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Girth 5 Graphs from Elliptic Semiplanes

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Abstract. For $3 \leq k \leq 20$ with $k \neq 4, 8, 12$, all the smallest currently known k–regular graphs of girth 5 have the same orders as the girth 5 graphs obtained by the following construction: take a (not necessarily Desarguesian) elliptic semiplane $\mathcal{S}$ of order $n-1$ where $n = k - r$ for some $r \geq 1$; the Levi graph $\Gamma(\mathcal{S})$ of $\mathcal{S}$ is an n–regular graph of girth 6; parallel classes of $\mathcal{S}$ induce co–cliques in $\Gamma(\mathcal{S})$, some of which are eventually deleted; the remaining co–cliques are amalgamated with suitable r–regular graphs of girth at least 5. For $k > 20$, this construction yields some new instances underbidding the smallest orders known so far.

Keywords: (k,5)-cages, girth 5 graphs, elliptic semiplanes, Hughes planes.

MSC 2000 classification: 05C35 (05B25, 05C38, 51E30)

1 Introduction and Preliminaries

Old and new results in *Graph Theory* will be proved using methods from *Finite Geometries*. For basic notions we refer to [3] and [7], respectively. A (k, g)–*cage* is a k–regular graph of girth g of minimum order. Surveys on cages can be found in [9], [13], and [30]. Eight $(k, 5)$–cages are known:

k	order	$\lvert \mathrm{Aut} \rvert$	cage due to	reference(s)
3	10	120	Petersen	[21]
4	19	24	Robertson	[23]
5	30	20	Robertson, Wegner	[24], [28]
		30	Foster	cf. [30]
		96	Yang & Zhang, Meringer	[19], [31]
		120	Robertson, Wegner	[24], [28]
6	40	480	O'Keefe & Wong	[20], [29]
7	50	252,000	Hoffman & Singleton	[12]

For $k \geq 8$, the orders of $(k, 5)$–cages are not known. A rough lower bound is $k^2 + 1$. In 1960 Hoffman & Singleton [12] showed that this bound is sharp if and only if $k = 2, 3, 7$, and (possibly) 57. Some refinements concerning lower bounds are due to [4], [8], and [17], cf. also [9]. Upper bounds are given by the orders

[i]Dedicated to Prof. Norman L. Johnson on the occasion of his 70^{th} Birthday

$rec(k,5)$ of the *smallest currently known k–regular graphs of girth* 5. In [9], Exoo & Jajcay survey the state of the art and give detailed descriptions of the current record holders for $k \leq 20$:

k	lower bound	upper bound $rec(k,5)$	supported by graphs due to	references	comment on con–struction	v_{k-r}
8	67	80	Royle, Jørgensen	[26], [14]		
9	86	96	Jørgensen	[14]		48_7
10	103	126	Exoo	[10]		63_8
11	124	156	Jørgensen	[14]		78_9
12	147	203	Exoo	[10]		
13	174	240	Exoo	[10]		120_{11}
14	199	288	Jørgensen	[14]	deletion	
15	230	312	Jørgensen	[14]	deletion	
16	259	336	Jørgensen	[14]		168_{13}
17	294	448	Schwenk	[27]	deletion	
18	327	480	Schwenk	[27]	deletion	
19	364	512	Schwenk	[27]		256_{16}
20	403	576	Jørgensen	[14]		288_{17}

"Deletion" refers to a standard technique, which has been re–invented several times and described in different languages, see also Section 6.

For $k = 7, 9, 10, 11, 13, 16$, 19, and 20, the girth 5 graphs listed above have a number of vertices which is *just twice the number v of points of some* elliptic semiplane *with $k - r$ points on each line*, namely:

k	7	9	10	11	13	16	19	20
r	2	2	2	2	2	3	3	3
configuration type v_{k-r}	25_5	48_7	63_8	78_9	120_{11}	168_{13}	256_{16}	288_{17}
semiplane type	C	L	L	D	L	L	C	L

In this paper, we convert this observation into a unifying construction principle. We start with Levi graphs of elliptic semiplanes. Construction 2 transforms these n–regular graphs of girth 6 into $(n + r)$–regular graphs: this will be done by *suitably* amalgamating copies of small r–regular graphs Π and Λ of girth ≥ 5. Theorem 7 guarantees that the amalgams have girth 5. Sections 3, 4, and 5 deal with the challenging task of finding such suitable pairs. As to orders, our results tie with the smallest currently known instances and furnish some new examples for $k > 20$.

