On the number of k-gons in finite projective planes

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Abstract. Let $\pi = \pi_q$ denote a finite projective plane of order q, and let $G = Levi(\pi)$ be the bipartite point-line incidence graph of π. For $k \geq 3$, let $c_{2k}(\pi)$ denote the number of cycles of length $2k$ in G. Are the numbers $c_{2k}(\pi)$ the same for all π_q? We prove that this is the case for $k = 3, 4, 5, 6$ by computing these numbers.

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1 Introduction

Our work concerns the number of certain substructures in finite projective planes. We will need the following definitions and notations. Any graph-theoretic notion not defined here may be found in Bollobás [2]. All of our graphs are finite, simple and undirected. If $G = (V, E) = (V(G), E(G))$ is a graph, then the *order* of G is $|V|$, the number of vertices of G, and the *size* of G is $|E|$, the number of edges in G. Each edge of G is thought as a 2-subset of V. An edge $\{x, y\}$ will be denoted by xy or yx. A vertex v is *incident* with an edge e if $v \in e$.

We say that a graph $G' = (V', E')$ is a *subgraph of* G, and denote it by $G' \subset G$, if and only if $V' \subset V$ and $E' \subset E$. For a vertex $v \in V$, $N(v) = N_G(v) =$

$\{u \in G : uv \in E\}$ denotes the *neighborhood* of v, and $d(v) = d_G(v) = |N(v)|$ – the *degree* of v. If the degrees of all vertices of G are equal d, G is called *d-regular*. For a graph F, we say that G is *F-free* if G contains no subgraph isomorphic to F.

For $k \geq 3$, we define a *k-cycle* as the graph with a vertex-set $\{x_1, \ldots, x_k\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\}$. Any subgraph of G isomorphic to a k-cycle is called a *k-cycle in G*. The number of k-cycles in G is denoted by $c_k(G)$. The *girth* of a graph G containing cycles, denoted by $g = g(G)$, is the length of a shortest cycle in G. Let A and B be two disjoint nonempty subsets of vertices of a graph G whose union is $V(G)$, and let every edge from $E = E(G)$ have one endpoint in A and another in B. Then G is called *bipartite* and we denote it by $G(A, B; E)$. When $|A| = |B| = n$, we refer to G as an n by n bipartite graph.

All notions of incidence geometry not defined here may be found in [8]. A *partial plane* $\pi = (\mathcal{P}, \mathcal{L}; I)$ is an incidence structure with a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and a symmetric binary relation of incidence $I \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ such that any two distinct points are on at most one line, and every line contains at least two points. The definition implies that any two lines share at most one point. We will often identify lines with the sets of points on them. We say that a partial plane is a *subplane* of π, denoted $\pi' \subset \pi$, if and only if $\mathcal{P}' \subset \mathcal{P}, \mathcal{L}' \subset \mathcal{L}$, and $I' \subset I$. If there is a line containing two distinct points X and Y, we denote it by XY or YX. For $k \geq 3$, we define a *k-gon* as a partial plane with k distinct points $\{P_1, P_2, \ldots P_k\}$, with k distinct lines $\{P_1P_2, P_2P_3, \ldots, P_{k-1}P_k, P_kP_1\}$, and with point and line being incident if and only if the point is on the line. A subplane of π isomorphic to a k-gon is called a *k-gon in π*. Let $c_k(\pi)$ denote the number of k-gons in π. The *Levi graph* of a partial plane π is its point-line bipartite incidence graph $Levi(\pi) = Levi(\mathcal{P}, \mathcal{L}; E)$, where $Pl \in E$ if and only if point P is on line l. The Levi graph of any partial plane is 4-cycle-free. Clearly, there exists a bijection between the set of all k-gons in π and the set of $2k$-cycles in $Levi(\pi)$. Hence, $c_k(\pi) = c_{2k}(Levi(\pi))$.

A *projective plane of order* $q \geq 2$, denoted π_q, is a partial plane with every point on exactly $q+1$ lines, every line containing exactly $q+1$ points, and having four points such that no three of them are collinear. It is easy to argue that π_q contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Let $n_q = q^2 + q + 1$. It is easy to show that an equivalent definition of π_q in terms of bipartite graphs, is a partial plane whose Levi graph is a $(q + 1)$-regular graph of girth 6 and diameter 3. Projective planes π_q are known to exist only when the order is a prime power q. If $q \geq 9$ is a prime power but not a prime, there are always non-isomorphic planes of order q, and their number grows fast with q.

