Note di Matematica
Note Mat. 29 (2009), suppl. n. 1, 121-134
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v29n1supplp121

Some sporadic translation planes of order 11^2

Vito Abatangelo [i]
Department of Mathematics,
Politecnico of Bari (Italy)
abatvito@poliba.it

Gábor Korchmáros [i]
Department of Mathematics and Computer Science,
University of Basilicata (Italy)
gabor.korchmaros@unibas.it

Bambina Larato [i]
Department of Mathematics,
Politecnico of Bari (Italy)
larato@poliba.it

Abstract. In [29], the authors constructed a translation plane Π of order 11^2 arising from replacement of a sporadic chain F' of reguli in a regular spread F of $PG(3,11)$. They also showed that two more non isomorphic translation planes, called Π_1 and Π_{13}, arise respectively by derivation and double derivation in $F \setminus F'$ which correspond to a further replacement of a regulus with its opposite regulus and a pair of reguli with their opposite reguli, respectively. In [8], the authors proved that the translation complement of Π contains a subgroup isomorphic to $\mathrm{SL}(2,5)$. Here, the full collineation group of each of the planes Π, Π_1 and Π_{13} is determined.

Keywords: Translation plane, Replacement, Collineation, Chain of circles

MSC 2000 classification: primary 51E15, secondary 14H25

Introduction

The idea to construct translation planes from the Desarguesian plane of the same square order by (multiple) derivation was developed by Ostrom in the late sixties. In terms of the regular spread F of $\mathrm{PG}(3,q)$ corresponding to the Desarguesian plane of order q^2, a multiple derivation consists in replacing a set of pairwise disjoint reguli of F by their opposite ones. For odd q, Bruen [14] introduced a similar construction from a chain of reguli, that is, from a set $\mathcal{R}$ of $\frac{1}{4}(q+1)(q+3)$ reguli in F such that

(i) any two reguli in $\mathcal{R}$ have two common lines;

(ii) no three reguli in $\mathcal{R}$ have a common line.

[i]This work is partially supported by M.I.U.R. Grant Prin 2008/10

If F' is the set of $\frac{1}{4}(q+1)(q+3)$ lines of the reguli in $\mathcal{R}$, Bruen's construction consists in replacing F' with lines from the opposite reguli of those in $\mathcal{R}$. For this purpose, half of the lines of each opposite regulus has to be taken in such a way that the resulting set of $\frac{1}{8}(q+1)(q+3)$ lines are pairwise disjoint. If this happens, $(F \setminus F') \cup G'$ is a spread of $\mathrm{PG}(3,q)$, and the arising translation plane Π of order q^2 is a *β-derived plane* Π. Of course, any set of pairwise disjoint reguli in F which are also disjoint from $\mathcal{R}$ gives rise to a multiple derivation, and the arising translation plane is a *δ-derived plane* of Π.

In the last twenty years many translation planes have been constructed by δ-derivation for small values of q, see [4, 6, 7, 11, 15, 23, 27]. There are known 20 chains of reguli, all for $q \le 37$, see [8, 16]. The non-existence result for $q = 41, 43, 47, 49$ obtained by exhaustive computer search in [16], provide some evidence that perhaps all chains of reguli are known.

It should be noticed that Bruen's idea was developed further by G. Ebert who introduced the more general concept of a nest of reguli. In terms of nests, a chain of reguli is a nest of minimum size. Replacements of nets have been under investigation giving a wider opportunity to obtain interesting sporadic translation planes, see [9, 10, 12, 19, 30]. Collineation groups of β-derived planes were investigated in [1–3, 5, 8].

In this paper, we discuss the collineation groups of the β-derived plane Π of order 11^2 and the two non-isomorphic δ-derived planes of Π constructed in [29]. The chain of reguli giving rise to Π was obtained from the chain of circle in the inversive plane of order 11 constructed in [26]. Some collineations of Π and Π_i inherited from the Desarguesian plane of order 11^2 were also determined. In [8], the authors proved that the translation complement of Π contains a subgroup isomorphic to $\mathrm{SL}(2,5)$. Our aim is to determine the full collineation group of Π. The main results are stated in the following theorems.