2 From Semiplanes to Graphs of Girth 5

Recall that an incidence structure $\mathcal{I} = (\mathfrak{P}, \mathfrak{L}, |)$ (in the sense of [7] or [11]) is said to be a *partial plane* if two distinct points are incident with at most one line. A v_k *configuration* or a *configuration of type* v_k is a partial plane consisting of v points and v lines such that each point and each line are incident with k lines and k points, respectively. A finite *elliptic semiplane of order* $k-1$ is a v_k configuration satisfying the following axiom of parallels: for each *anti–flag* $p_1 \nmid l_1$, i. e. a non–incident point line pair (p_1, l_1), there exists at most one line l_2 incident with p_1 and *parallel* to l_1 (i.e. there is no point incident with both l_1 and l_2) and at most one point p_2 incident with l_1 and *parallel* to p_1 (i.e. there is no line incident with both p_1 and p_2). A *Baer subset* of a finite projective plane $\mathcal{P}$ is either a Baer subplane $\mathcal{B}$ or, for a distinguished point–line pair (p_0, l_0), the union $\mathcal{B}(p_0, l_0)$ of all lines and points incident with p_0 and l_0, respectively. We shall write $\mathcal{B}(p_0|l_0)$ or $\mathcal{B}(p_0 \nmid l_0)$, according as $p_0|l_0$ or not. It was already known to Dembowski [7] that elliptic semiplanes are obtained by deleting a Baer subset from a projective plane $\mathcal{P}$. We call any such elliptic semiplane *Desarguesian* if $\mathcal{P}$ is so. Dembowski proved the following partial converse:

1 Theorem. *If $\mathcal{S} = (\mathfrak{P}, \mathfrak{L}, |)$ is an elliptic semiplane of order $\nu = n-1$ (i.e. with $n = \nu + 1$ points on each line), then all the parallel classes in $\mathfrak{P}$ and $\mathfrak{L}$ have the same size, say m. Moreover, m divides $n(n-1)$, the total number of points (lines) is $n(n-1)+m$, and exactly one of the following cases holds true:*

semi–plane type	m	*construction from a projective plane $\mathcal{P}$ of order n*	*configuration type*
(improper)	1	$\mathcal{S} = \mathcal{P}$	$(\nu^2+\nu+1)_{\nu+1}$
C	n	$\mathcal{S} = \mathcal{P} - \mathcal{B}(p_0\|l_0)$	$(n^2)_n$
L	$n-1$	$\mathcal{S} = \mathcal{P} - \mathcal{B}(p_0 \nmid l_0)$	$(n^2-1)_n$
D	$n-\sqrt{n}$	$\mathcal{S} = \mathcal{P} - \mathcal{B}$	$(n^2-\sqrt{n})_n$
B	$< n-\sqrt{n}$		

If $\mathcal{S}$ is proper, parallelism partitions $\mathfrak{P}$ into $\mu := \frac{n(n-1)+m}{m}$ parallel classes $\mathfrak{p}_i$ with $i \in I$, say, and dually $\mathfrak{L}$ into μ parallel classes $\mathfrak{l}_j$ with $j \in I$.

The semiplane types refer to contributions by Cronheim [6], Lüneburg [18], Dembowski [7], and Baker [2]. Dembowski left the existence of elliptic semiplanes of type B as an open problem. In 1977 Baker [2] found such an elliptic semiplane, which has 45 points, order $\nu = 6$, and parallel class size $m = 3$.

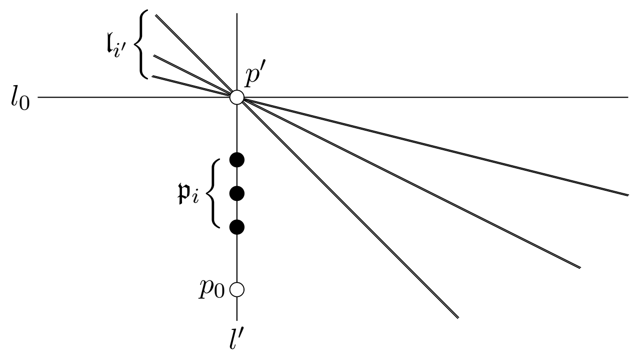
2 Definition. We extend the concept of parallelism in a v_k configuration and call two flags $(p_1 \,|\, l_1)$ and $(p_2 \,|\, l_2)$ with $p_1 \neq p_2$ and $l_1 \neq l_2$ parallel if both $\{p_1, p_2\}$ and $\{l_1, l_2\}$ make up pairs of parallel elements.