Let H and F be two graphs, and let $\mathcal{F}$ be a family of F-free graphs. For

every $G \in \mathcal{F}$, one can count the number of isomorphic copies of a graph H in G, and then try to maximize (or minimize) this number over all graphs in $\mathcal{F}$. Such optimization problems have been investigated at length by numerous researchers. The Turán-type problems are the most familiar instances of these problems, where $H = K_2$, and they are discussed in detail in the book by Bollobás [1], and surveys Füredi [15], Simonovits [23], Bondy [4]. For problems of other types, see, e.g., Fisher [14], Győri, Pach and Simonovits [17], Fiorini and Lazebnik [12,13], De Winter, Lazebnik and Verstraëte [9]. To mention another example, Erdős [11] conjectured that the maximum number of cycles of length five in an n-vertex triangle-free graph is achieved by the blowup of a pentagon (see Győri [16], Bollobás and Győri [3] for details).

Several of these problems lead to $Levi(\pi_q)$'s as extremal objects. For every $q \geq 2$, let $\mathcal{F}_q$ be the family of all 4-cycle free n_q by n_q bipartite graphs. If π_q exists, then $Levi(\pi_q) \in \mathcal{F}_q$. Then every $Levi(\pi_q)$ has the greatest number of edges among all graphs in $\mathcal{F}$, as was shown by Reiman [22] (or in [1]). Also every $Levi(\pi_q)$ has the greatest number of 6-cycles among all graphs in $\mathcal{F}$, and, if $q \geq$ 157 – the greatest number of 8-cycles (see [13] and [9], respectively). In all these results, *every* extremal graph must be $Levi(\pi_q)$ for *every* (existing) π_q. Using the geometric terminology, the latter two results give extremal characterizations of projective planes as the partial planes on n_q points with a maximum number of triangles (3-gons), or with the maximum number of quadrilaterals (4-gons). This implies, in particular, that the numbers $c_3(\pi_q)$ and $c_4(\pi_q)$ are the same for all projective planes of order q.

This paper was motivated by the following questions.

Q1: Assuming q is large compared to k, which graphs from $\mathcal{F}_q$ contain the greatest number of $2k$-cycles, or equivalently, which partial planes on n_q points have the greatest number of k-gons?

Q2: Do all projective planes of order q contain the same number of $2k$-cycles?

Unfortunately, we do not have any new results concerning **Q1.** It was conjectured in [9], that the classical (Desarguesian) projective plane $PG(2,q)$, or its Levi graph, is the only extremal object. We believe that the conjecture is true.

With respect to **Q2**, we found closed form expressions for the numbers of pentagons (5-gons) and hexagons (6-gons) in every π_q using only synthetic techniques. Hence, these numbers are the same for all π_q. The results are presented in the following theorem, with those for the numbers of triangles and quadrilaterals included for completeness.

1 Theorem. Let q be a prime power, and let $n_q = q^2 + q + 1$. Then for every projective plane π_q of order $q \geq 2$ the following holds.

(i) $c_3(\pi_q) = \frac{1}{6}q^2 n_q(n_q - 1)$.

(ii) $c_4(\pi_q) = \frac{1}{8}q^2 n_q(n_q - 1)(q - 1)^2$.

(iii) $c_5(\pi_q) = \frac{1}{10}q^2 n_q(n_q - 1)(q - 1)^2(n_q - q)$.

(iv) $c_6(\pi_q) = \frac{1}{12}q^2 n_q(n_q - 1)(q - 1)^2(q + 2)(q^3 - 2q^2 - q + 3)$.

A question similar to **Q2**, concerning the number of closed walks of length k in π_q, can be answered affirmatively. Let us explain this. Consider the Levi graph G of π_q. An $x_0 - x_k$ walk W in a graph is an alternating sequence of vertices and edges, say $x_0, e_1, x_1, e_2, \ldots, e_k, x_k$ where $e_i = x_{i-1}x_i, 1 \leq i \leq k$, where neither vertices nor edges are necessarily distinct. The length of the walk is its number of edges. We say that a walk $W = x_0 x_1 \ldots x_k$ is *closed* if $x_0 = x_k$. Suppose $V(G) = \{1, 2, \ldots, v\}$. Let $A = A(G) = (a_{ij})$ be the adjacency matrix of G: a $v \times v$ matrix where rows and columns of G are labeled by the vertices of G, with $a_{ij} = 1$ if and only if vertices i and j are adjacent, and $a_{ij} = 0$ otherwise. Let $A^l = (a_{ij}^{(l)})$. Then $a_{ij}^{(l)}$ is equal to the number of $i - j$ walks in G. Therefore $\text{trace}(A^l)$ is equal to the number of all closed walks in G of length l. The trace of a square matrix is equal to the sum of all its eigenvalues. The eigenvalues of G are well-known, and are the same for all π_q. Therefore $\text{trace}(A^l)$ is the same for all π_q as the sum of the lth powers of these eigenvalues. This implies that the number of closed walks of length k in π_q is the same for all π_q. The details can be found in [2] and Biggs [6].

