretic inclusion. In particular, the line incident with distinct points P and Q is denoted by PQ.

2.1 Ovals and line-ovals

An *oval* of Π is a set of $q+1$ points, no three collinear. Let $\mathcal{O}$ be an oval. A line ℓ is called *exterior*, *tangent* or *secant* to $\mathcal{O}$ according as ℓ meets $\mathcal{O}$ in 0, 1 or 2 points. Easy counting arguments show that

(a) on each point of $\mathcal{O}$ there is one tangent line;

(b) the number of tangent lines is $q+1$;

(c) the number of secant lines is $q(q+1)/2$;

(d) the number of exterior lines is $q(q-1)/2$.

We will be mostly interested in the case of planes of even order. In this case, all the tangent lines to the oval $\mathcal{O}$ pass through the same point N, called the *nucleus* (or else the *knot*) of the oval. The set $\Omega = \mathcal{O} \cup \{N\}$ is called a *hyperoval.* For the theory of ovals we refer to [12] and [19].

There is a very simple characterization of hyperovals, which uses counting arguments.

1 Proposition. *Let Ω be a set of $q+2$ points in a projective plane of even order q, and let t_0 be the number of lines of the plane that do not intersect Ω. Then Ω is a hyperoval if and only if $t_0 \geq q(q-1)/2$.*

For the proof see [23] or [24].

The dual definition of an oval is that of line-oval: a set of $q+1$ distinct lines of Π, no three of which are concurrent. Clearly, all properties of ovals translate to line-ovals.

In case of projective planes of even order, if $\mathcal{O}$ is a line-oval, there exists a unique line ℓ_∞ such that on each of its points there is only one line of $\mathcal{O}$. This line ℓ_∞ is called the *nucleus*, or also the *dual nucleus.* Denote by $\mathfrak{A} = \Pi^{\ell_\infty}$ the affine plane obtained from Π by deleting the line ℓ_∞ and its points, and by $B(\mathcal{O})$ the set of affine points which are on the lines of $\mathcal{O}$. From properties (a), (b) and (c) above, it is easy to prove that

(1) each point of $B(\mathcal{O})$ belongs to two lines of $\mathcal{O}$;

(2) $|B(\mathcal{O})| = q(q+1)/2$;

(3) if ℓ is an affine line not belonging to $\mathcal{O}$, then $|\ell \cap B(\mathcal{O})| = q/2$.

2.2 Spread sets of matrices

Let (V, β) be a $2n$-dimensional symplectic space over the finite field $K = \mathrm{GF}(q)$ (we let q to be even or odd). We illustrate the use we will make of coordinates. Let S_0 and S_∞ be totally isotropic n-subspaces such that $V = S_0 \oplus S_\infty$. The bases $B_0 = (v_1, \ldots, v_n)$ of S_0 and $B_\infty = (w_1, \ldots, w_n)$ of S_∞ are called *dual bases* if $\beta(v_i, w_j) = \delta_{ij}$, for all $i, j = 1, \ldots, n$. The basis $B = B_0 \cup B_\infty$ is called a *symplectic basis* of V, and vector-coordinates with respect to this basis are called *symplectic coordinates.* If B is a symplectic basis, then S_0 and S_∞ identify with K^n and V identifies with $K^n \times K^n$. We will represent elements of K^n as $n \times 1$ matrices, so that vectors of V are assigned coordinates of type $\binom{X}{Y}$, where $X = (x_1, \ldots, x_n)^t$ and $Y = (y_1, \ldots, y_n)^t$ are elements of K^n (symbol ^{t} denotes transposition). With respect to this basis, S_0 has "the equation" $Y = O$ and S_∞ has "the equation" $X = O$ (here O denotes the $n \times 1$

zero matrix). Finally, the matrix representing β in this basis is

$$\begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} .$$

2 Proposition. *With respect to the symplectic basis $B = B_0 \cup B_\infty$ there is a bijection between the family of all totally isotropic n-subspaces of V intersecting S_∞ only in the zero vector and the space of all $n \times n$ symmetric matrices with entries in K.*

Proof. (See also [25, Proposition 2.1]) Let S be a totally isotropic n-subspace such that $S \cap S_\infty = \{0\}$. There is a homogeneous linear system which represents S:

$$AX + BY = O \tag{1}$$

for suitable $n \times n$ matrices A and B. Since S_∞ is represented by $X = O$, the condition $S \cap S_\infty = \{0\}$ gives $\det B \neq 0$. So system (1) is equivalent to

$$Y = M_S X \tag{2}$$

where $M_S = -B^{-1}A$.

We prove that M_S is symmetric. As S is totally isotropic,

$$(H^t, (M_S H)^t) \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} \begin{pmatrix} H' \\ M_S H' \end{pmatrix} = 0 \tag{3}$$

for all $H, H' \in K^n$. Therefore

$$H^t (M_S - M_S^t) H' = 0 \,, \text{ for all } H, H' \in K^n \tag{4}$$

and so $M_S = M_S^t$.

It is easy to see that the map $S \mapsto M_S$ is bijective. $\boxed{QED}$

3 Corollary. *Let Σ be a symplectic spread and let S_0 and S_∞ be two distinct components of Σ. Pick a symplectic basis $B = B_0 \cup B_\infty$. Then to the set $\Sigma \setminus \{S_\infty\}$ there corresponds bijectively a set $\mathcal{M}$ of $n \times n$ symmetric matrices over K such that*

(1) the zero matrix O_n is in $\mathcal{M}$; and

(2) if A and B are in $\mathcal{M}$ and $A \neq B$, then $A - B$ is non-singular.

4 Definition. The set $\mathcal{M}$, as defined in the above corollary, is called a *symmetric spread set* for Σ (with respect to the symplectic basis B).

We will denote by $\mathfrak{A}(\mathcal{M})$ the symplectic translation plane determined by the spread set $\mathcal{M}$.

5 Remark. (1) In the following when we say "*$\mathcal{M}$ is a symmetric spread set of order q^n*" we mean that $\mathcal{M}$ is a set of q^n symmetric $n \times n$ matrices over $K = \mathrm{GF}(q)$ verifying (1) and (2) of Corollary 3.

(2) Note that if $\mathcal{M}$ is a symmetric spread set of order q^n, then also $A^t\mathcal{M}A = \{A^tMA \mid M \in \mathcal{M}\}$ is a symmetric spread set of order q^n for every $A \in \mathrm{GL}(n,q)$. Moreover the associated planes $\mathfrak{A}(\mathcal{M})$ and $\mathfrak{A}(A^t\mathcal{M}A)$ are isomorphic via the isomorphism given by the matrix $\begin{pmatrix} A^{-1} & O_n \\ O_n & A^t \end{pmatrix}$.

The class of spread sets is closely related to the class of *regular sets* of linear maps. A subset $\mathcal{R}$ of $\mathrm{GL}(V)$, where V is a finite dimensional K-vector space, is called *regular* if for any $v, w \in V \setminus \{0\}$ there is precisely one $\lambda \in \mathcal{R}$ such that $\lambda(v) = w$. Clearly, if $\mathcal{M}$ is a spread set, then the set $\mathcal{M}^* = \mathcal{M} \setminus \{O\}$ is a regular set. It is also easy to prove a converse statement.

We determine which conditions are imposed on a symmetric spread set in order that the corresponding plane be desarguesian.