3 Lemma. *Let $\mathcal{S}$ be an elliptic semiplane of type C, D, or L. For all $i, j \in I$, the m^2 point–line–pairs $(p, l) \in \mathfrak{p}_i \times \mathfrak{l}_j$ either fall into precisely m pairwise non–parallel flags and $m^2 - m$ anti–flags or all of them are anti–flags.*

Proof. First we show that there are at most m flags in each $\mathfrak{p}_i \times \mathfrak{l}_j$: suppose that $p \,|\, l$ is such a flag; since the points in $\mathfrak{p}_i$ and the lines in $\mathfrak{l}_j$ are parallel in pairs, a second flag $p' \,|\, l'$ in $\mathfrak{p}_i \times \mathfrak{l}_j$ can exist only if $p \neq p'$ and $l \neq l'$; this, in turn, implies that $p \,|\, l$ and $p' \,|\, l'$ are parallel flags; the statement follows by induction on the number of flags.

Now we distinguish three cases: if $\mathcal{S}$ is of type C, we count n^2 points and n^2 lines. Both sets fall into $\mu = n$ parallel classes of $m = n$ elements each. Hence there are n^2 sets $\mathfrak{p}_i \times \mathfrak{l}_j$, each containing at most n flags. On the other hand, the n^2 points of $\mathcal{S}$, each incident with n lines, make a total number of n^3 flags. Thus each set $\mathfrak{p}_i \times \mathfrak{l}_j$ contains exactly n flags.

In an elliptic semiplane $\mathcal{S}$ of type L, the point and line sets have $n^2 - 1$ elements. They are partitioned into $\mu = n + 1$ parallel classes of $m = n - 1$ elements each. Hence there are $(n + 1)^2$ sets $\mathfrak{p}_i \times \mathfrak{l}_j$, each containing at most $n - 1$ flags. Since $\mathcal{S} = \mathcal{P} - \mathcal{B}(p_0 \nmid l_0)$, the points in the parallel class $\mathfrak{p}_i$ are incident with some line l' of $\mathcal{P}$ passing through p_0. The line l' meets l_0 in some point p'. The lines of $\mathcal{P}$ passing through p' other than l_0 make up a parallel class of $\mathcal{S}$, say $\mathfrak{l}_{i'}$. Obviously, there is no flag at all in $\mathfrak{p}_i \times \mathfrak{l}_{i'}$.



Hence, for each parallel class $\mathfrak{p}_i$ of points there is exactly one parallel class $\mathfrak{l}_{i'}$ of lines such that there are m^2 anti–flags in $\mathfrak{p}_i \times \mathfrak{l}_{i'}$. Analogously for each parallel class $\mathfrak{l}_j$ of lines. This implies that there are $(n + 1)^2 - (n + 1) = n^2 + n$ sets $\mathfrak{p}_i \times \mathfrak{l}_j$, each containing at most $n - 1$ flags. On the other hand, the $n^2 - 1$ points of $\mathcal{S}$, each incident with n lines, make a total number of $n^3 - n$ flags. Thus each set $\mathfrak{p}_i \times \mathfrak{l}_j$ with $j \neq i'$ contains exactly $n - 1$ flags, while $\mathfrak{p}_i \times \mathfrak{l}_{i'}$ contains only anti–flags.

If $\mathcal{S}$ is of type D, an analogous reasoning shows that for a fixed parallel class $\mathfrak{p}_i$ of points there are precisely $\sqrt{n}$ parallel classes $\mathfrak{l}_{i_r}$ with $r = 1, \ldots, \sqrt{n}$ such that $\mathfrak{p}_i \times \mathfrak{l}_{i_r}$ contains only anti–flags, while the other sets $\mathfrak{p}_i \times \mathfrak{l}_j$ with $j \neq i_r$ contain $m = n - \sqrt{n}$ flags each. QED

4 Definition. Let $\mathcal{S} = (\mathfrak{P}, \mathfrak{L}, |)$ be an elliptic semiplane with parallel class size m. Fix an m–subset $(G, +)$ of some group $(G', +)$. Extend the labelling for the parallel classes by the set I to a labelling for the elements in $\mathfrak{P}$ and $\mathfrak{L}$ by double indices, say $p_{i,s} \in \mathfrak{p}_i \subseteq \mathfrak{P}$ and $l_{j,t} \in \mathfrak{l}_j \subseteq \mathfrak{L}$ with $s, t \in G$. We will refer to $(i; s)$ and $[j; t]$ as the G–coordinates of $p_{i,s}$ and $l_{j,t}$, respectively. In the case $I = G$, we shall substitute the semicolon with a comma and write (i, s) and $[j, t]$.

5 Corollary. *Being parallel in pairs, the m flags (if any) belonging to $\mathfrak{p}_i \times \mathfrak{l}_j$ induce a permutation*

$$\sigma_{ij} : \begin{cases} G & \longrightarrow \quad G \\ s & \longmapsto \quad t \quad \textit{if and only if} \quad p_{i,s} \,|\, l_{j,t} \end{cases}$$

of the elements in G. QED

Denote by $K_m(G)$ the the complete graph K_m on the vertex set G. Recall that the *Cayley colour* of an edge $\{v, w\}$ in $K_m(G)$ is $\pm(v - w)$.