In the next sections we provide proofs of our main results. Section 2 gives the synthetic proof of Theorem 1, and Section 3 gives some general results about embedded cycles. The last section gives some final thoughts for future work.

2   Proof of Theorem 1

The number of n-cycles in a projective plane of order q can be counted using elementary techniques and using only synthetic properties. Recall that $n_q = q^2 + q + 1$ is the number of points in a projective plane $\pi = \pi_q$. We wish to count the number of k-gons in π for small values of k. This can be achieved as follows. We count the number of (ordered) k-tuples of points, say $[A_1, A_2, \ldots, A_k]$, where each k-tuple represents the points around a k-gon, where "around" means that $A_1 A_2, A_2 A_3, \ldots, A_{k-1}A_k, A_k A_1$ are the lines of the k-gon. This puts some collinearity restrictions on the points of the k-tuple. For instance, any three consecutive point around a k-gon are necessarily non-collinear.

For completeness, we begin with the number of 3-gons and 4-gons in a projective plane π_q, and we will see that these numbers are easy to count. For

5-gons and 6-gons, we will frequently consider when points lie (or do not lie) on one of three non-concurrent lines, a triangle. We say that a point X is *on* $\triangle ABC$, if points A,B, C are not collinear, and X is a point on a line AB, or BC, or CA.

1 Lemma. *For every prime power q, and every projective plane π_q of order q, $c_3(\pi_q) = \frac{1}{6}q^2 n_q(n_q - 1)$.*

PROOF. Any triangle of π is determined by 3 non-collinear points. The number of ways to choose 3 points of π is $\binom{n_q}{3}$ and the number of ways to choose three collinear points of π is $n_q\binom{q+1}{3}$. Hence, the total number of triangles in π is $\binom{n_q}{3} - n_q\binom{q+1}{3} = \frac{1}{6}q^2 n_q(n_q - 1)$. $\boxed{QED}$

2 Lemma. *For every prime power q, and every projective plane π_q of order q, $c_4(\pi_q) = \frac{1}{8}q^2 n_q(n_q - 1)(q - 1)^2$.*

PROOF. We count the number of ways to choose an ordered 4-tuple $[A, B, C, D]$ and then divide by eight for automorphisms of the 4-gon. The first three points can be chosen easily. There are n_q choices for A, $n_q - 1$ choices for B, and $n_q - (q + 1)$ choices for C since C cannot lie on the line AB. The last point, D, cannot be collinear with AB, AC or BC. The number of points not on $\triangle ABC$ is $n_q - 3q$. Hence, there are $n_q(n_q - 1)(n_q - q - 1)(n_q - 3q)$ such ordered 4-tuples. Simplifying and dividing by eight gives us the desired formula. $\boxed{QED}$

3 Lemma. *For every prime power q, and every projective plane π_q of order q, $c_5(\pi_q) = \frac{1}{10}n_q(n_q - 1)(n_q - q)(q^2 - q)^2$.*

PROOF. We count pentagons by first counting the number of 5-tuples $[A, B, C, D, E]$ of points of π satisfying the conditions. We do this count in five steps. First note that there are n_q choices for A, $n_q - 1$ choices for B (since B is distinct from A), and $n_q - (q + 1)$ choices for C (since C is not on the line AB). We now choose D and E in two steps.

(1) **Choose D.** Clearly D cannot be on the line BC. However, D can be on either of the lines AB or AC. We have two disjoint cases.

    a. **The point D is on $\triangle ABC$.** In this case, D can be any one of the points on AB or AC, except for A, B or C. There are $2(q - 1)$ such points.

    b. **The point D is not on $\triangle ABC$.** There are $q + 1$ points on AB, q more points on BC (not counting B which was already eliminated) and $q - 1$ more points on BC (not counting B or C), for a total of $3q$ points on the $\triangle ABC$. Hence, there are $n_q - 3q$ choices for D.

(2) **Choose E.** We have 2 cases depending on how D was chosen above.

a. **Suppose D is on $\triangle ABC$.** Then, by the non-collinearity conditions, E cannot lie on AB, AD or CD. But since D is on the triangle, either $AD = AB$ or $AD = CD$, as pictured.
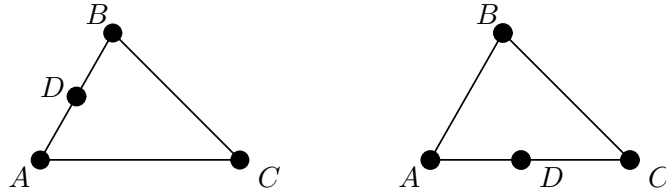
Hence, E can be any point in π except for those lying on two lines. There are $2q + 1$ points covered by these lines. Hence, there are $n_q - 2q - 1$ choices for E.

b. **Suppose D is not on $\triangle ABC$.** In this case, E again cannot be on either AB, AC or AD. But these three lines are distinct and cover $3q$ points. Hence, there are $n_q - 3q$ choices for E.