6 Proposition. *Let $\mathcal{M}$ be a symmetric spread set. Then $\mathcal{M}$ is a field if and only if $\mathcal{M}$ consists of commuting elements. In this case K^n is a 1-dimensional vector space over $\mathcal{M}$ and the plane defined by $\mathcal{M}$ is desarguesian.*

PROOF. (See also [13] and [25]) We need only to prove that if $\mathcal{M}$ is commutative, then $\mathcal{M}$ is a field. Since $\mathcal{M}^*$ acts regularly on $K^n \setminus \{0\}$, then, by Schur's Lemma (see [20, Proposition 1.1, p. 643]), the centralizer of $\mathcal{M}^*$ in $M_n(K)$ is a field (it is isomorphic to the kernel of the plane $\mathfrak{A}(\mathcal{M})$) and, by hypothesis, it contains $\mathcal{M}$. Therefore $\mathcal{M}$ coincides with its centralizer and thus is a field. The last statement is now clear. $\boxed{QED}$

There is also a converse in case q even.

7 Proposition. *Let q be even. If $\mathcal{M}$ is a symmetric spread set of order q^n such that $I_n \in \mathcal{M}$ and $\mathfrak{A}(\mathcal{M})$ is desarguesian, then $\mathcal{M}$ is a field.*

PROOF. The desarguesian plane $\mathfrak{A}(\mathcal{M})$ admits a group of shears with axis $X = O$ of order q^n and, for example, the involution $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$. Therefore $\mathcal{M}$ is an additive group such that $M \in \mathcal{M}^*$ if and only if $M^{-1} \in \mathcal{M}^*$. By [9], $\mathcal{M}$ is a field. $\boxed{QED}$

2.3 Completely regular line-ovals

From now on in this paper we assume q **even**, that is $q = 2^d$, where $d \geq 2$. Only in two points of this paper we will depart from this assumption.

If (V, β) is a symplectic space over the field $K = \mathrm{GF}(q)$, let Q be a quadratic form whose polar form is β:

$$\beta(u,v) = Q(u+v) - Q(u) - Q(v)$$

for all $u, v \in V$. We denote by $S(Q)$ the set of *singular vectors* of the quadratic form Q (we include also the zero vector):

$$S(Q) := \{v \in V \mid Q(v) = 0\}\,.$$

Let $\mathcal{S}$ be the K-vector space of all symmetric $n \times n$ matrices. Its dimension is $n(n+1)/2$. Therefore

$$|\mathcal{S}| = q^{n(n+1)/2}\,. \tag{5}$$

Let $\mathcal{A}$ be the K-vector space of all skew-symmetric $n \times n$ matrices. Since we are in characteristic 2, then $\mathcal{A}$ is a subspace of $\mathcal{S}$ (in characteristic 2 a skew-symmetric matrix is a symmetric matrix with zero diagonal). Note that

$$|\mathcal{A}| = q^{n(n-1)/2}\,. \tag{6}$$

8 Definition. *The diagonal map* is the map $\mathrm{d}\colon \mathcal{S} \to K^n$, which associates to every symmetric matrix M the vector $\mathrm{d}(M)$ whose components are the square root those of the diagonal of M in their natural order.

The map d plays a relevant role in [6]. The next two propositions are easy to prove.

9 Proposition. *The map* d *is semilinear, with companion automorphism* $x \mapsto \sqrt{x}$*, for all* $x \in K$*. Its kernel is* $\mathcal{A}$*.*

10 Proposition. *Let A be an $n \times n$ matrix. For every symmetric $n \times n$ matrix M the following identities hold:*

$$A\,\mathrm{d}(M) = \mathrm{d}(AMA^t)\,. \tag{7}$$

$$X^t M X = (X^t \mathrm{d}(M))^2\,,\ \textit{for all } X \in K^n\,. \tag{8}$$

We denote by $\mathrm{T}_a\colon K \to \mathrm{GF}(2)$ the *absolute trace map* of the field K:

$$\mathrm{T}_a(x) = \sum_{i=0}^{d-1} x^{2^i}\,, \quad \text{for all } x \in K\,.$$

By restriction of scalars V can be considered as a GF(2)-space, with symplectic form $\mathrm{T}_a \circ \beta$. It is easy to see that if Q is a quadratic form polarizing to β, then $\mathrm{T}_a \circ Q$ is a quadratic form of V as GF(2)-space, polarizing to $\mathrm{T}_a \circ \beta$.

The following theorem is proven in [24]. In view of its importance for the rest of this paper we give here a very short proof.

11 Theorem. *Let Σ be a symplectic spread of a symplectic space (V,β) over $K = \mathrm{GF}(q)$ and let $\mathfrak{A}(\Sigma)$ be the corresponding symplectic plane of even order q^n. Fix symplectic coordinates in such a way that β is represented by the matrix $\begin{pmatrix} O_n & I_n \\ I_n & O_n \end{pmatrix}$ and let $\mathcal{M}$ be a symmetric spread set for Σ. Let Q be the quadratic form $Q\binom{X}{Y} = X^tY$. Then the set of lines*

$$\mathcal{O} = \{X = O\} \cup \{Y = MX + \mathrm{d}(M)\,,\ M \in \mathcal{M}\} \tag{9}$$

is a line-oval such that

$$B(\mathcal{O}) = S(\mathrm{T}_a \circ Q)\,. \tag{10}$$

Proof. Every element of $B(\mathcal{O})$ is a singular vector for $\mathrm{T}_a \circ Q$. For

$$\begin{aligned}(\mathrm{T}_a \circ Q)\begin{pmatrix} X \\ MX + \mathrm{d}(M)\end{pmatrix} &= \mathrm{T}_a(X^tMX + X^t\mathrm{d}(M)) \\ &= \mathrm{T}_a((X^t\mathrm{d}(M))^2 + X^t\mathrm{d}(M)) = 0\end{aligned}$$

because of (8) and Hilbert's theorem 90 ([20, Theorem 6.3, p. 290]). Therefore $B(\mathcal{O}) \subseteq S(\mathrm{T}_a \circ Q)$. Now

$$|S(\mathrm{T}_a \circ Q)| = \frac{q^n(q^n+1)}{2}$$

(see [30, Theorem 11.5, p. 140]). Hence

$$|B(\mathcal{O})| \le \frac{q^n(q^n+1)}{2}\,.$$

The complement of $B(\mathcal{O})$ has size $\ge \frac{q^n(q^n-1)}{2}$ and by the dual statement of Proposition 1 the set of lines $\mathcal{O}$ is a line-oval. Thus $|B(\mathcal{O})| = \frac{q^n(q^n+1)}{2}$ and so $B(\mathcal{O}) = S(\mathrm{T}_a \circ Q)$. $\boxed{QED}$

This theorem not only states that every symplectic translation plane of even order admits a line-oval, but also provides us with a simple description of it, that allows a general construction of symplectic translation planes of even order. This construction relies on the following observation and some of its consequences.