6 Definition. Let r be a fixed positive integer with $r \leq \frac{m-1}{2}$. A pair of subgraphs Π, Λ of the complete graph $K_m(G)$ on G is said to be suitable (with respect to the permutations σ_{ij}) if

(i) Π and Λ are both r–regular, of order m, and of girth at least 5;

(ii) the Cayley colours of Π and Λ are σ_{ij}–disjoint, i.e. $\{s, v\} \in E(\Pi)$ and $\{t, w\} \in E(\Lambda)$ imply $s^{\sigma_{ij}} - v^{\sigma_{ij}} \neq \pm(t - w)$ for all $i, j \in I$.

The *Levi graph* $\Gamma(\mathcal{S})$ of $\mathcal{S} = (\mathfrak{P}, \mathfrak{L}, |)$ is the graph with vertex set $\mathfrak{P} \cup \mathfrak{L}$, the edges being the flags of $\mathcal{S}$, cf. e.g. [5]. It is well known that $\Gamma(\mathcal{S})$ is an n–regular bipartite graph of girth 6 and order $2m\mu$.

Construction. Let Π and Λ be a pair of suitable subgraphs of $K_m(G)$. Take μ copies of both Π and Λ and label them by the elements of the index set I. Amalgamate the Levi graph $\Gamma(\mathcal{S})$ and the families $\{\Pi_i : i \in I\}$ and $\{\Lambda_j : j \in I\}$ by identifying the following vertices with each other:

$$\begin{array}{ccccccc} \Gamma(\mathcal{S}) & \ni & p_{i,s} & \longleftrightarrow & s & \in & \Pi_i \\ \Gamma(\mathcal{S}) & \ni & l_{j,t} & \longleftrightarrow & t & \in & \Lambda_j \end{array} \quad \text{for all } i, j \in I,\ s, t \in G\,.$$

Denote the resulting amalgam by $\mathcal{S}(\Pi, \Lambda)$.

7 Theorem. *The amalgam $\mathcal{S}(\Pi, \Lambda)$ is an $(n + r)$–regular simple graph of girth 5 and order $2\mu m$.*

Proof. The amalgam is a simple graph since the additional edges arising from the families $\{\Pi_i : i \in I\}$ and $\{\Lambda_j : j \in I\}$ connect vertices belonging to one and the same bipartition class of $\Gamma(\mathcal{S})$. Degree and order of the amalgam can easily be checked. The amalgamation cannot produce 3–cycles; 4–cycles, however, might come into being.

So we have to show that this does not happen. Any two distinct vertices $p_{i,s}$ and $p_{i',v}$ of $\Gamma(\mathcal{S})$ are connected by some edge of Π_i only if $i' = i$, i.e. they arise from two points belonging to the same pencil $\mathfrak{p}_i$. Parallel points of $\mathcal{S}$ give rise to vertices at distance 4 from each other in the Levi graph $\Gamma(\mathcal{S})$ since there exist lines, say l and l', intersecting in some point p'' of $\mathcal{S}$ such that

$$p_{i,s},\ l,\ p'',\ l',\ p_{i,v}$$

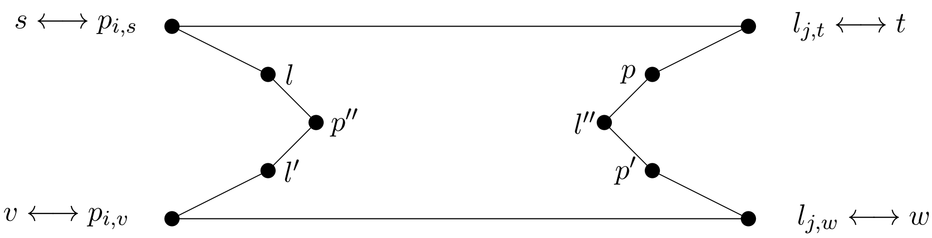
is a shortest path from $p_{i,s}$ to $p_{i,v}$ in $\Gamma(\mathcal{S})$. If s and v are joined by an edge in Π_i, we obtain the 5–cycle

$$p_{i,s},\ l,\ p'',\ l',\ p_{i,v} \longleftrightarrow v,\ s \longleftrightarrow p_{i,s}$$

in $\mathcal{S}(\Pi, \Lambda)$. Analogously, any two distinct vertices $l_{j,t}$ and $l_{j',w}$ of $\Gamma(\mathcal{S})$ are connected by some edge of Λ_j only if $j' = j$, i.e. they arise from two lines belonging to the same pencil $\mathfrak{l}_j$. A dual argument as above works for the vertices $l_{j,t}$ and $l_{j,w}$, eventually giving rise to a 5–cycle

$$l_{j,t},\ p,\ l'',\ p',\ l_{j,w} \longleftrightarrow w,\ t \longleftrightarrow l_{j,t}$$

in $\mathcal{S}(\Pi, \Lambda)$.