In summary, we have a total of

$$n_q(n_q - 1)(n_q - q - 1)\left[(n_q - 3q)(n_q - 3q) + 2(q - 1)(n_q - 2q - 1)\right]$$

different ordered 5-tuples. Substituting for n_q, dividing by 10, and simplifying gives us the desired formula. $\boxed{QED}$

Counting the number of 6-gons in π_q is naturally a more involved problem. The techniques used above can adapt to this more complicated situation, but care must be taken to cover all possible cases. It would be nice to find a more elegant technique for performing this count. However, we emphasize that our method is completely synthetic and therefore does not rely on anything more than the order of the plane.

4 Lemma. *For every prime power q, and every projective plane π_q of order q, $c_6(\pi_q) = \frac{1}{12} n_q (n_q - 1)(q^2 - q)^2(q + 2)(q^3 - 2q^2 - q + 3)$.*

PROOF. Again, we count ordered 6-tuples of points $[A, B, C, D, E, F]$ where any three consecutive points are non-collinear and then divide by 12, the number of automorphisms any given tuple. In addition, we need the lines of the 6-gon to be distinct. The non-collinearity condition for consecutive points around the k-gon take care of this for 5-gons. However, for 6-gons, we must require that the lines AB and DE are distinct, or equivalently, the points A, B, D and E are non-collinear. Similarly, we require $\{B, C, E, F\}$ and $\{C, D, F, A\}$ to be sets of non-collinear points. We will see that this creates many cases that need to be considered separately.

The first three points of our 6-tuple can be chosen among all ordered pairs of non-collinear points. Hence, we have n_q choices for A, $n_q - 1$ choices for B, and $n_q - (q + 1)$ choices for C. To continue, we consider three different cases. Either D is on AB, D is on AC, or D is not on $\triangle ABC$.

(1) **D is on AB**

In this case, there are $q - 1$ choices for D. To count the number of choices for E and F, we again break it into cases. Note that E cannot lie on AB (as this would force A, B, E, and F to be collinear), and F cannot lie on AB. Furthermore, E and F cannot both lie BC or AB because of the conditions. Hence, we have the following cases based on the locations of E and F.

    a. **E is on AC and F is on BC.** Here we have $(q - 1)$ choices for E since E is any point on AC except for A or C. The point F is now any point on BC except for B, C or the point of intersection between BC and DE (since D, E and F cannot be collinear). Hence, there are $q - 2$ choices for F. So we have $(q - 1)(q - 2)$ choices for E and F in this case.
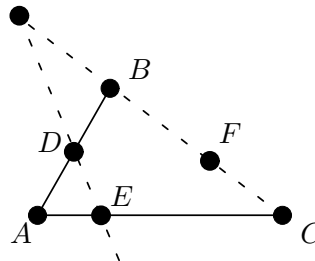

    b. **E is on BC and F is on AC.** This case is very similar to case 1 above and gives the same final count of $(q - 1)(q - 2)$.

    c. **E is on BC and F is not on $\triangle ABC$.** In this case, we again have $(q - 1)$ choices for E. The point F now cannot lie on neither the triangle $\triangle ABC$ by the case assumption, nor on either of the lines AE or DE (since no three consecutive points of the 6-gon are collinear).
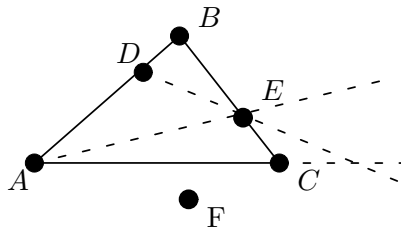
There are $n_q - 3q$ points off $\triangle ABC$. Of these points, $q - 1$ are on the line AE and $q - 2$ are on the line DE. Note that for DE, we must eliminate the points D, E, and the intersection point with the line AC. In summary, we have $q - 1$ choices for E and $n_q - 3q - (q-1) - (q-2)$ choices for F giving us a total of $(q-1)(q^2 - 4q + 4)$ choices for E and F in this case.

d. **E is on AC and F is not on $\triangle ABC$.** Here we have $q - 1$ choices for E. The last point F is not on the triangle $\triangle ABC$ and is also not on the line DE. The line DE meets the triangle in 3 points. Hence, we have $n_q - 3q - (q - 2)$ choices for F. This gives us a total of $(q-1)(q^2 - 3q + 3)$ choices for E and F in this case.