1 Theorem. *The translation complement $\mathcal{G}$ of the β-derived plane of order 11^2 constructed in* [29] *contains a normal subgroup $S \cong \mathrm{SL}(2,5)$. Furthermore,*

(i) $|\mathcal{G}| = 7200$*;*

(ii) *$\mathcal{G}$ has a cyclic subgroup Z_{60} of order 60 containing the scalar group of $\mathcal{G}$ of order 10;*

(iii) *The centralizer $\mathcal{E}$ of Z_{60} in G has order 3600 and contains $\mathcal{S}$;*

(iv) *The subgroup $\langle \mathcal{S}, Z_{60} \rangle$ is the central product of $\mathcal{S}$ and Z_{60};*

(v) *$\mathcal{G}$ is not a splitting extension of $\mathcal{S}$.*

2 Theorem. *With the above notation,*

(i) *The translation complement* $\mathcal{G}_1$ *of the* δ*-derived plane* Π_1 *of* Π *constructed in* [29] *has order* 240*, and it contains a normal cyclic subgroup* Z_{60} *of order* 60 *such that* $\mathcal{G}_1/Z_{60}$ *is an elementary abelian group of order four. Also, the collineation group induced by* $\mathcal{G}_1$ *on the spread associated to* Π_1 *is isomorphic to* $D_6 \times C_2$*.*

(ii) *The translation complement* $\mathcal{G}_{13}$ *of the double* δ*-derived plane* Π_{13} *of* Π *constructed in* [29] *has order* 120 *and it contains a normal cyclic subgroup* Z_{60} *of order* 60*. Also, the collineation group induced by* $\mathcal{G}_{13}$ *induced on the spread associated to* Π_{13} *is isomorphic to* D_6*.*

The classification of translation planes of order q^2 whose translation complements contain a subgroup isomorphic to $\mathrm{SL}(2,5)$ is an on-going project. Apart from the case where q is a power of 5, the classification has been achieved so far for $q \leq 9$, see [17, 20, 21]. The plane Π provides an example for $q = 11$.

The general theory of translation planes is found in [13], see also [18]. For properties of quadrics in $\mathrm{PG}(3,q)$, see [22].

1 A chain of circles in $\mathrm{PG}(3,11)$ and its collineation group

In $\mathrm{PG}(3,11)$, the chain $\mathcal{C}$ of circles constructed in [26] is represented on the elliptic quadric Q of equation $x^2 + y^2 = zt$ by the seven conics which are the sections of the planes

$$\pi_\infty : y = 0, \quad \pi_0 : x = 0, \quad \pi_1 : z = x + t, \quad \pi_2 : z = 3x + 9t,$$
$$\pi_3 : z = 9x + 4t, \quad \pi_4 : z = 5x + 3t, \quad \pi_5 : z = 4x + 5t.$$

Any two of these conics meet in two points, but any three of them have empty intersection. The poles of the above planes are the points

$$P_\infty = (0,1,0,0), \quad P_0 = (1,0,0,0), \quad P_1 = (6,0,10,1), \quad P_2 = (7,0,2,1),$$
$$P_3 = (10,0,7,1), \quad P_4 = (8,0,8,1), \quad P_5 = (2,0,6,1),$$

and they form a set $\mathcal{P}$ that contains six points namely $P_0, P_1, \ldots P_5$ from the plane π_∞ while the remaining point P_∞ in $\mathcal{P}$ is the pole of π_∞.

3 Proposition. *The full collineation group* Γ *preserving* $\mathcal{C}$ *has order* 120 *and it contains a normal subgroup isomorphic to* $PSL(2,5)$*.*

Proof. Obviously π_∞ is left invariant by Γ, and hence P_∞ is fixed by Γ. The axial symmetry ξ with center P_∞ and axis π_∞ that preserves Q is associated to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously, ξ is the unique non-trivial collineation in Γ fixing $\mathcal{P}$ point-wise. Let $\bar{\Gamma}$ be the permutation group on $\mathcal{P}' = \{P_0, \ldots, P_5\}$ induced by Γ. Then $\bar{\Gamma} = \Gamma/\langle\sigma\rangle$.

It is straightforward to show that both Q and $\mathcal{P}$ are left invariant by the collineations M_1, M_2 of $\mathrm{PG}(3, 11)$ represented by the matrices

$$\begin{pmatrix} 3 & 0 & 7 & 5 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 6 & 0 & 10 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 7 & 5 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 9 \\ 6 & 0 & 5 & 4 \end{pmatrix},$$

respectively. They act on $\mathcal{P}'$ as the permutations

$$\mu_1 : (P_0P_1P_3)\ (P_2P_5P_4), \qquad \mu_2 : (P_0P_1P_4P_5P_3)(P_2),$$

and generate a permutation group isomorphic to $\mathrm{PSL}(2, 5)$. On the other hand $\bar{\Gamma}$ is a subgroup of the collineation group in π_∞ which preserves the conic C_∞ cut out by Q. Therefore, $\bar{\Gamma}$ is isomorphic to a subgroup of $\mathrm{PGL}(2, 11)$ containing $\mathrm{PSL}(2, 5)$. From [28, Theorem 14.6], such a subgroup is $\mathrm{PSL}(2, 5)$. Therefore, $\bar{\Gamma} = \langle\mu 1, \mu 2\rangle \cong \mathrm{PSL}(2, 5)$. In particular, $|\Gamma| = 2|\bar{\Gamma}| = 120$. $\boxed{QED}$