12 Theorem. *Let $\mathcal{M}$ be a symmetric spread. Let $M, N \in \mathcal{M}$. Then* $\mathrm{d}(M) = \mathrm{d}(N)$ *if and only if $M = N$.*

Proof. Let $\mathcal{O}$ be the line-oval of $\mathfrak{A}(\mathcal{M})$, as described in Theorem 11. If $\mathrm{d}(M) = \mathrm{d}(N)$ for some $M \ne N$, then the three lines of $\mathcal{O}$

$$X = O\,,\ Y = MX + \mathrm{d}(M) \text{ and } Y = NX + \mathrm{d}(N)$$

would have the common point $(O, \mathrm{d}(M))$, which is absurd, as $\mathcal{O}$ is a line-oval. $\boxed{QED}$

13 Corollary. *The map* $\mathrm{d}\colon \mathcal{M} \to K^n$*, restricted to* $\mathcal{M}$*, is bijective.*

14 Corollary. *Each element of* $\mathcal{M}$ *belongs to precisely one coset of* $\mathcal{S}/\mathcal{A}$*.*

15 Corollary. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be symmetric spread sets for symplectic spreads of the same symplectic space. If* $M \in \mathcal{M}$*,* $N \in \mathcal{N}$ *and* $\mathrm{d}(M) = \mathrm{d}(N)$*, then* $M - N \in \mathcal{A}$*.*

As a consequence, given a symmetric spread set $\mathcal{M}$, any other symmetric spread set can be constructed by adding to each matrix M of $\mathcal{M}$ a suitable skew-symmetric matrix A_M (depending on M). In particular, let $\mathcal{D}$ be a *desarguesian* symmetric spread set (the corresponding plane is desarguesian). Any other symmetric spread set $\mathcal{M}$ is of type

$$\mathcal{M} = \{D + A_D \mid D \in \mathcal{D}\,,\ A_D \in \mathcal{A}\}\,, \tag{11}$$

where the skew-symmetric matrices A_D have to be chosen so that $\mathcal{M}$ be really a spread set. Therefore the construction of any symplectic translation plane can begin with a desarguesian plane; then we "modify" some components of the desarguesian spread, so that we get a new symplectic spread. Clearly, this sort of construction requires a criterion in order to be sure that we are *effectively* constructing a spread. We call this construction a *modification process.*

2.4 The modification process

Examples of the modification process can be given using coordinatization of translation planes by quasifields. We treat the case of finite quasifield and finite translation planes. As a general reference see [7]. For a while we let q to be even or odd.

16 Definition. A *quasifield* Q is a nonempty finite set equipped with two binary operations: addition $+$ and multiplication $*$, such that $(Q, +)$ is an abelian group, whose identity is denoted by 0, and, for all $x, y, z \in Q$,

(i) there is a multiplicative identity, denoted by 1, that is, $x * 1 = 1 * x = x$;

(ii) $(x + y) * z = x * z + y * z$ (left distributivity);

(iii) $x * y = x * z \implies x = 0$ or $y = z$;

(iv) $x * y = 0 \iff x = 0$ or $y = 0$.

The *kernel* $K(Q)$ of Q consists of all the elements $k \in Q$ such that $k * (x * y) = (k * x) * y$ and $k * (x + y) = k * x + k * y$ for all $x, y \in Q$. The kernel $K(Q)$ is a field and Q is a left vector space over $K(Q)$.

If one does not require (i), then Q is called a *prequasifield.*

Any finite field is clearly a quasifield, coincident with its kernel. It can be proven that Q has order p^e, where p is a prime, and that it is not restrictive to assume that Q coincides (as a set) with the finite field $F = \mathrm{GF}(p^e)$ and that Q and F have the same addition. In other words, every finite quasifield can be obtained from a finite field F preserving the addition $+$ and defining a new multiplication $*$ related to the field addition by (i), (ii), (iii) and (iv) of the foregoing definition. With this identification the prime field of F and that of $K(Q)$ are the same. Moreover, $K(Q)$ is a subfield of F and F can be viewed as a $K(Q)$-vector space. We will follow this representation of quasifields, and the notation $Q = (F, +, *)$ will indicate that the quasifield Q has been constructed from the field F with a new moltiplication $*$. In the following the multiplication of the field F will be denoted by juxtaposition.

A translation plane $\mathfrak{A}(Q)$ is obtained from the quasifield $Q = (F, +, *)$, having $F \times F$ as point-set, and as line-set the following subsets of $F \times F$:

$$\{(a, y) \mid y \in F\} \text{ and } \{(x, x * m + b) \mid x \in F\} \text{ for all } a, b, m \in F\,.$$

As is usual, lines are described in terms of equations:

$$x = a \text{ and } y = x * m + b \text{ for all } a, b, m \in F\,.$$

The *order* of $\mathfrak{A}(Q)$ is $|F|$ and the spread $\Sigma = \Sigma(Q)$ defining $\mathfrak{A}(Q)$ consists of the lines through the origin 0:

$$\Sigma = \{x = 0\} \cup \{y = x * m\}_{m \in F}\,. \tag{12}$$

The points at infinity will be denoted by (∞), corresponding to the line $x = 0$, and (m), with $m \in F$, corresponding to the line $y = x * m$.

The map $P_m \colon F \to F$, where $m \in F^* = F \setminus \{0\}$, defined by $P_m(x) = x * m$ for all $x \in F$ is bijective (since F is finite and because of (iii) above) and is a $K(Q)-$linear map. The map P_0 is the zero map. For $k \in K(Q)^*$ the map P_k defines the collineation λ_k of $\mathfrak{A}(Q)$ letting $\lambda_k(x, y) = (k * x, k * y)$, for all $(x, y) \in F \times F$. The set $\{\lambda_k \mid k \in K(Q)\}$ constitutes the kernel of $\mathfrak{A}(Q)$.

In what follows it will sometimes be more convenient to work with prequasifields than quasifields. This is by no means restrictive, since by modifying the multiplication the prequasifield can be turned in a quasifield in such a way that the respective associated translation planes are the same.

Let $Q = (F, +, *)$ be a prequasifield. We assume that $F = \mathrm{GF}(q^n)$ is an algebraic extension of $K = \mathrm{GF}(q)$ of degree n, so that K is a subfield of F. In particular F is a K-vector space of dimension n. We also assume the following hypothesis:

(1) $K \subseteq K(Q)$; and

(2) $(kx) * y = k(x * y)$ for all $k \in K$ and $x \in F$.

In view of this hypothesis the map $P_m \colon x \mapsto x * m$ is also a K-linear map of the K-space F, for all $m \in F$.

Let $\mathrm{T} \colon F \to K$ be the trace map of the field F (relative to K):

$$\mathrm{T}(x) := \sum_{i=0}^{n-1} x^{q^i} \text{ for all } x \in F\,.$$

The map T is a K-linear map, which is onto. The map $\langle , \rangle \colon F \times F \to K$, such that, for all $x, y \in F$,

$$\langle x, y \rangle := \mathrm{T}(xy) \tag{13}$$

is a symmetric non-singular bilinear form on the K-vector space F, called an *inner product*. An endomorphism $S \colon F \to F$ is called *symmetric* if S coincides with its adjoint with respect to $\langle , \rangle$.

There is an orthonormal basis that let us identify F, equipped with this inner product, with K^n, equipped with its usual dot product (see [20, Theorem 5.2 and Corollary 5.3]).

17 Definition. A prequasifield $Q = (F, +, *)$ is called *symplectic* if the following condition holds for all $x, y, z \in F$:

$$\mathrm{T}(x(y * z)) = \mathrm{T}(y(x * z))\,. \tag{14}$$

The following proposition, whose proof is very easy, explains the use of the term "symplectic quasifield":

18 Proposition. *Equip $F \times F$ with the alternating bilinear form*

$$\beta((x_1, y_1), (x_2, y_2)) := \mathrm{T}(x_1 y_2 - x_2 y_1)\,. \tag{15}$$

Then the spread $\Sigma(Q)$ of $F \times F$ associated to the quasifield $Q = (F, +, *)$, as defined in (12), is symplectic if and only if Q satisfies condition (14). Moreover, if Q is symplectic, then each map P_m is a symmetric endomorphism of the $K-$space F, with respect to the inner product (13).