e. **F is on AC and E is not on $\triangle ABC$.** This case is the same as case (c) above and gives us the same count of $(q-1)(q^2 - 4q + 4)$ choices for E and F.

f. **F is on BC and E is not on $\triangle ABC$.** For this case, we count the number of choices for F first. If F is on BC, there are $q - 1$ choices for F. Now, the point E cannot be on the triangle $\triangle ABC$, but also cannot be on the lines AF, CD or DF. The picture is the same as in case (c) above except that E and F are interchanged. There are $q - 1$ points on AF off $\triangle ABC$, and additional $q - 2$ points on DF off $\triangle ABC$ (D, F, and the intersection with AC) and $q - 2$ points on CD off $\triangle ABC$ (C, D, and the intersection with BC) for a total of $n_q - 3q - (q-1) - (q-2) - (q-2) = q^2 - 5q + 6$ choices for F. Hence, we have $(q-1)(q^2 - 5q + 6)$ choices for E and F in this case.

g. **Neither E nor F is on $\triangle ABC$.** Our final case is more difficult to count. First note that E can be any point off $\triangle ABC$ and not on the line CD. This gives us $n_q - 3q - (q-1)$ choices for E, as pictured on the left. Now, the point F is not on the triangle $\triangle ABC$, and is also not on any of the lines AE or DE, as pictured on the right.
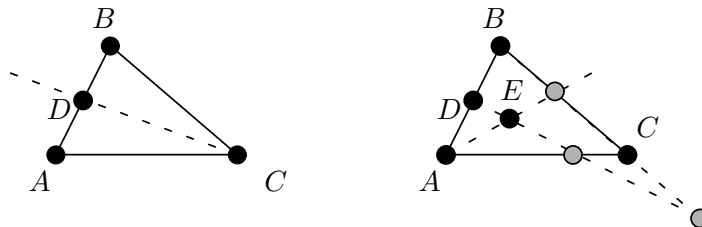
There are $q - 2$ points of DE not intersecting lines of $\triangle ABC$ which are unavailable. Then there are $q - 2$ additional points of AE not available. This gives us a total of $n_q - 3q - (q-2) - (q-2)$ choices

for F. Hence, in this case, we have $(n_q - 3q - (q-1))(n_q - 3q - (q-2) - (q-2)) = (q^2 - 3q + 2)(q^2 - 4q + 5)$ choices for E and F.

As the above 7 cases are exhaustive, their sum tells us the number of ordered 6-tuples forming a hexagon of π if we assume that the point D lies on the line AB. Factoring out the common factor of $q - 1$ gives us a sum of

$$(q-1)(q^3 - 2q^2 - q + 3).$$

(2) **D is on AC**

As in case one, there are $q-1$ choices for D in this case. As before, we can break into cases very similar to those above. The roles of E and F switch in some cases, but the techniques, and the final answer, are all the same. Hence, the total number of ways to count 6-tuples forming a hexagon with the condition that point D lies on AC is

$$(q-1)(q^3 - 2q^2 - q + 3).$$
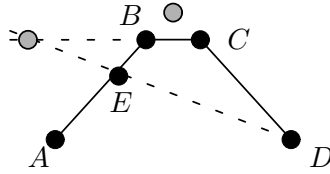
(3) **D is not on $\triangle ABC$**

In this final case, D can be any point off the triangle $\triangle ABC$. The lines AB and CD now intersect in a new point, say P, and the points B, C and P form a triangle.
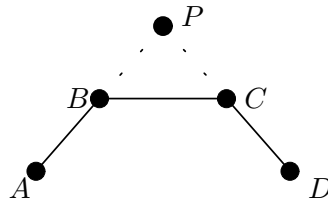
We again have several cases. First we note some natural conditions on the location of E and F. The point E cannot be on CD and the point F cannot be on AB. Also, E and F cannot both lie on BC. This gives us the following cases.

    a. **E is on AB and F is on BC.** We have $q - 2$ choices for E since E is a point on AB, except for A, B and P. Similarly, there are $q - 2$ choices for F. Hence, we have $(q-2)(q-2)$ choices for E and F in this case.

    b. **E is on AB and F is on CD.** This case can be counted just as in case (a) and gives us $(q-2)(q-2)$ choices for E and F.

c. **E is on BC and F is on CD.** Here we choose F first to obtain the same count of $(q-2)(q-2)$ choices for E and F.

d. **E is on AB and F is not on AB, BC or CD.** As before, there are $q-2$ choices for E in this case. The point F cannot be on $\triangle BCP$ which covers $n_q - 3q$ points. But F must also not lie on the line DE which intersects our triangle in $q-2$ points, D, E, and the intersection with BC, as pictured.