Each of the planes

$$\rho_1 : z = 6y + 10t, \quad \rho_2 : z = 5y + 10t,$$
$$\rho_3 : z = 5x + 6y + t, \quad \rho_4 : z = 5x + 5y + t.$$

meets Q in a conic that is disjoint from the conics cut out by the planes $\pi_\infty, \pi_0, \ldots, \pi_5$. Furthermore the conics cut out by ρ_1 and ρ_2 meet in two points, and this holds true when ρ_1 and ρ_2 are replaced by ρ_3 and ρ_4. However, the conics cut out by ρ_1 and ρ_3 (or ρ_4) are also disjoint, and this holds true when ρ_1 is replaced by ρ_2. The poles of these planes are

$$L_1 = (0, 3, 1, 1), L_2 = (0, 8, 1, 1), L_3 = (8, 3, 10, 1), L_4 = (8, 8, 10, 1).$$

Note that the symmetry ξ changes L_1 with L_2 and L_3 with L_4.

4 Proposition. *The subgroup Γ_1 of Γ which fixes L_1 is an elementary abelian group of order* 4.

Proof. The line ℓ through P_∞ and L_1 is left invariant by Γ_1. Therefore, the polar line ℓ' of ℓ with respect to the quadric Q lies on π_∞ and is also left invariant by Γ_1. Since ℓ meets Q in two distinct points, namely $Q_1(0,1,1,1)$ and $Q_2 = (0,10,1,1)$, the line ℓ' is an external line to the conic C_∞ of π_∞ cut out by Q. Since the unique element of Γ which fixes π_∞ point-wise is σ but L_1 is not fixed by σ, Γ_1 acts on Π_∞ faithfully. It turns out that Γ_1 is isomorphic to a subgroup of $\mathrm{PGL}(2,11)$ preserving C_∞ and an external line to C_∞. By [25], Γ_1 is a subgroup of a dihedral group of order 24.

The involutory collineations σ_1 and σ_2 of $\mathrm{PG}(3,11)$ represented by the matrices

$$\begin{pmatrix} 3 & 0 & 8 & 3 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 10 & 2 \\ 6 & 0 & 2 & 10 \end{pmatrix}, \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

fix L_1 and preserve Q and $\mathcal{P}$. They generate an elementary abelian group of order 4. Let $\bar{\sigma_1}$ and $\bar{\sigma_2}$ be the permutations on $\mathcal{P}'$ induced by σ_1 and σ_2, respectively. Then

$$\bar{\sigma_1} = (P_0P_1)(P_3P_4), \qquad \bar{\sigma_2} = (P_2P_5)(P_3P_4).$$

and $\Delta = \langle \sigma_1, \sigma_2 \rangle$ is an elementary abelian group of order 4 contained in $\Gamma \cong \mathrm{PGL}(2,5)$. From the classification of subgroups of $\mathrm{PGL}(2,5)$, if Δ is a proper subgroup of Γ_1 then Γ_1 is isomorphic to $\mathrm{PGL}(2,3)$. On the other hand, as we have already proven, Γ_1 is subgroup of a dihedral group D_{12} of order 24. But D_{12} has only two subgroups of order 12, one is cyclic the other is a dihedral; both non isomorphic to $\mathrm{PSL}(2,3)$. This shows that $\Delta = \Gamma_1$ and the assertion follows. $\boxed{QED}$

5 Proposition. *The subgroup Γ_3 of Γ which fixes L_3 is an elementary abelian group of order 4.*

Proof. The above proof also works for L_3 when σ_2 are replaced by the involutory collineation σ_3 represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{pmatrix}.$$

$\boxed{QED}$

6 Proposition. *The subgroup Γ_{13} of Γ which preserves the set $\{L_1, L_3\}$ has order two.*

PROOF. From Propositions 4 and 5, the subgroup of Γ_{13} fixes both L_1 and L_3 is generated by σ_1. Assume on the contrary that Γ_{13} has a collineation σ that changes L_1 with L_3. Obviously, σ is either an involution or $\sigma^2 = \sigma_1$. The line ℓ through L_1 and and L_3 meets the plane π_∞ in the point $\bar{U} = (7, 0, 1, 0)$ which is an external point to the conic C_∞. Clearly, $\bar{U}$ is fixed by σ. The plane through P_∞ and ℓ meets π_∞ in a line $\bar{\ell}$ left invariant by σ. The line $\bar{\ell}$ passes through $\bar{U}$ and the projections onto π_∞ of the points L_1 and L_3 from P_∞, that is, the points $\bar{L_1} = P_\infty L_1 \cap \pi_\infty$ and $\bar{L_3} = P_\infty L_3 \cap \pi_\infty$ lie on $\bar{\ell}$. Obviously, σ changes $\bar{L_1}$ with $\bar{L_3}$.