If $p_{i,s} \mid l_{j,t}$ and $p_{i,v} \mid l_{j,w}$, Corollary 5 implies $s^{\sigma_{ij}} = t$ as well as $v^{\sigma_{ij}} = w$, i.e. $s^{\sigma_{ij}} - v^{\sigma_{ij}} = t - w$. Since Π and Λ make up a suitable pair with respect to σ_{ij}, the edge $\{s, v\}$ can become an edge of Π, only if $\{t, w\}$ does not appear as an edge of Λ, and analogously, $\{t, w\}$ can become an edge of Λ, only if $\{s, v\}$ does not appear as an edge of Π. Thus the amalgam does not contain 4–cycles. QED

The following three Sections (one for each type of elliptic semiplanes) will deal with the challenging task of finding such suitable pairs.

3 Elliptic Semiplanes of Type C

In this Section, we use non–homogeneous coordinates over some algebraic structure such that lines are given by equations $y = x \cdot a + b$. Typically we may choose quasifields. Under this rather general hypothesis, Construction 2 yields several non–isomorphic graphs with the same parameters $k = n + r$ and $2\,\mu\, m$.

Let $\mathcal{C} = (\mathfrak{P}, \mathfrak{L}, |)$ be an elliptic semiplane of type C obtained from a translation plane $\mathcal{T}$ over a quasifield $(\mathfrak{Q}, +, \cdot)$ of order a prime power $n = q$ by deleting a Baer subset $\mathcal{B}(p|l)$. Introduce non–homogeneous coordinates in $\mathcal{T}$, following Hall's method ([11], see also [7]) such that $p = (\infty)$ and $l = [\infty]$. Then the points and lines of $\mathcal{C}$ have coordinates (a, b) and $[\alpha, \beta]$, respectively, with $a, b, \alpha, \beta \in \mathfrak{Q}$, and incidence is given by the rule

$$(a, b) \,|\, [\alpha, \beta] \quad \text{if and only if} \quad a \cdot \alpha + \beta = b\,.$$

Two points or two lines of $\mathcal{C}$ are parallel if and only if their first coordinates coincide: in $\mathcal{T}$, two distinct points $(a, b), (a, b')$ are joined by the line $[a]$ and two distinct lines $[\alpha, \beta], [\alpha, \beta']$ meet in the point (α), both belonging to $\mathcal{B}(p|l)$. Hence

$$\mathfrak{p}_a := \{p_{a,b} = (a, b) \,:\, b \in \mathfrak{Q}\} \quad \text{and} \quad \mathfrak{l}_\alpha := \{l_{\alpha,\beta} = [\alpha, \beta] \,:\, \beta \in \mathfrak{Q}\}$$

are the pencils of pairwise parallel points and lines, respectively, and we may choose $I := \mathfrak{Q}$ as well as $(G, +) := (\mathfrak{Q}, +)$.

8 Proposition. *Let r be a positive integer with $r \leq \frac{q-1}{2}$. Let Π and Λ be two subgraphs of $K_q(\mathfrak{Q})$, which are both r–regular, of order q, and of girth at least 5. Then Π and Λ are suitable if they have disjoint Cayley colours, i. e. $\{a, b\} \in E(\Pi)$ and $\{c, d\} \in E(\Lambda)$ always imply $a - b \neq \pm(c - d)$.*

Proof. The rule characterizing incidence in terms of the above coordinates implies that, for all $a, \alpha \in \mathfrak{Q}$, the permutation $\sigma_{a,\alpha}$ acts by (right) addition (say):

$$\sigma_{a,\alpha} : \begin{cases} \mathfrak{Q} & \longrightarrow \quad \mathfrak{Q} \\ b & \longmapsto \quad \beta = b - a \cdot \alpha \end{cases}$$

Hence $\sigma_{a,\alpha}$ leaves the Cayley colours of the edges of $K_q(\mathfrak{Q})$ invariant, i.e.

$$v^{\sigma_{a,\alpha}} - w^{\sigma_{a,\alpha}} = v - a \cdot \alpha - w + a \cdot \alpha = v - w$$

for all distinct $v, w \in K_q(\mathfrak{Q})$. Thus "$\sigma_{a,\alpha}$–disjoint Cayley colours" mean just "disjoint Cayley colours." QED