Hence, we have $n_q - 3q - (q-2)$ choices for F. This gives us a total of $(q-2)(q^2 - 3q + 3)$ choices for E and F.

e. **E is on BC and F is not on AB, BC or CD.** Things get trickier at this point. The problem arises because E could possibly lie on the line AD, as pictured.
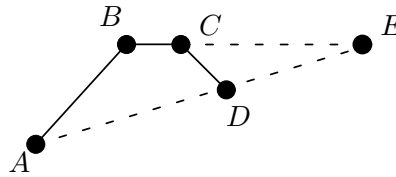

If E is the intersection point of BC and AD, then F can be any point off of the four lines AB, BC, CD, and AD. There are $n_q - 3q - (q-2)$ such points. On the other hand, if E is not on the line AD, then there are $q-2$ choices for E. The point F is now any point off $\triangle BCP$ and also off the lines AE and DE, both of which meet $\triangle BCP$ in three distinct points. Hence, the number of choices for F is $n_q - 3q - (q-2) - (q-2)$. In summary, the number of choices for E and F in this case is $(q^2 - 3q + 3) + (q-2)(q^2 - 4q + 5) = q^3 - 5q^2 + 10q - 7$.

f. **F is on BC and E is not on AB, BC or CD.** This case is very much the same as the previous case, with the roles of E and F reversed. We count F first and again, the total number of choices for E and F is $q^3 - 5q^2 + 10q - 7$.

g. **F is on CD and E is not on AB, BC or CD.** In this case, we count F first. There are $q-2$ choices for F since F is on CD but is not any of the points B, C or P. Now, E is off the lines AB, BC and

CD by the case assumption and is also off the line AF. Hence, there are $n_q - 3q - (q - 2) = q^2 - 3q + 3$ choices for F. Therefore, the total number of choices for E and F in this case is $(q - 2)(q^2 - 3q + 3)$.

h. **E and F are both not on AB, BC or CD.** Naturally, this is the most difficult case to count. In order to facilitate the counting, we break this cases into disjoint subcases based on the location of the point E. In each case, we count the number of choices for E and F. Let Q be the intersection of the lines AC and BD.
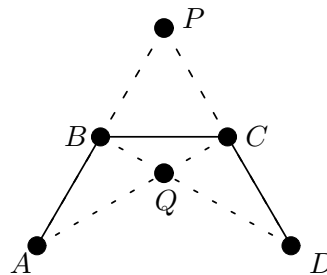
i. **Suppose $E = Q$.** Then, F is not on $\triangle BCP$ and is also not on the lines AC and BD. The line AC intersects $\triangle BCP$ in two points, and BD intersects $\triangle BCP$ in two points. Both of these lines contain E which need only be thrown out once. Hence, there are $n_q - 3q - (q - 1) - (q - 2)$ choices for F. Hence, the total number of choices for E and F is $q^2 - 4q + 4$.

ii. **Suppose E is on AC, but not equal to Q.** Then, there are $q - 2$ choices for E and $n_q - 3q - (q - 1) - (q - 3)$ choices for F, throwing out the additional points on lines AC and ED. This gives us $(q - 2)(q^2 - 4q + 5)$ choices for E and F in this case.

iii. **Suppose E is on BD, but not equal to Q.** This case parallels the previous case and again gives us $(q - 2)(q^2 - 4q + 5)$ choices for E and F.

iv. **Suppose E is on AD.** Then there are $q - 2$ choices for E, throwing out the intersections of AD with $\triangle BCP$. The point F cannot be on $\triangle BCP$ or the line AD. Hence, there are $n_q - 3q - (q - 2)$ choices for F. This gives us $(q - 2)(q^2 - 3q + 3)$ choices for E and F in this case.

v. **Finally, suppose that E is not on $\triangle BCP$ or either of the lines AC or BD.** There are then $n_q - 3q - (q - 2) - (q - 1) - (q - 2)$ choices for E by starting with the points off $\triangle BCP$ and throwing out the additional points on AD, AC and BD, respectively. Then, we have $n_q - 3q - (q - 2) - (q - 3)$ choices

for F by starting with the points off $\triangle BCP$ and throwing out the additional points on AE and DE. Hence, we have a total of $(q^2 - 5q + 6)(q^2 - 4q + 6)$ choices for E and F in this case.

These five cases sum to $(q - 2)(q^3 - 4q^2 + 8q - 7)$.