Now we return to the assumption q even, and note that our definition of symplectic quasifield is weaker than that of [16], in order to not distinguish between $[F : K] = n$ *odd* or *even*. In case n *odd* it can be proven [16, Proposition 3.10] that condition (14) is equivalent to

$$\mathrm{T}(x(x * s)) = \mathrm{T}(xs)^2\,. \tag{16}$$

We illustrate how to use the modification process in order to get a lot of examples.

Let $F = \mathrm{GF}(q^n) \supseteq K = \mathrm{GF}(q)$ (recall that now we are assuming q even). For each $s \in F$ define $D_s : F \to F$ by $D_s(x) = xs$, for all $x \in F$. This map is a symmetric endomorphism of F, as a K-vector space, that we call *multiplication by* s. It is represented in any orthonormal basis by a symmetric matrix. The set $\mathcal{D} = \{D_s \mid s \in F\}$ is a field isomorphic to the field F.

Let $Q = (F, +, *)$ be a symplectic prequasifield. Also the map

$$P_m : F \to F$$

given by $P_m(x) = x*m$ is a symmetric endomorphism of F. By the modification process, for each $s \in F$ there is $D_{(s)} \in \mathcal{D}$ and $A_{(s)} \in \mathcal{A}$ such that

$$P_s = D_{(s)} + A_{(s)}\,, \tag{17}$$

where we use the symbol $D_{(s)}$ to distinguish $D_{(s)}$ from the map $D_s : x \mapsto xs$, and the subscript indicates the dependence on s.

For every $s \in F^*$, the map $D_{(s)}$ belongs to $\mathcal{D}$, which is a field. So there is a unique $H_{(s)} \in \mathcal{D}$ such that

$$H_{(s)} D_s = D_{(s)}\,. \tag{18}$$

From this equation and (17) we get, for all $s \neq 0$,

$$P_s = H_{(s)} D_s + A_{(s)}\,. \tag{19}$$

In case $s = 0$, we put $P_0 = 0$, $D_{(0)} = 0$ and $A_{(0)} = 0$.

Since $x * s = P_s(x)$ and $xs = D_s(x)$, we obtain

$$x * s = P_s(x) = H_{(s)} D_s(x) + A_{(s)}(x) = H_{(s)}(xs) + A_{(s)}(x)\,. \tag{20}$$

In this expression, $H_{(s)}$ is an invertible (for $s \neq 0$) symmetric endomorphism, while $A_{(s)}$ is skew-symmetric. Thus we have the following necessary conditions: for all $s, x, y \in F$

$$\mathrm{T}(xH_{(s)}(y)) = \mathrm{T}(yH_{(s)}(x)) \tag{21}$$

$$\mathrm{T}(xA_{(s)}(x)) = 0 \tag{22}$$

We give a few examples of this construction.

Referring back to (20), all the examples will have $H_{(s)} = D_s$. So the prequasifields we will describe have multiplication $*$:

$$x * s = xs^2 + A_{(s)}(x)\,.$$

The problem is to find suitable skew-symmetric endomorphisms $A_{(s)}$. To this end, we need of the following observations.

Let $F = \mathrm{GF}(q^n) \supseteq F' = \mathrm{GF}(q^{n'}) \supseteq K = \mathrm{GF}(q)$ be fields with nn' odd and with corresponding trace maps $\mathrm{T}' : F \to F'$ and $\mathrm{T} : F \to K$.

19 Lemma. *If $z \in F$ and $u \in F'$, then*

(i) $\mathrm{T}(\mathrm{T}'(z)) = \mathrm{T}(z)$,

(ii) $\mathrm{T}(uz) = \mathrm{T}(u\mathrm{T}'(z))$, *and*

(iii) $\mathrm{T}'(u) = u$ *and* $\mathrm{T}(1) = 1$.

PROOF. See [17, Lemma 2.14]. $\boxed{QED}$

Following [17], the notation $(F_i)_0^n$, where n is **odd**, stands for a tower of fields

$$F = F_0 \supset F_1 \supset \cdots \supset F_n \supseteq K = \mathrm{GF}(q)\,,$$

with corresponding trace maps $\mathrm{T}_i\colon F \to F_i$. The following proposition is the source of a lot of examples.

20 Proposition. *Let $(F_i)_0^n$ be a tower of fields. Set $\lambda_0 = 1$; let $\lambda_i \in F_i^*$ and $\zeta_i \in F$ be arbitrary for $1 \le i \le n$; and for $0 \le i \le n$ write $c_i = \prod_{j=0}^{i} \lambda_j$. Define $Q = (F, +, *)$ by*

$$\begin{aligned} x * y = xy^2 &+ \sum_{i=1}^{n} [c_{i-1} y \mathrm{T}_i(c_{i-1}xy) + c_i y \mathrm{T}_i(c_i xy)] \\ &+ \sum_{i=1}^{n} [c_{i-1} y \mathrm{T}_i(x\zeta_i) + \zeta_i y \mathrm{T}_i(c_{i-1}xy)]\,. \end{aligned} \tag{23}$$

Then Q is a prequasifield coordinatizing a symplectic translation plane.

This is [18, Proposition 2.19]. The second and third summand of the above formula give the map $A_{(y)}$. Using Lemma 19, it is easy to verify that $A_{(y)}$ is skew-symmetric.

Different choices of elements λ_i and ζ_i in (23) produce different symplectic planes. Here are some examples.

(1) *Semifield planes.* If all λ_i are 1 and $\zeta_i \in F^*$, presemifields are obtained. These are extensively investigated in [15].

(2) *Nearly flag-transitive planes.* A translation plane is called *nearly flag-transitive* if admits a collineation group fixing two points on the line at infinity and transitive on the remaining ones. Referring to the multiplication as defined in (23), if all ζ_i are 0, then the plane coordinatized by the prequasifield admits the group of collineations $(x, y) \mapsto (s^{-1}x, sx)$, where $s \in F^*$, fixing two points on the line at infinity and cyclically permuting the remaining ones; this situation is investigated in [32].

3 Flag-transitive planes

In this section we illustrate a general construction of all known symplectic translation planes of even order admitting a flag-transitive group. We follow, with some minor modification, the construction given in [15, II]. We first show how symplectic subplanes of a symplectic plane can be constructed.