9 Remark. Construction 2 furnishes k–regular graphs of girth 5, some of whose orders tie with or even improve the known upper bounds $rec(k, 5)$:

k	q	r	order of $\mathcal{C}(\Pi, \Lambda)$	known upper bound	first constructed by	reference(s)
7	5	2	**50**	**50**	Hoffman & Singleton	[12], Ex. 10
9	7	2	98	**96**	Jørgensen	[14]
10	8	2	128	**126**	Exoo	[10]
11	9	2	162	**124**	Jørgensen	[14]
13	11	2	242	**240**	Exoo	[10]
15	13	2	338	**230**	Jørgensen	[14]
19	16	3	**512**	**512**	Schwenk	[27], Ex. 11
19	17	2	578	**512**	Schwenk	[27]
21	19	2	722	**684**	Jørgensen	[14]
36	32	4	**2048**	2448	new	Ex. 12

In the third column, r indicates the (highest) feasible degrees for suitable graphs Π and Λ of girth ≥ 5 on q vertices. Graphs of degree an odd number have even order. This well known fact gives rise to a handicap: an odd value for r is eligible only if q is even. For $q = 32$, one might think of $r = 5$, but the feasibility of $\mathcal{C}(\Pi, \Lambda)$ remains an open problem.

For the following examples, we chose $\mathfrak{Q}$ to be the finite field $\mathbb{F}_q$ of prime power order $q \geq 5$.

10 Example. Solutions for $r = 2$ and the prime numbers $q = 5, 7, 11, 13, 17, 19$ are quite obvious: $(\mathbb{F}_q, +)$ is cyclic and we can choose Π and Λ to be the q–cycles with edge sets

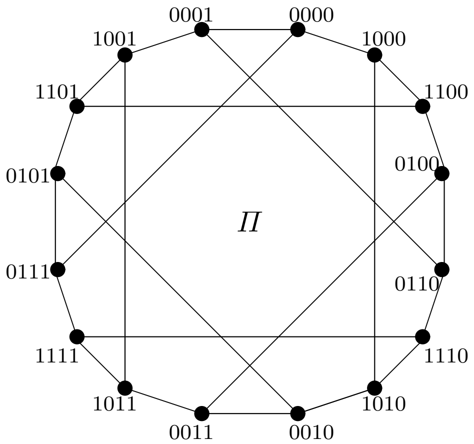
$$E(\Pi) = \{\{i, i+1\} \, : \, i \in \mathbb{F}_q\} \quad \text{and} \quad E(\Lambda) = \{\{i, i+2\} \, : \, i \in \mathbb{F}_q\}\,,$$

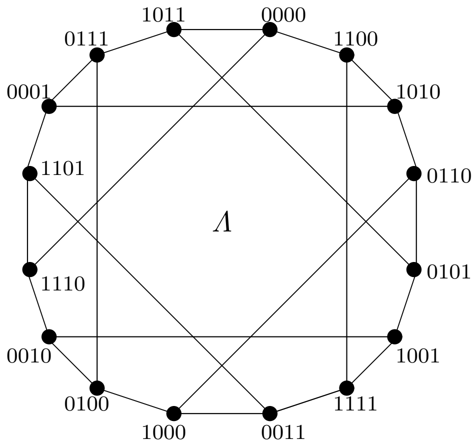
made up by edges of Cayley colours ± 1 and ± 2, respectively.

11 Example. Let $r = 3$ and $q = 16$. Denote the elements of $(\mathbb{F}_{16}, +) \cong ((\mathbb{F}_2)^4, +)$ by $defg$ instead of (d, e, f, g) where $d, e, f, g \in \mathbb{F}_2$. We take over an idea of Schwenk's [27] (cf. also [9, p. 39]). We choose the following two copies Π and Λ of the so–called Möbuis–Kantor graph (i.e. the Levi graph of the unique 8_3 configuration) as cubic subgraphs of $K_{16}((\mathbb{F}_2)^4)$. Being Levi graphs, both Π and Λ have girth 6. The Cayley colours of Π and Λ lie in

$$\{1000, 0100, 0010, 0001, 0111\} \quad \text{and} \quad \{1100, 0110, 0011, 1011, 1110\}\,,$$

respectively.