If σ is an involution then it also acts faithfully on π_∞ as involutory collineation $\bar{\sigma}$ preserving C_∞. The center of $\bar{\sigma}$ is $\bar{U}$. Since $\bar{U}$ is an external point to the conic C_∞, $\bar{\sigma}$ induces an odd permutation on C_∞. On the other hand, Γ_1 acts on C_∞ as $\mathrm{PSL}(2, 5)$ which is a simple group. Therefore, it consists of even permutations. This contradiction shows that σ is not an involution.

If $\sigma^2 = \sigma_1$ then σ has order 4 and acts faithfully on π_∞ preserving the conic C_∞. On the other hand, Γ_1 acts on C_∞ as $\mathrm{PSL}(2, 5)$ which contains no elements of order 4. Again a contradiction which completes the proof. $\boxed{QED}$

2 The β-derived plane Π and its translation complement

We use the Bruck-Bose model to represent a translation plane Π of order 11^2. In the four-dimensional projective space $\mathrm{PG}(4, 11)$ we take a three-dimensional projective subspace $\mathrm{PG}(3, 11)$ and consider the four-dimensional affine space $\mathrm{AG}(4, 11)$ obtained from $\mathrm{PG}(4, 11)$ deleting $\mathrm{PG}(3, q)$.

The points of Π are those of $\mathrm{AG}(4, 11)$. If Σ is a spread of $\mathrm{PG}(3, 11)$, then the "lines" of Π are those two-dimensional subspaces of $\mathrm{AG}(4, 11)$ which meet $\mathrm{PG}(3, 11)$ in a line of Σ. The lines of Π which pass through a given point $O \in \mathrm{AG}(4, 11)$ are the components of Π, so that the lines of Π are the components together with their images under the translations of $\mathrm{AG}(4, 11)$.

The translation complement $\mathcal{G}$ of Π is the subgroup of affinities of $\mathrm{AG}(4, 11)$ whose elements fix O and preserve the components of Π. If $\mathcal{D}$ is the dilation group of $AG(4, 11)$ fixing O, then the factor group $\mathcal{G}/\mathcal{D}$ is the collineation group of Σ.

Let F be the regular spread of $\mathrm{PG}(3, 11)$. We may assume a projective frame (x, y, z, t) in $\mathrm{PG}(3, 11)$ such that F consists of the line

$$r_\infty = \langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle$$

together with the lines

$$r_{a,b} = \langle (a, -b, 0, 1), (b, a, 1, 0) \rangle$$

with a, b ranging over GF(11).

The chain of reguli arising from the chain of circles discussed in Section 1 consists of those reguli of F which are contained in the following seven quadrics given by their equations:

$$\begin{aligned} Q_\infty &: xz - yt = 0, \\ Q_0 &: xt + yz = 0, \\ Q_1 &: x^2 + y^2 + 10z^2 + 10t^2 + 10xt + 10yz = 0, \\ Q_2 &: x^2 + y^2 + 2z^2 + 2t^2 + 8xt + 8yz = 0, \\ Q_3 &: x^2 + y^2 + 7z^2 + 7t^2 + 2xt + 2yz = 0, \\ Q_4 &: x^2 + y^2 + 8z^2 + 8t^2 + 6xt + 6yz = 0, \\ Q_5 &: x^2 + y^2 + 6z^2 + 6t^2 + 7xt + 7yz = 0. \end{aligned}$$

The lines of the above reguli are as follows, where n ranges over GF(11):

$$\begin{aligned} R_\infty &= \{y = nt, y = nz\} \cup \{t = 0,\ z = 0\}; \\ R_0 &= \{x = nz, y = nt\} \cup \{z = 0,\ t = 0\}; \\ R_1 &= \{x - ny + (2 + 6n)z + (5 + 9n)t = 0,\ nx + y + (5 + 9n)z + \\ &\quad + (9 + 5n)t = 0\} \cup \{10y + 6z + 9t = 0, x + 9z + 5t = 0\}; \\ R_2 &= \{x - ny + (6 + 7n)z + (4 + 5n)t = 0,\ nx + y + (4 + 5n)z + \\ &\quad + (5 + 4n)t = 0\} \cup \{10y + 7z + 5t = 0, x + 5z + 4t = 0\}; \\ R_3 &= \{x - ny + (7 + 10n)z + (1 + 4n)t = 0,\ nx + y + (1 + 4n)z + \\ &\quad + (4 + n)t = 0 \cup \{10y + 10z + 4t = 0, x + 4z + t = 0\}; \\ R_4 &= \{x - ny + (10 + 8n)z + (3 + n)t = 0,\ nx + y + (3 + n)z + \\ &\quad + (1 + 3n)t = 0\} \cup \{10y + 8z + t = 0, x + z + 3t = 0\}; \\ R_5 &= \{x - ny + (8 + 2n)z + (9 + 3n)t = 0,\ nx + y + (9 + 3n)z + \\ &\quad + (3 + 9n)t = 0\} \cup \{10y + 2z + 3t = 0, x + 3z + 9t = 0\}. \end{aligned}$$