21 Theorem. *Let $\mathfrak{A}$ be a symplectic translation plane of even order q^{2n} admitting a Baer involution α. Then* $\mathrm{Fix}(\alpha)$*, the set of fixed points and fixed lines of α, is a Baer subplane $\mathfrak{A}_0$ which has order q^n and is still symplectic.*

Proof. Let $\mathcal{O}$ be a completely regular line-oval of $\mathfrak{A}$. If $\mathfrak{A}_0$ is the Baer subplane determined by α, then the fixed lines of $\mathcal{O}$ form a line-oval of $\mathfrak{A}_0$. We claim that $\mathcal{O}_0$ is completely regular. We directly apply the definition (see [24]). Let $(P) \in \ell_\infty$ be a point at infinity fixed by α and let x and y be two fixed lines through (P). Since $\mathcal{O}$ is completely regular there is a unique line z through (P) such that $\{x, y, z\}$ is a regular triple; this means that every line ℓ not through (P) meets x, y, z in points A, B, C of which at least one is in $B(\mathcal{O})$. We prove that z is a fixed line. In fact, $\{x, y, z\}$ is a regular triple if and only if $\{\alpha(x), \alpha(y), \alpha(z)\} = \{x, y, \alpha(z)\}$ is a regular triple. Since z, the third line of the triple, is unique, we get $\alpha(z) = z$. Therefore the sub-line-oval $\mathcal{O}_0$ is completely regular and so because of [26, Theorem 4.7] the plane $\mathfrak{A}_0$ is symplectic. $\boxed{QED}$

We now construct the following nearly flag-transitive symplectic translation plane of even order q^{2n}, where n is odd. Let $F = \mathrm{GF}(q^{2n})$ and $K = \mathrm{GF}(q^2)$. Let $k \in K \setminus \{0, 1\}$ such that $k^q = k$. Define on F the following multiplication $*$: for all $x, y \in F$

$$x * y := (1 + k)xy^2 + k \sum_{i=1}^{n-1} x^{q^{2i}} y^{q^{2i}+1} .$$

Referring to the preceding section

$$H_{(y)}(x) = (1 + k)xy^2 \text{ and } A_{(y)}(x) = k \sum_{i=1}^{n-1} x^{q^{2i}} y^{q^{2i}+1} .$$

While it is easy to verify that $H_{(y)}$ is a symmetric linear map, it is long and tedious to prove that $A_{(y)}$ is a skew-symmetric linear map (it is fundamental that n is odd), and that $Q = (F, +, *)$ is a prequasifield. Also, some calculations (or some minor modifications of the construction given in [15, II, section 6]) prove that the map $\alpha\colon F \times F$, such that $\alpha(x, y) = (y^{q^n}, x^{q^n})$ for all $x, y \in F$, is a Baer involution which interchanges the component $x = 0$ with $y = 0$. The plane $\mathfrak{A}(Q)$ is nearly flag-transitive. In fact, the map g_a, defined for every $a \in F^*$ by $g_a(x, y) = (ax, a^{-1}y)$ is a collineation fixing the lines $x = 0$ and $y = 0$. The group

$G = \{g_a \mid a \in F^*\}$ is cyclic and permutes the points at infinity distinct from (∞) and (0). The centralizer of α is the group $G_0 = C_G(\alpha) = \{g_a \in G \mid a^{q^{n+1}} = 1\}$. It is a cyclic group of order q^{n+1}. Because of Theorem 21 the subplane $\mathfrak{A}_0$ determined by the Baer involution α is symplectic and admits the flag-transitive group T_0G_0, where T_0 is the translation group of $\mathfrak{A}_0$.

The following are very natural questions, occurred during conversations with W. Kantor.

(1) Can the above construction be inverted? That is, does every symplectic flag-transitive plane of even order q^n and kernel $\mathrm{GF}(q)$, with n odd, arise in this way? A similar question is stated in [10].

(2) Can this construction be extended to symplectic flag-transitive planes of even order q^n and kernel containing $\mathrm{GF}(q)$, with n **even**?

4 The case of planes of order q^2

In this section we consider symplectic translation planes of order q^2, with kernel containing $K = \mathrm{GF}(q)$. For a while we let q to be even or odd. Let $\mathcal{M}$ be a symmetric spread set of order q^2 (see Remark 5, item (1)). Since $\mathcal{M}^* = \mathcal{M}\setminus\{O\}$ is a regular set, then given $(u, w) \in K^2 \setminus \{(0, 0)\}$ there is a unique matrix $M \in \mathcal{M}^*$ such that

$$M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}u\\w\end{pmatrix}.$$

Because of this bijection between $K^2 \setminus \{(0, 0)\}$ and $\mathcal{M}^*$, there exists some function $f\colon K^2 \to K$, such that

(1) $f(0, 0) = 0$; and

(2) $\mathcal{M} = \left\{\begin{pmatrix}u & w\\ w & f(u,w)\end{pmatrix} \mid u, v \in K\right\}$.

Using again the regularity of $\mathcal{M}^*$, for every $(a, b), (a', b')$ belonging to $K^2 \setminus \{(0, 0)\}$ there is a unique $M = \begin{pmatrix}u & w\\ w & f(u,w)\end{pmatrix} \in \mathcal{M}^*$, such that

$$\begin{pmatrix}u & w\\ w & f(u,w)\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a'\\b'\end{pmatrix}.$$

This leads to the system

$$\begin{cases} ua + wb = a' \\ wa + f(u,w)b = b' \end{cases}.$$

If $b = 0$ (thus $a \neq 0$), we get the unique solution $u = a'/a$, $w = b'/a$. If $b \neq 0$ (in this case a can be 0), we obtain the equation

$$-\frac{a^2}{b^2}u + f(u, -\frac{a}{b}u + \frac{a'}{b}) = \frac{b'}{b} - \frac{a'a}{b^2}$$

which admits a unique solution. This is the same as requiring that the function on K (letting $s = a/b$ and $t = a'/b$)

$$u \mapsto -s^2u + f(u, -su + t)$$

is a permutation for all $s, t \in K$. We have thus proved the following theorem.

22 Theorem. *Let $\mathcal{M}$ be a symmetric spread set of order q^2. Then there exists a function $f\colon K \times K \to K$, such that*

(1) $f(0, 0) = 0$;

(2) $\mathcal{M} = \left\{ \begin{pmatrix} u & w \\ w & f(u, w) \end{pmatrix} \mid u, v \in K \right\}$;

(3) for every $s, t \in K$, the function $\gamma_{s,t}\colon K \to K$, defined letting

$$\gamma_{s,t}(u) := -s^2u + f(u, -su + t)\,, \text{ for all } u \in K\,, \tag{24}$$

is a permutation of K.

Conversely, to every function $f\colon K \times K \to K$, such that (1) *and* (3) *hold, it is uniquely associated a symmetric spread set $\mathcal{M}$, defined as in* (2).

We call the function f the *associated function* of the spread set $\mathcal{M}$.

23 Remark. Putting $u = -x$, $s = a$ and $t = -b$, the permutation given in (24) is the same as that found, using different methods, in [1]. Note also that starting from the condition $M\binom{0}{1} = \binom{u}{w}$ one obtains the following representation for the symmetric spread set

$$\left\{ \begin{pmatrix} g(u, w) & u \\ u & w \end{pmatrix} \mid u, w \in K \right\} ,$$

where $g\colon K \times K \to K$ is a map such that $g(0, 0) = 0$ and the function $\rho_{s,t}\colon K \to K$,

$$\rho_{s,t}(w) = -s^2u + g(-sw + t, w)\,, \text{ for all } w \in K\,,$$

is a permutation for all $s, t \in K$.

24 Example. (see also [3]) The Kantor–Knuth spread set

$$\mathcal{K} = \left\{ \begin{pmatrix} w & au^\sigma \\ u & w \end{pmatrix} \mid u, w \in K \right\}$$

where $K = \mathrm{GF}(q)$, q is odd, a is a non-square, and $\sigma \in \mathrm{Aut}(K)$, defines a symplectic spread with respect to the symplectic form whose matrix is

$$\begin{pmatrix} O_2 & J \\ -J & O_2 \end{pmatrix},$$

where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Applying a basis change in K^2 (or by multiplying J with each matrix of $\mathcal{K}$) we get a symmetric spread set:

$$\mathcal{M} = J\mathcal{K} = \left\{ \begin{pmatrix} u & w \\ w & au^\sigma \end{pmatrix} \mid u, v \in K \right\}$$

which defines the same translation plane. In this example, the associated function is $f(u, w) = au^\sigma$. Moreover, this example proves that in the odd case there is, in general, no bijection between the set of diagonal vectors of the matrices of $\mathcal{M}$ and the set of vectors of K^2.