12 Example. Let $r = 4$ and $q = 32$. As before, denote the elements of $(\mathbb{F}_{32}, +) \cong ((\mathbb{F}_2)^5, +)$ by $defgh$ instead of (d, e, f, g, h). A suitable pair Π and Λ of subgraphs in $K_{32}((\mathbb{F}_2)^5)$ can be constructed as follows. Choose Π to be the Levi graph of the elliptic semiplane $\mathcal{C}_{16}$ of type C on 16 vertices given by entries **1** in the following incidence table M. This table can be found in [1, p. 182]: the transformation of coordinates

$$(a, b) \quad \text{and} \quad [\alpha, \beta] \quad \text{with} \quad a, b, \alpha, \beta \in \mathbb{F}_4 = \{0, 1, x, \overline{x} = x + 1\}$$

into elements of $\mathbb{F}_{32} = \{defgh \,:\, d, e, f, g, h \in \mathbb{F}_2\}$ is given by the rules

$$(e1 + fx, g1 + hx) = 0efghy \quad \text{and} \quad [e1 + fx, g1 + hx] = 1efgh\,.$$

	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0 0 0 0 0	**1**	*o*	*o*	*o*	**1**	*o*	**0**	*o*	**1**	**0**	*o*	*o*	**1**	**0**	*o*	*o*
0 0 0 1 0	*o*	**1**	*o*	*o*	*o*	**1**	*o*	**0**	**0**	**1**	*o*	*o*	**0**	**1**	*o*	*o*
0 0 0 0 1	*o*	*o*	**1**	*o*	**0**	*o*	**1**	*o*	*o*	*o*	**1**	**0**	*o*	*o*	**1**	**0**
0 0 0 1 1	*o*	*o*	*o*	**1**	*o*	**0**	*o*	**1**	*o*	*o*	**0**	**1**	*o*	*o*	**0**	**1**
0 1 0 0 0	**1**	*o*	**0**	*o*	*o*	**1**	*o*	*o*	*o*	*o*	**1**	**0**	*o*	**0**	*o*	**1**
0 1 0 1 0	*o*	**1**	*o*	**0**	**1**	*o*	*o*	*o*	*o*	*o*	**0**	**1**	**0**	*o*	**1**	*o*
0 1 0 0 1	**0**	*o*	**1**	*o*	*o*	*o*	*o*	**1**	**1**	**0**	*o*	*o*	*o*	**1**	*o*	**0**
0 1 0 1 1	*o*	**0**	*o*	**1**	*o*	*o*	**1**	*o*	**0**	**1**	*o*	*o*	**1**	*o*	**0**	*o*
0 0 1 0 0	**1**	**0**	*o*	*o*	*o*	*o*	**1**	**0**	*o*	*o*	*o*	**1**	*o*	**1**	*o*	**0**
0 0 1 1 0	**0**	**1**	*o*	*o*	*o*	*o*	**0**	**1**	*o*	*o*	**1**	*o*	**1**	*o*	**0**	*o*
0 0 1 0 1	*o*	*o*	**1**	**0**	**1**	**0**	*o*	*o*	*o*	**1**	*o*	*o*	*o*	**0**	*o*	**1**
0 0 1 1 1	*o*	*o*	**0**	**1**	**0**	**1**	*o*	*o*	**1**	*o*	*o*	*o*	**0**	*o*	**1**	*o*
0 1 1 0 0	**1**	**0**	*o*	*o*	*o*	**0**	*o*	**1**	*o*	**1**	*o*	**0**	*o*	*o*	**1**	*o*
0 1 1 1 0	**0**	**1**	*o*	*o*	**0**	*o*	**1**	*o*	**1**	*o*	**0**	*o*	*o*	*o*	*o*	**1**
0 1 1 0 1	*o*	*o*	**1**	**0**	*o*	**1**	*o*	**0**	*o*	**0**	*o*	**1**	**1**	*o*	*o*	*o*
0 1 1 1 1	*o*	*o*	**0**	**1**	**1**	*o*	**0**	*o*	**0**	*o*	**1**	*o*	*o*	**1**	*o*	*o*

It is usually formulated as an exercise to show that the block matrix $\left(\begin{smallmatrix} 0 & M \\ M^T & 0 \end{smallmatrix}\right)$ is an adjacency matrix for the Levi graph $\Pi := \Gamma(\mathcal{C}_{16})$ of girth 6. The Cayley colours of Π lie in

$$\{10000, 10001, 10010, 10011, 10100, 10111, 11000, 11010, 11100, 11101\}\,.$$

To construct Λ, we start with the 16_3 configuration $\mathcal{A}$ whose incidence matrix is obtained from the above table by substituting 1 for **0** and 0 for all the other entries (**1** or o), respectively. The Levi graph $\Gamma(\mathcal{A})$ is a cubic bipartite graph of girth 6 and Cayley colours belonging to

$$\{10110, 11001, 11011, 11110, 11111\}\,.$$

Then Λ is obtained from $\Gamma(\mathcal{A})$ by adding 16 further edges, namely the ones joining the first and second, the third and fourth, ..., the 15^{th} and 16^{th} vertices of type $0efgh$ on the one hand, and the first and forth, the second and third, the fifth and eighth, ..., the 13^{th} and 16^{th}, the 14^{th} and 15^{th} vertices of type $1efgh$ on the other hand. The additional edges have Cayley colours in $\{00010, 00011\}$. A computer verification (using [16]) shows that Λ is a rigid 4–regular graph of girth 5, whose edge set partitions into two Hamilton cycles.