From the definition of a chain of reguli, each of these lines appears twice showing that their total number equals 42 when each is counted once. We will denote their union by F'.

Each quadric has an opposite regulus, listed below:

$$\begin{aligned} R'_\infty &= \{x = ny, t = nz\} \cup \{y = 0,\ z = 0\}; \\ R'_0 &= \{x = ny, z = 10nt\} \cup \{y = 0,\ t = 0\}; \\ R'_1 &= \{x + 10ny + 8nz + 3t = 0, nx + y + 7z + 7nt = 0 \\ &\quad \cup \{10y + 8z = 0, x + 7t = 0\}; \end{aligned}$$

$$\begin{aligned}
R'_2 =&\{x+10ny+2nz+9t=0, nx+y+10z+10nt=0\\
&\cup\{10y+2z=0, x+10t=0\};\\
R'_3 =&\{x+10ny+6nz+5t=0, nx+y+8z+8nt=0,\\
&\cup\{10y+6z=0, x+8t=0\};\\
R'_4 =&\{x+10ny+7nz+4t=0, nx+y+2z+2nt=0,\\
&\cup\{10y+7z=0, x+2t=0\};\\
R'_5 =&\{x+10ny+10nz+t=0, nx+y+6z+6nt=0,\\
&\cup\{10y+10z=0, x+6t=0\}.
\end{aligned}$$

For each opposite regulus R'_i we take a half regulus $\frac{1}{2}\,R'_i$ as follows: the lines from R'_∞ for $n=0,2,5,6,9$ together with the line $\{y=0,\ z=0\}$ and the lines from R'_i for $n=1,3,4,7,8,10$ when $i=0,1,\dots,5$.

Let G' be the union of the above seven half reguli. Then the following result holds, see [29, Theorem 2 (i)].

7 Theorem. *The set* $(F\setminus F')\cup G'$ *is a spread of* $\mathrm{PG}(3,11)$.

Let Π be the translation plane arising from the spread $(F\setminus F')\cup G'$. Our aim is to determine the translation complement $\mathcal{G}$ of Π.

For this purpose we use the affine frame in $\mathrm{AG}(4,11)$ with point-coordinates (x,y,z,t) such that the line through the origin $O=(0,0,0,0)$ and the point $P=(x,y,z,t)$ has its infinite point $P_\infty\in\ PG(3,11)$ with homogeneous coordinates $(x,y,z,t,0)$.

The dilations with center in O have matrix representation of type $D(d)=d\,I$ where I is the identity matrix of order four, and d ranges over $\mathrm{GF}(11)$. They fix $\mathrm{PG}(3,11)$ point-wise and are in G. The affinities other than the dilations which preserve every component of Π have matrix representation of type $M(m)D(d)$ where

$$M(m)=\begin{pmatrix} m & 10 & 0 & 0\\ 1 & m & 0 & 0\\ 0 & 0 & m & 1\\ 0 & 0 & 10 & m\end{pmatrix}$$

with $m\in\mathrm{GF}(11)$. Those with $m\in\{0,2,5,6,9\}$ also takes any line in $\frac{1}{2}\,R'_i$ to a line in the same $\frac{1}{2}\,R'_i$. These produce a cyclic group Z_{60} of order 60 consisting of $M(m)$ and $D(d)$ where m ranges over $\{0,2,5,6,9\}$ and d over $\mathrm{GF}(11)$. A generator of Z_{60} is $M(6)$.

Now, we exhibit a subgroup $\mathcal{S}\cong\mathrm{SL}(2,5)$ of G. Let

$$S_1=\begin{pmatrix} 6 & 0 & 0 & 3\\ 0 & 6 & 3 & 0\\ 0 & 4 & 4 & 0\\ 4 & 0 & 0 & 4\end{pmatrix},\quad S_2=\begin{pmatrix} 0 & 0 & 0 & 7\\ 0 & 0 & 7 & 0\\ 0 & 3 & 0 & 0\\ 3 & 0 & 0 & 0\end{pmatrix}.$$

Then $S_1^3 = S_2^4 = (S_1S_2)^{10} = I$. Therefore, $\mathcal{S} = \langle S_1, S_2\rangle \cong SL(2,5)$. A direct computation shows that both $S_1, S_2 \in \mathcal{G}$.