We investigate in which cases a symmetric spread set of order q^2 is desarguesian, to wit the corresponding plane is desarguesian. A very general and simple result is the following proposition.

25 Proposition. *A symmetric spread set $\mathcal{M}$ of order q^2 containing the identity matrix I is desarguesian if and only if its associated function f is of type $f(u, w) = u + tw$, for all $u, w \in K$, and some non-zero $t \in K$.*

Proof. In view of Proposition 6 it suffices to find which conditions are imposed on f when $\mathcal{M}$ consists of commuting matrices. We have

$$\begin{pmatrix} u & w \\ w & f(u,w) \end{pmatrix} \begin{pmatrix} \overline{u} & \overline{w} \\ \overline{w} & f(\overline{u},\overline{w}) \end{pmatrix} = \begin{pmatrix} \overline{u} & \overline{w} \\ \overline{w} & f(\overline{u},\overline{w}) \end{pmatrix} \begin{pmatrix} u & w \\ w & f(u,w) \end{pmatrix}$$

if and only if

$$u\overline{w} + wf(\overline{u},\overline{w}) = w\overline{u} + \overline{w}f(u,w)$$

for all $u, w, \overline{u}, \overline{w} \in K$. In particular, putting $\overline{w} = 1$ and $\overline{u} = 0$, we get

$$f(u, w) = u + f(0, 1)w$$

for all $u, v \in K$, so that f is of type required. The converse is a simple calculation. $\boxed{QED}$

Now we return to the even case. In this case, symmetric spread sets can be put in a more convenient form. Let $\mathcal{M}$ be a symmetric spread set. There is a bijection between the set of diagonal vectors (u, v) of the matrices of $\mathcal{M}$ and the set of vectors of $K \times K$ (see Corollary 13). Thus we can assume as variables the diagonal entries (u, v), so that the third entry w of each matrix of $\mathcal{M}$ must be some function of (u, v). Let us define $w := \varphi(u, v)$, where $\varphi\colon K \times K \to K$ is such that $\varphi(u, v) = 0$. Since $\mathcal{M}^*$ is a regular set, a similar reasoning as that in the proof of Theorem 22 gives that the function

$$u \mapsto su + \varphi(u, s^2u + t)\,, \tag{25}$$

as well

$$v \mapsto sv + \varphi(s^2v + t, v)\,, \tag{26}$$

are permutations of K for all $s, t \in K$. In the even case we call the map φ *the associated function* of the symmetric spread set $\mathcal{M}$. Moreover, we can assume that the symmetric spread set of order q^2 contains the identity matrix I. For if φ is the associated function, it is easy to verify that there are $a, b \in K^*$ such that $\varphi(1/a^2, 1/b^2) = 0$; put $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and consider the symmetric spread set $A\mathcal{M}A$. This spread set contains the identity matrix and defines a plane isomorphic to $\mathfrak{A}(\mathcal{M})$ (see Remark 5, item (2)).

26 Proposition. *A symmetric spread set $\mathcal{M}$ of order q^2 and containing the identity matrix I is a field if and only if φ is of type $\varphi(u, v) = h(u + v)$, for all $u, v \in K$, and some non-zero $h \in K$ of absolute trace* 1.

The proof is completely similar to that of Proposition 25.

4.1 Symmetric spread sets and ovoids

Another interesting fact we want to investigate is the close relation between symmetric spread sets of order q^2 and ovoids of $\mathrm{PG}(3, q)$. Represent the K-vector space of symmetric 2×2 matrices as the 3-dimensional affine space $\mathrm{AG}(3, q)$, by identifying the matrix $\begin{pmatrix} u & w \\ w & v \end{pmatrix}$ with the affine point (u, v, w). Consider the following "quadratic map"

$$\chi\colon\ \mathrm{AG}(3, q) \to \mathrm{AG}(3, q)$$

such that $\chi(x, y, z) = (xy + z^2, x, y)$. The spread set $\mathcal{M}$, represented by the "affine surface" $z = \varphi(x, y)$, is mapped onto the set of q^2 affine points of the ovoid $\Omega(\mathcal{M})$ of $\mathrm{PG}(3, q)$:

$$\Omega(\mathcal{M}) = \{(xy + \varphi(x, y)^2, x, 1, y) \mid x, y \in K\} \cup \{(1, 0, 0, 0)\}\,.$$

(In the projective space, point coordinates are assigned up to a nonzero scalar.) Desarguesian symmetric spread sets are sent by the map χ onto elliptic quadrics, and conversely. The use of such a quadratic map is not completely new. It has been used, for example, by Glynn in [8]. Anyway, it is an application of the Klein correspondence in a disguised form (see [25] for further details).

Ovoids of $\mathrm{PG}(3,q)$, and thus symplectic translation planes of order q^2 with kernel containing $\mathrm{GF}(q)$, are rare. Only two families are known: elliptic quadrics, to which there correspond desarguesian planes, and Tits ovoids, whose corresponding symplectic planes are the Lüneburg planes (see [21]). Therefore to have a criterion which allows us to establish when a plane (an ovoid) is desarguesian (an elliptic quadric) is very important. In view of these considerations we prove some results that provide such a criterion. As we will see, the key result we use is Brown's theorem [4], which states that *every ovoid of* $\mathrm{PG}(3,q)$*, q even, containing a conic section is an elliptic quadric.*

First we need some definitions. Let Σ be a symplectic spread of the symplectic space (V,β), where V has dimension 4 over $K=\mathrm{GF}(q)$, and let $\mathfrak{A}(\Sigma)$ be the corresponding symplectic translation plane of even order q^2. A *totally isotropic Baer subplane* of $\mathfrak{A}(\Sigma)$ is a Baer subplane whose set of points coincides with a totally isotropic 2-subspace. Such a Baer subplane has order q and is desarguesian. It is not difficult to prove that every totally isotropic 2-subspace, which is not a component of the spread, is indeed a totally isotropic Baer subplane, and that on each of them it is induced by a completely regular line-oval of the plane $\mathfrak{A}(\Sigma)$ a (not necessarily completely regular) line-oval. We investigate what happens when the induced line-oval is completely regular.