We now add together the results of cases (a) - (h). Their sum is $q^4 - 3q^3 - q^2 + 3q$. For each of these cases, there are $n_q - 3q$ choices for D. Hence, the number of ways to select D, E and F when D is off $\triangle ABC$ is

$$(q - 1)^2(q^4 - 2q^3 - q^2 + 3q).$$

We are now ready to assemble the final count. In cases 1 and 2 when D lies on a line of the $\triangle ABC$, there are $q-1$ choices for D and a total of $2(q^3 - 2q^2 - q + 3)$ choices for E and F. Hence, the total number of choices for D, E and F is

$$2(q - 1)^2(q^3 - 2q^2 - q + 3) + (q - 1)^2(q^4 - 2q^3 - q^2 + 3q)$$

or

$$(q - 1)^2(q^4 - 5q^2 + q + 6).$$

which factors as

$$(q - 1)^2(q + 2)(q^3 - 2q^2 - q + 3).$$

Now recall that there are n_q choices for A, $n_q - 1$ choices for B, and $n_q - (q+1)$ choices for C. Putting all of these cases together and dividing by 12 gives

$$\frac{1}{12}(q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q + 2)(q^3 - 2q^2 - q + 3).$$

$\boxed{QED}$

We summarize the results of the previous lemmas which gives us the complete result of Theorem 1.

1 Theorem. Let q be a prime power, and let $n_q = q^2 + q + 1$. Then for every projective plane π_q of order $q \geq 2$ the following holds.

(i) $c_3(\pi_q) = \frac{1}{6}q^2 n_q(n_q - 1)$.

(ii) $c_4(\pi_q) = \frac{1}{8}q^2 n_q(n_q - 1)(q - 1)^2$.

(iii) $c_5(\pi_q) = \frac{1}{10}q^2 n_q(n_q - 1)(q - 1)^2(n_q - q)$.

(iv) $c_6(\pi_q) = \frac{1}{12}q^2 n_q(n_q - 1)(q - 1)^2(q + 2)(q^3 - 2q^2 - q + 3)$.

3   Other results

It is desirable to know not just how many cycles of certain lengths exist in the Levi graph of a given plane, but whether other factors apart from the order q will affect this number. For instance, does every plane of order q contain embedded cycles of length $2n_q$ in its Levi graph? For large q, the positive answer follows from the fact that the Levi graph of a projective plane is pseudo-random. See [19] by Krivelevich and Sudakov for the details. Can anything be said about the number of cycles of a particular length, even if the exact number of such cycles is unknown?

First, one might ask whether or not cycles of certain lengths exist at all in the Levi graph for a given plane π_q. For lengths $2k \geq 6$ the following theorem due to Bondy and Simonovits [5] gives us the result we expect.

5 Theorem. *Suppose $k \geq 3$. There exists a constant $c = c(k)$ such that for all n, every graph G of order n and size at least $cn^{1+1/k}$ contains a $2k$-cycle.*

Moreover, the best known constant $c(k) = 8(k-1)$ was provided by Verstraëte [24]. For the Levi graphs of π_q, $n = 2n_q$, and $e = (q+1)n_q > (1/(2\sqrt{2})) \cdot n^{1+1/2}$. Clearly, for every fixed $k \geq 3$, there exists $q_0 = q_0(k)$ such that for all $q > q_0$, $e(Levi(\pi_q)) > 8(k-1)n^{1+1/k}$. This proves that the Levi graphs of π_q, for all $q > q_0(k)$ always contains a $2k$-cycle, since, for $k \geq 3$, $(1/(2\sqrt{2}))n^{1+1/2} > 8(k-1)n^{1+1/k}$. Clearly $q_0(k)$ is easy to estimate. It is of magnitude $\sqrt{k}$. Therefore the existence of $2k$-cycles in the Levi graphs of π_q is not clear only for small q, $q < q_0(k)$. We believe that for this range they also exist.

We turn our attention now to some specific examples, applications of our formulae, and some more general results for large classes of planes.

6 Example. Every Desarguesian projective plane π admits a function σ that permutes all its lines, and all its points, in a cycle of length n_q. The group generated by σ is called a Singer group of π. We will now use this cyclic group to create a maximal cycle in π; we start with any line l_1 of π and we consider $l_2 = \sigma(l_1)$, the intersection of these two lines yields a point p_1, similarly we create $p_2 = l_2 \cap \sigma(l_2)$, and so on until we get $p_{n_q} = l_{n_q} \cap l_1$. Note that the set $\{p_1, p_2, \cdots, p_{n_q}\}$ is actually the orbit of p_1 under the Singer group, which has exactly n_q elements. Hence, we obtain a C_{n_q} with vertices all points of π and lines all lines of π.