Furthermore, since S_1 and S_2 commute with Z_{60} and $\mathcal{S} \cap Z_{60} = \{I, -I\}$, we have that $\mathcal{U} = \langle \mathcal{S}, Z_{60}\rangle$ has order 3600 and is the central product of $\mathcal{S}$ and Z_{60}.

Actually, $\mathcal{G}$ is larger than $\mathcal{U}$, as $\mathcal{G}$ also contains

$$V = \begin{pmatrix} 0 & 0 & 8 & 4 \\ 0 & 0 & 7 & 8 \\ 4 & 9 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix}$$

with $V \notin \mathcal{U}$, and $V^2 \in \mathcal{U}$. Therefore, $\mathcal{G}$ has order at least 7200. This implies that the spread $(F \setminus F') \cup G'$ is left invariant by a collineation group $\mathcal{G}'$ of $\mathrm{PG}(3,11)$ of order at least 720.

To complete the proof of Theorem 1 we only need the following result.

8 Proposition. *The collineation group $\mathcal{G}'$ of $\mathrm{PG}(3,11)$ which preserves the spread $(F \setminus F') \cup G'$ has order at most 720.*

PROOF. $\mathcal{G}$ induces a collineation group $\mathcal{G}'$ of $\mathrm{PG}(3,11)$ preserving the spread $(F \setminus F') \cup G'$. By a result of Bruen [6, Theorem 3.5], $\mathcal{G}'$ induces a collineation group preserving the corresponding chain $\mathcal{C}$ of circles. The kernel of this representation is the cyclic group of order at most 6 which preserves F line-wise. From Proposition 3, $|\mathcal{G}'| \leq 720$. $\boxed{QED}$

3 δ-derived planes of Π and their translation complements

The four quadrics of $\mathrm{PG}(3,11)$ arising from the circles cut out on Q by the planes ρ_i with $i = 1,2,3,4$ are

$$\begin{aligned} H_1 :& x^2 + y^2 + z^2 + t^2 + 5xz + 6yt = 0, \\ H_2 :& x^2 + y^2 + z^2 + t^2 + 6xz + 5yt = 0, \\ H_3 :& x^2 + y^2 - z^2 - t^2 + 5xz + 6xt + 6yz + 6yt = 0, \\ H_4 :& x^2 + y^2 - z^2 - t^2 + 6xz + 6xt + 6yz + 5yt = 0, \end{aligned}$$

and their reguli contained in F consists of the following lines

$$\begin{aligned} T_1 =& \{(2n+10)x + (5n+9)y + z + nt = 0, \\ & (3n+10)x + (4n+3)y + (10n+10)z + (10n+1)t = 0\} \\ & \cup \{2x + 5y + t = 0, 3x + 4y + 10z + 10t = 0\}; \end{aligned}$$

$$\begin{aligned}
T_2 =&\{(5n+9)x+(2n+10)y+nz+t=0,\\
&(4n+3)x+(3n+10)y+(10n+1)z+(10n+10)t=0\}\\
&\cup\{5x+2y+z=0, 4x+3y+10z+10t=0\};\\
T_3 =&\{(n+10)x+(n+1)y+10z+2nt=0,\\
&(n+1)x+(10n+1)y+9nz+10t=0\}\\
&\cup\{x+y+2t=0, x+10y+9z=0\};\\
T_4 =&\{(n+1)x+(n+10)y+2nz+10t=0,\\
&(n+10)x+(10n+10)y+z+2nt=0\}\\
&\cup\{x+y+2z=0, x+10y+2t=0\}
\end{aligned}$$

where n ranges over $\mathrm{GF}(11)$. The opposite reguli consist of the following lines

$$\begin{aligned}
T_1' =&\{(3n+9)x+(4n+6)y+10nz+(10n+10)t=0,\\
&(10n+1)x+(3n+2)y+(10n+10)z+nt=0\}\\
&\cup\{3x+4y+10z+10t=0,\ 10x+3y+10z+t=0\};\\
T_2' =&\{(4n+6)x+(3n+9)y+(10n+10)z+10nt=0,\\
&(3n+2)x+(10n+1)y+nz+(10n+10)t=0\}\\
&\cup\{4x+3y+10z+10t=0,\ 3x+10y+z+10t=0\};\\
T_3' =&\{(n+10)x+(n+1)y+2z+2nt=0,\\
&(10n+10)x+(n+10)y+10nz+t=0\}\\
&\cup\{x+y+2t=0,\ 10x+y+10z=0\};\\
T_4' =&\{(n+1)x+(n+10)y+10nz+10t=0,\\
&(10n+1)x+(n+1)y+2z+9nt=0\}\\
&\cup\{x+y+10z=0,\ 10x+y+9t=0\}.
\end{aligned}$$