27 Theorem. *Let $\mathfrak{A}(\Sigma)$ be a symplectic translation plane of even order q^2, with kernel containing $K=\mathrm{GF}(q)$, and $\mathcal{O}$ a completely regular line-oval. Then $\mathfrak{A}(\Sigma)$ is desarguesian if and only if there is a totally isotropic Baer subplane $\mathfrak{A}_0$, such that the induced line-oval $\mathcal{O}_0=\mathfrak{A}_0\cap\mathcal{O}$ is completely regular.*

Proof. Let $\mathfrak{A}_0$ be a totally isotropic Baer subplane. Then $\mathfrak{A}_0$ meets nontrivially $q+1$ components of Σ. Pick two of them, say S_0 and S_∞, and fix symplectic coordinates $\binom{X}{Y}=(x_0,x_1,x_2,x_3)^t$, such that S_0 has equations $Y=O$, S_∞ has equations $X=O$, the symplectic form has matrix $\begin{pmatrix}O_2 & I_2\\ I_2 & O_2\end{pmatrix}$ and the plane admits a symmetric spread set $\mathcal{M}$ with associated function φ. Since $\mathfrak{A}_0$ contains the 1-dimensional subspaces $\mathfrak{A}_0\cap S_0$ and $\mathfrak{A}_0\cap S_\infty$, then $\mathfrak{A}_0$ is represented by equations

$$\begin{cases} sx_0+x_1=0\\ x_2+sx_3=0\end{cases},$$

for some $s \in K$, or else by equations

$$\begin{cases} x_0 + tx_1 = 0 \\ tx_2 + x_3 = 0 \end{cases},$$

for some $t \in K$. The two cases are similar. So we can assume for $\mathfrak{A}_0$ the first set of equations. Let

$$\mathcal{O} = \{X = O\} \cup \{Y = MX + \mathrm{d}(M) \mid M \in \mathcal{M}\}$$

be a completely regular line-oval of $\mathfrak{A}(\Sigma)$ (see Theorem 11). The spread induced on the subplane $\mathfrak{A}_0$ is

$$\Sigma_0 = \left\{ \begin{cases} x_0 = 0 \\ x_1 = 0 \\ x_2 + sx_3 = 0 \end{cases} \right\} \bigcup \left\{ \begin{cases} x_1 = sx_0 \\ x_2 = sx_3 \\ x_3 = (\varphi(s^2v, v) + sv)x_0 \end{cases}, \ v \in K \right\}$$

and the line-oval induced on $\mathfrak{A}_0$ is

$$\mathcal{O}_0 = \left\{ \begin{cases} x_0 = 0 \\ x_1 = 0 \\ x_2 + sx_3 = 0 \end{cases} \right\} \bigcup \left\{ \begin{cases} x_1 = sx_0 \\ x_2 = sx_3 \\ x_3 = (\varphi(s^2v, v) + sv)x_0 + \sqrt{v} \end{cases}, \ v \in K \right\}.$$

The subplane $\mathfrak{A}_0$ is desarguesian. Assume that the induced line-oval $\mathcal{O}_0$ is completely regular. By [22, Corollary 1] $\mathcal{O}_0$ is a line-conic (its dual is a conic). Using the linear map

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s & 1 & 0 & 0 \\ 0 & 0 & 1 & s \end{pmatrix}$$

the affine plane $\mathfrak{A}_0$ is mapped onto the subspace $x_2 = 0\,, x_3 = 0$, and thus is isomorphic to the desarguesian affine plane $\mathrm{AG}(2, q)$, with affine coordinates (x, y). The line-conic $\mathcal{O}_0$ is equivalent to the following line-conic of $\mathrm{AG}(2, q)$:

$$\{x = 0\} \cup \{y = (sv + \varphi(s^2v, v))x + \sqrt{v} \mid v \in K\}\,.$$

Since the map $v \mapsto sv + \varphi(s^2v, v)$ is a permutation (see equation (26)), it follows that $s + \varphi(s^2, 1) \neq 0$. The above line-conic is affinely equivalent to the line-conic

$$\mathcal{O}_0' = \{x = 0\} \cup \left\{ y = \left(\frac{sv + \varphi(s^2v, v)}{s + \varphi(s^2, 1)} \right) x + \sqrt{v} \mid v \in K \right\}.$$

Passing to homogeneous coordinates (z_0, z_1, z_2), such that $x = z_1/z_0$ and $y = z_2/z_0$, the line-conic becomes

$$\mathcal{O}'_0 = \{z_1 = 0\} \cup \left\{ (\sqrt{v})\, z_0 + \left(\frac{sv + \varphi(s^2 v, v)}{s + \varphi(s^2, 1)} \right) z_1 + z_2 = 0 \mid v \in K \right\} .$$

By duality we get the conic

$$\mathcal{L} = \{(0,1,0)\} \cup \left\{ \left(\sqrt{v}\, , \frac{sv + \varphi(s^2 v, v)}{s + \varphi(s^2, 1)}\, , 1 \right) \mid v \in K \right\} .$$

Finally, by suitably interchanging the coordinates and putting $t = \sqrt{v}$, we get the conic

$$\mathcal{C} = \{(0,0,1)\} \cup \left\{ \left(1\, , t\, , \frac{st^2 + \varphi(s^2 t^2, t^2)}{s + \varphi(s^2, 1)} \right) \mid t \in K \right\} .$$

This conic $\mathcal{C}$ has as nucleus the point $(0,1,0)$ and contains the points $(0,0,1)$, $(1,0,0)$ and $(1,1,1)$. Therefore (see [11]) it must be in its standard form, which is of type $\{(0,0,1\} \cup \{(1,t,t^2) \mid t \in K\}$. Thus, in our case,

$$\frac{st^2 + \varphi(s^2 t^2, t^2)}{s + \varphi(s^2, 1)} = t^2\, , \text{ for all } t \in K\, .$$

Hence, $\varphi(s^2 t^2, t^2) = \varphi(s^2, 1) t^2$, for every $t \in K$, or in terms of the old variable v,

$$\varphi(s^2 v, v) = \varphi(s^2, 1) v\, , \text{ for all } v \in K\, .$$

Therefore the spread set $\mathcal{M}$ contains the following subset of size q:

$$\left\{ \begin{pmatrix} s^2 v & \varphi(s^2, 1) v \\ \varphi(s^2, 1) v & v \end{pmatrix} \mid v \in K \right\} ,$$

which is mapped by the quadratic map χ onto the set

$$\{(s^2 + \varphi(s^2, 1)^2) v^2, s^2 v, 1, v) \mid v \in K\}$$

of q affine points of the associated ovoid $\Omega(\mathcal{M})$. These points, together with the point at infinity $(1,0,0,0)$, constitute a conic, which is the section of the ovoid by the plane $x_1 = s^2 x_3$. By Brown's theorem [4] the ovoid is an elliptic quadric, and thus $\mathfrak{A}(\mathcal{M})$ is a desarguesian plane.

The converse is immediate. $\boxed{QED}$

28 Remark. In a weaker form the same theorem is proved in [25, Theorem 3.8], where the Klein correspondence and Brown's theorem are the key tools for the proof. In the Klein correspondence every regulus of $\mathrm{PG}(3,q)$ corresponds with a conic section of the Klein's quadric (see [12]), and a regulus contained in a symplectic spread corresponds with a conic section of the associated ovoid. Therefore

29 Corollary. *A symplectic plane of even order q^2, with kernel containing $K = \mathrm{GF}(q)$, is desarguesian, if and only if a symplectic spread defining it contains a regulus.*

This corollary is the main result of [28].

We list some interesting consequences.

30 Corollary. *Let $\mathcal{M}$ be a symmetric spread set of even order q^2 with associated function φ. Then $\mathcal{M}$ is a field of order q^2 if and only if, for some $s \in K$, $\varphi(s^2 v, v) = \varphi(s^2, 1)v$, for all $v \in K$.*

31 Corollary. *Let $\mathfrak{A}(\Sigma)$ be a symplectic translation plane of even order q^2. Then $\mathfrak{A}(\Sigma)$ admits a totally isotropic Baer subplane fixed by a Baer involution if and only if $\mathfrak{A}$ is desarguesian.*

Proof. It follows from Theorem 21 and the above theorem. $\boxed{QED}$

32 Remark. This corollary is a different, but equivalent, formulation of a theorem of Brown [5].

33 Corollary. *Let $\mathcal{M}$ be a symmetric spread set of even order q^2 with associated function φ. Then $\mathcal{M}$ is a field if and only if $\varphi(u,v) = \varphi(v,u)$ for all $u, v \in K$.*

Proof. It suffices to note that the map from K^4 to K^4

$$(x_0, x_1, x_2, x_3) \mapsto (x_1, x_0, x_3, x_2)$$

is a Baer involution, fixing the totally isotropic 2-subspace $x_0 = x_1$, $x_2 = x_3$, if and only if $\varphi(u,v) = \varphi(v,u)$ for all $u, v \in K$. $\boxed{QED}$

In [14] Johnson and Vega give a non-existence condition for symplectic translation plane of order q^n, proving that any such a plane can only admit affine homologies of order dividing $q - 1$. We have the following theorem.