7 Example. Applying the formulas of Section 2 to the Fano plane (or $PG(2,2)$), the number of 3-gons is exactly 28, the number of 4-gons is 21, the number of 5-gons is 84, and the number of 6-gons is 56. Finally, because of Example 6 there is at least one 7-gon in $PG(2,2)$. In fact, a quick *Magma* [7] computation shows that there are exactly 24 7-gons in $PG(2,2)$. No other k-gons can be embedded in this plane.

We note that in the previous example whenever we counted the number of embedding of a k-gon, $k \leq 6$, we got a multiple of $7 = 2^2 + 2 + 1$. This observation inspires the following result.

8 Lemma. *Let c_k be a k-gon embedded in a projective (or affine) plane π of order q that admits a cyclic collineation group H of order m acting freely on the points of π. Then, $m \mid c_k(\pi)$ whenever $gcd(k, m) = 1$.*

PROOF. Clearly H acts on the set of embeddings of c_k in π. If any element of H fixed the set of vertices of an embedding of c_k, then H acting freely on points of π would imply that m and k have a common factor. Hence, H acts freely on the embeddings of c_k, the result follows.   QED

9 Corollary. *Assume that a k-gon is embedded in an (affine) translation plane α of order q^2, and that q is a power of a prime p with $p \nmid k$, then $q^2 \mid c_k(\alpha)$.*

10 Remark. If a projective plane of order q admits a Singer cycle and c_k is a k-gon with $gcd(k, n_q) = d$, then the Singer cycle containing a subgroup of order n_q/d and $gcd(k, n_q/d) = 1$ forces that n_q/d divides $c_k(\pi)$. This implies that, for instance, if π were a Desarguesian plane of order q such that $gcd(k, n_q) = 1$ then $n_q \mid c_k(\pi)$

Now we look at how small the order of a plane could be to allow the embedding of a k-cycle.

11 Lemma. *If a projective plane Π contains a k-arc then there is a k-cycle embedded in Π with vertices the points of the k-arc.*

12 Corollary. *Any k-cycle can be embedded in some projective plane.*

PROOF. It is known that the maximal arcs in a projective plane of order q have size $q + 1$ (if q is odd) or $q + 2$ (if q is even). Since a sub-arc of an arc is also an arc, then a k-cycle can be embedded in any projective plane of order q, and q is less than $k - 1$ if q is odd, or less than $k - 2$ if q is even.   QED

4   Concluding remarks

We would like to point out that there was some computational investigation of the number of 7-gons in non-isomorphic planes. It is well-known that there are four projective planes of order 9, up to isomorphism: the Desarguesian plane, the Hall plane, the dual of the Hall plane, and the Hughes plane. Moreover, 9 is the smallest order in which we find non-Desarguesian planes. The Hughes plane has much less symmetry than the Desarguesian plane and, as such, it serves as a good place to search for structural differences from $PG(2, 9)$. It was discovered through some involved computation that the number of 7-gons in the Hughes

plane of order 9 is the same as the number of 7-gons in the Desarguesian plane $PG(2,9)$. This was not completely surprising to us.

It would be most interesting to know whether the number of k-gons embedded in a projective plane is a function of the order only. Perhaps the number of cycles of length 10 would be different for non-isomorphic planes since the famous *Desargues's configuration* contains exactly this number of points. We note that counting the number of 10-gons in small order planes is computationally infeasible at the present time. However the authors have not reached a consensus whether the answer to question **Q2** is positive.

There are other questions that we are led to by our present discussion. One easy observation is that if a k-gon c_k is embedded in a projective plane π then c_k can also be embedded in any other plane that is isomorphic to π or that has (an isomorphic copy of) π as a subplane. Since $\pi \cong \Pi$ implies $c_k(\pi) = c_k(\Pi)$, we can choose the presentation we please for the planes to study. Also, note that a collineation of π might create new embeddings of c_k in π. This yields the interesting issue of looking at the orbits under the collineation group of π of the set of embeddings of a k-gon in π.

It seems obvious that the close relation between affine and projective planes should yield a relation between the number of k-gons embedded in an affine plane α and the number of k-gons in a projective extension of α. So far that is not known.

Q3: Is it true that two affine planes have the same number of k-gons if and only if so do their projective extensions?

These results are of great interest when one wants to study embeddings of graphs in translation planes, as most of the tools used in the study of translation planes are 'affine'.

Finally, the idea of looking at cycles embedded in projective planes, was motivated by the question of a possibility of embedding of a finite partial plane in a finite projective plane. It seems that in print the question was posed independently by Erdős [10] and Welsh [25]. On the other hand, it was shown by Hall [18] that any finite partial plane can be embedded in an infinite projective plane. For some partial results on this problem, see a monograph by Metsch [20], and a recent paper by Moorhouse and Williford [21].

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