Now we prove Theorem 2. Let $\mathcal{G}_1$ be the translation complement of the translation plane Π_1 arising from the spread $F_1=(F\setminus(F'\cup T_1))\cup(G'\cup T_1')$. Then $\mathcal{G}_1$ is a subgroup of $\mathcal{G}$ since F_1 arises from F by δ-derivation. Obviously $\mathcal{G}_1$ contains $M(6)$. Arguing as in the proof of Proposition 2, the factor group $\mathcal{G}_1/Z_{60}$ turns out to be a collineation group of $\mathrm{PG}(3,11)$ which preserves Q and $\mathcal{P}$ while fixes L_1. From Proposition 4, $\mathcal{G}_1$ has order at most 240.

A straightforward computation shows that $\mathcal{G}_1$ also contains both

$$N_1=\begin{pmatrix}0&0&10&0\\0&0&0&1\\10&0&0&0\\0&1&0&0\end{pmatrix},\quad N_3=\begin{pmatrix}5&8&2&4\\8&6&7&2\\2&7&5&3\\4&2&3&6\end{pmatrix}.$$

Furthermore, $N_1^2 = I$ and $N_3^2 = -I$, and $N_1N_3 = N_3N_1$. Hence, $\langle N_1, N_3\rangle$ is the direct product $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_3$ where $\mathcal{N}_1 = \langle N_1\rangle$ and $\mathcal{N}_3 = \langle N_3\rangle$. From this the factor group $\mathcal{G}_1/Z_{60}$ has order at least 4. Therefore, the order of $\mathcal{G}_1$ is exactly 240, and $\mathcal{G}_1/Z_{60}$ is an elementary abelian group of order 4.

The collineation group $\bar{\mathcal{G}}_1$ induced by $\mathcal{G}_1$ has order 24 and contains the collineations $\bar{W}_1, \bar{W}_2, \bar{W}_3$ associated to the matrices

$$W_1 = \begin{pmatrix} 3 & 5 & 0 & 0\\ 6 & 3 & 0 & 0\\ 0 & 0 & 3 & 6\\ 0 & 0 & 5 & 3 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 1 & 3 & 0\\ 1 & 0 & 0 & 3\\ 3 & 0 & 0 & 10\\ 0 & 3 & 10 & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 9\\ 2 & 0 & 0 & 0\\ 0 & 9 & 0 & 0 \end{pmatrix},$$

respectively. Here

$$W_1^6 = W_2^2 = W_2W_1W_2W_1 = -I,\ W_3^2 = W_3W_1W_3W_1^{-1} = 4I,\ W_3W_2W_3W_2 = 7I.$$

Therefore $\bar{\mathcal{G}}_1 = \langle \bar{W}_1, \bar{W}_2, \bar{W}_3\rangle \cong D_6 \times Z_2$. This completes the proof of (i) of Theorem 2.

Let $\mathcal{G}_{13}$ be the translation complement of the translation plane Π_{13} arising from the spread $F_{13} = (F\setminus(F'\cup T_1\cup T_3))\cup(G'\cup T_1'\cup T_3')$. Then $\mathcal{G}_{13}$ is a subgroup of $\mathcal{G}$ since F_{13} arises from F by double δ-derivation. Arguing as before the factor group $\mathcal{G}_1/Z_{60}$ turns out to be a collineation group of $\mathrm{PG}(3,11)$ which preserves Q, $\mathcal{P}$ and the pair $\{L_1, L_3\}$. From Proposition 6, $\mathcal{G}_{13}$ has order at most 120. Since $\mathcal{G}_{13}$ also contains N_3, the order of $\mathcal{G}_{13}$ is exactly 120, and $\mathcal{G}_{13}/Z_{60}$ is a group of order 2.

The collineation group $\bar{\mathcal{G}}_{13}$ induced by $\mathcal{G}_{13}$ has order 24 and contains the collineations associated to the matrices W_1 and W_2. For the above discussion, $\bar{\mathcal{G}}_{13} = \langle \bar{W}_1, \bar{W}_2\rangle \cong D_6$.

This completes the proof of Theorem 2.

References

[1] L. M. Abatangelo: *Fibrations of* PG(3, q) *equipped with special families of reguli,* Rend. Sem. Mat. Brescia, **7** (1984), 3–11.

[2] L. M. Abatangelo: *On the automorphism groups of β-derived planes,* J. Geom. **25** (1985), 19–29.