34 Theorem. *Let $\mathfrak{A}$ be a symplectic translation plane of even order q^2, with kernel containing $K = \mathrm{GF}(q)$. If $\mathfrak{A}$ admits an affine homology of order $q - 1$, then the plane is desarguesian.*

Proof. Fix symplectic coordinates. Then the plane is represented by a symmetric spread set $\mathcal{M}$ with associated function φ. We can assume that $\mathcal{M}$

contains the identity matrix I and that the affine homology has axis $Y = O$ and co-axis $X = O$. Therefore the homology is represented by the 4×4 matrix

$$\begin{pmatrix} I_2 & O_2 \\ O_2 & A \end{pmatrix},$$

where A is a 2×2 matrix such that $AM \in \mathcal{M}$ for all $M \in \mathcal{M}$. In particular then $A \in \mathcal{M}$, as $I \in \mathcal{M}$, and AM is a symmetric matrix. Since A and M are symmetric, the matrix AM is symmetric if and only if A and M commute. If

$$A = \begin{pmatrix} a & \varphi(a,b) \\ \varphi(a,b) & b \end{pmatrix} \text{ and } M = \begin{pmatrix} u & \varphi(u,v) \\ \varphi(u,v) & v \end{pmatrix},$$

then A and M commute if and only if

$$a\varphi(u,v) + v\varphi(a,b) = u\varphi(a,b) + b\varphi(u,v)$$

for all $u, v \in K$. Thus

$$(a+b)\varphi(u,v) = \varphi(a,b)(u+v)\,. \tag{27}$$

There are then two possibilities: either

(1) $a \neq b$; or

(2) $a = b$ and $\varphi(a,b) = 0$.

In case (1) we get

$$\varphi(u,v) = \frac{\varphi(a,b)}{a+b}(u+v)\,, \text{ for all } u, v \in K\,.$$

Thus φ is of type $h(u+v)$, and so $\mathcal{M}$ is desarguesian, because of Proposition 26.

In case (2), $\mathcal{M}$ contains a scalar matrix $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ of order $q-1$. Therefore the associate function of $\mathcal{M}$ satisfies $\varphi(v,v) = 0$, and so $\mathcal{M}$ is desarguesian, because of Corollary 30 when $s = 1$. $\boxed{QED}$

Finally, we prove a well known fact about symplectic semifield planes. May be also the proof we give is known, but we are unable to find a reference.

35 Theorem. *Let $\mathfrak{A}$ be a symplectic semifield plane of even order q^2 with kernel containing* $\mathrm{GF}(q)$. *Then $\mathfrak{A}$ is desarguesian.*

PROOF. Since $\mathfrak{A}$ is a semifield plane, it admits a group G of order q^2, consisting of shears with axis a component, say S_∞, of the defining spread Σ. Fix symplectic coordinates $\binom{X}{Y}$, in such a way that S_∞ is $X = O$ and that the symplectic form has matrix $\begin{pmatrix} O_2 & I_2 \\ I_2 & O_2 \end{pmatrix}$. Let $\mathcal{M}$ be a symplectic spread set such that $\mathfrak{A} = \mathfrak{A}(\mathcal{M})$. We can assume that $\mathcal{M}$ contains the identity matrix I. The group G consists of linear maps and is represented by 4×4 matrices of type

$$\begin{pmatrix} I_2 & O_2 \\ M & I_2 \end{pmatrix}, \text{ where } M \in \mathcal{M}.$$

Therefore $\mathcal{M}$ is an additive group of order q^2 and if φ is its associated function, such that $\varphi(1,1) = 0$, then

$$\varphi(u,v) = \varphi(u,0) + \varphi(0,v) = H(u) + L(v),$$

where H and L are additive maps on K. Using the quadratic map χ, we get that the associated ovoid contains the following ovals, which are the section by the planes $x_3 = 0$ and $x_1 = 0$, respectively,

$$\{(H(u)^2, u, 1, 0) \mid u \in K\} \cup \{(1,0,0,0)\}$$

and

$$\{(L(v)^2, 0, 1, v) \mid v \in K\} \cup \{(1,0,0,0)\},$$

which are respectively projectively equivalent to the ovals of $\mathrm{PG}(2,q)$

$$\mathcal{T}_1 \colon \{(1, u, H(u)^2) \mid u \in K\} \cup \{(0,0,1)\}$$

and

$$\mathcal{T}_2 \colon \{(1, v, L(v)^2) \mid v \in K\} \cup \{(0,0,1)\}.$$

These ovals are translation ovals, since the maps $u \mapsto H(u)^2$ and $v \mapsto L(v)^2$ are additive. Since both ovals have as nucleus the point $(0,1,0)$ and as an axis the line $x_0 = 0$, by [11]

$$H(u)^2 = a^2 u^{2^n}, \quad \text{and} \quad L(v)^2 = b^2 v^{2^m}$$

for some $a, b \in K^*$ and some positive integers n and m such that $(n, q-1) = (m, q-1) = 1$. The subset of $\mathcal{M}$, obtained putting $u = v$,

$$\left\{ \begin{pmatrix} u & au^{2^{n-1}} + bu^{2^{m-1}} \\ au^{2^{n-1}} + bu^{2^{m-1}} & u \end{pmatrix} \mid u \in K \right\}$$

is mapped by χ onto the affine points of the following oval, which is the section of $\Omega(\mathcal{M})$ by the plane $x_1 = x_3$,

$$\{(u^2 + a^2u^{2^n} + b^2u^{2^m}, u, 1, u) \mid u \in K\} \cup \{(1, 0, 0, 0)\}\,.$$

This oval is equivalent to the oval

$$\{(1, u, a^2u^{2^n} + b^2u^{2^m})\} \cup \{(0, 0, 1)\}$$

of $\mathrm{PG}(2, q)$ havig as nucleus the point $(0, 1, 0)$. It is a translation oval, since the map $u \mapsto a^2u^{2^n} + b^2u^{2^m}$ is additive. Again by [11], necessarily $n = m$. So

$$\varphi(u, u) = (a + b)u^{2^{n-1}} \text{ for all } u \in K\,.$$

As $\varphi(1, 1) = 0$, so $a = b$. Therefore $\varphi(u, u) = 0$ for all $u \in K$, and so $\mathcal{M}$ is a field, because of Corollary 30, and the plane is desarguesian. $\boxed{QED}$

In view of the previous results that emphasize the role played by Brown's theorem, we state the following problem, which must be compared with a similar problem posed in [25, Section 3, problem 3.3]:

Problem. Prove that Theorem 27 implies Brown's theorem on conic sections.

We think that such a proof would provide new insights into the theory of symplectic translation planes of even order q^2.

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