[3] L. M. Abatangelo: *Affine homology groups of a β-derived translation plane,* Boll. Un. Mat. Ital. B (6) **5** (1986), 569–580.

[4] L.M. Abatangelo, V. Abatangelo: *On Bruen's planes of order* 25, Atti Sem. Mat. Fis. Univ. Modena **31** (1982), 348–369.

[5] L.M. Abatangelo: *On the collineation groups of a β-derived spread,* Res. Lecture Notes Math., Mediterranean, Rende Italy, (1991), 111–123.

[6] L.M. Abatangelo, V. Abatangelo, G. Korchmáros: *A translation plane of order* 25, J. Geom. **22** (1984), 108–116.

[7] V. Abatangelo: *A translation plane of order* 81 *and its full collineation group,* Bull. Austral. Math. Soc. **29** (1984), 19–34.

[8] V. Abatangelo, B. Larato: *Translation planes with an automorphism group isomorphic to* SL(2, 5)*,* Ann. Discrete Math., (North-Holland, Amsterdam-New York), **30** (1986), 1–7. North-Holland Math. Stud., 123, North-Holland, Amsterdam, (1986).

[9] R.D. Baker, G.L. Ebert: *A new class of translation planes*, Ann. Discrete Math., **37** 1988, 7–20.

[10] R.D. Baker, G.L. Ebert: *Nests of size $q-1$ and another family of translation planes*, J. London Math. Soc. (2) **38** (1988), 341–355.

[11] R.D. Baker, G.L. Ebert: *A Bruen chain for $q = 19$*, Des. Codes Cryptogr. **4** (1994), 307–312.

[12] R.D. Baker, G.L. Ebert, R. Weida: *Another look at Bruen chains*, J. Combin. Theory Ser. A **48** (1988), 77–90.

[13] M. Biliotti, V. Jha, N.L. Johnson, Foundations of translation planes, Monographs and Textbooks in Pure and Applied Mathematics, 243. Marcel Dekker, Inc., New York, 2001. xvi+542 pp.

[14] A.A. Bruen: *Inversive geometry and some new translation planes, I*, Geom. Dedicata **7** (1978), 81–98.

[15] M. Capursi: *Chains of circles obtainable by means of pseudoregular points with respect to a conic in a Galois plane*, Note Mat. **1** (1981), 113–126.

[16] I. Cardinali, N. Durante, T. Penttila, R. Trombetti: *Bruen chains over fields of small order*, Discrete Math. **282** (2004), 245–247.

[17] V. Jha, N.L. Johnson: *The translation planes of order 81 admitting* SL(2, 5)*,* Note Mat. **24** (2005), 59–73.

[18] N.L. Johnson, V. Jha, M. Biliotti: Handbook of finite translation planes, Pure and Applied Mathematics (Boca Raton), 289. Chapman & Hall/CRC, Boca Raton, FL, 2007. xxii+861 pp.

[19] N.L. Johnson, R. Pomareda: *Mixed nests*, J. Geom. **56** (1996), 59–86.

[20] N.L. Johnson, A. Prince: *The translation planes of order 81 that admits* SL(2, 5)*, generated by affine elations*, J. Geom. **84** (2005), 73–82.

[21] N.L. Johnson, A. Prince: *Orbit constructions for translation planes of order 81 admitting* SL(2, 5), Note Mat. **24** (2005), 41–57.

[22] J.W.P. Hirschfeld: Projective Geometries over Finite Fields, second ed., Oxford Univ. Press, Oxford, 1998, xiv+555 pp.

[23] O. Heden: *On Bruen chains*, Discrete Math. **146** (1995), 69–96.

[24] O. Heden, M. Saggese: *Bruen chains in* $\mathrm{PG}(3, p^k), k \geq 2$, Discrete Math. **214** (2000), 251–253.

[25] G. Korchmáros: *A geometric characterization of dihedral subgroups of order* $2(q+1)$ *of* $PGL(2, q)$, Geom. Dedicata, **9** (1980), 381–384.

[26] G. Korchmáros: *Example of a chain of circles on an elliptic quadric of* PG(3, q)*,* $q =$ 7, 11, J. Combin. Theory Ser. A **31** (1981), 98–100.

[27] B. Larato, G. Raguso: *Translation planes of order* 13^2, Riv. Mat. Univ. Parma (4) **10** (1984), 223–233.

[28] H. Lüneburg: Translation planes, Springer-Verlag, Berlin-New York, 1980. ix+278 pp.

[29] G. Pellegrino, G. Korchmáros: *Translation planes of order* 11^2, Ann. Discrete Math., (North-Holland, Amsterdam-New York), **14** (1982), 249–264.

[30] R.A. Weida: *An extension of Bruen chains*, Geom. Dedicata **30** (1989), 11–21.