4 Elliptic Semiplanes of Type L

In this Section, only Desarguesian semiplanes come into play since the application of Construction 2 fully relies on the facilities offered by homogeneous coordinates and the cyclic structure of the multiplicative group of finite fields. It will be convenient to identify the multiplicative group $\mathbb{F}_q^*$ with the additive group $\mathbb{Z}_{q-1}$ by the isomorphism

$$\iota \quad : \quad \begin{cases} \mathbb{F}_q^* & \longrightarrow \quad \mathbb{Z}_{q-1} \\ \epsilon^z & \longmapsto \quad z \end{cases}$$

for some fixed generator $\epsilon \in \mathbb{F}_q^*$. The projective line $PG(1,q)$ is represented by $\mathbb{F}_q \cup \{\infty\}$.

13 Lemma. *Let $\mathcal{L} = (P, L, |)$ be an elliptic semiplane of type L obtained from a Desarguesian projective plane $\mathcal{P}$ over a field $\mathbb{F}_q$ by deleting a Baer subset $\mathcal{B}(p \nmid l)$. Then points and lines are uniquely determined by* polar coordinates

$$(a; b) \;\; \textit{with} \;\; a \in \mathbb{F}_q \cup \{\infty\}\,,\; b \in \mathbb{Z}_{q-1}$$

and

$$[\alpha; \beta] \;\; \textit{with} \;\; \alpha \in \mathbb{F}_q \cup \{\infty\}\,,\; \beta \in \mathbb{Z}_{q-1}$$

respectively. Incidence is given by the rule:

$$(a;b) \mid [\alpha;\beta] \quad \textit{if and only if} \quad \epsilon^{\beta+b} = c_{a,\alpha} := \begin{cases} -\alpha & \textit{if } a=\infty, \alpha \neq \infty \\ -a & \textit{if } \alpha = \infty, a \neq \infty \\ -1 & \textit{if } \alpha = a = \infty \\ -1-\alpha a & \textit{otherwise} \end{cases}$$

Two points and two lines of $\mathcal{L}$ are parallel if and only if their first polar coordinates coincide.

PROOF. Introduce homogeneous coordinates in $\mathcal{P}$ such that $p \equiv (0:0:1)$ and $l = [0:0:1]$. Then the points of $\mathcal{L}$ are exactly the affine points of $\mathcal{P}$ other than the origin. Normalize either the second or the first coordinate to be 1 according as the first coordinate is zero or not. Thus we obtain

$$\{(0:1:c)\ :\ c \in \mathbb{F}_q^*\} \ \cup\ \{(1:a:c)\ :\ (a,c) \in \mathbb{F}_q^2 \text{ with } c \neq 0\}$$

as point set of $\mathcal{L}$. The lines of $\mathcal{L}$ are those affine lines of $\mathcal{P}$ whose affine equations read either $y = \alpha' x + \beta'$ with $\beta' \neq 0$ or $x = \mu'$ with $\mu' \neq 0$. In terms of homogeneous coordinates, these lines become either $[\alpha' : -1 : \beta']$ or $[-1 : 0 : \mu']$. Again we can normalize either the second or the first coordinate to be 1 according as the first coordinate is zero or not, and obtain

$$\{[0:1:\gamma]\ :\ \gamma \in \mathbb{F}_q^*\} \ \cup\ \{[1:\alpha:\gamma]\ :\ (\alpha,\gamma) \in \mathbb{F}_q^2 \text{ with } \gamma \neq 0\}$$

as line set of $\mathcal{L}$. Since the third coordinate is never 0, it can be written as a power of the generator ϵ. A 1-1 correspondence between homogeneous and polar coordinates is given by the following rules:

$$\begin{array}{ccccccc} (0:1:\epsilon^b) & \longleftrightarrow & (\infty;b) & \text{and} & (1:a:\epsilon^b) & \longleftrightarrow & (a;b) \\ [0:1:\epsilon^\beta] & \longleftrightarrow & [\infty;\beta] & \text{and} & [1:\alpha:\epsilon^\beta] & \longleftrightarrow & [\alpha;\beta] \end{array}$$

In terms of homogeneous coordinates, incidence holds if the usual dot product of the coordinates is zero; hence

$$\begin{array}{rcccl} (a;b) & \mid & [\alpha;\beta] & \Longleftrightarrow & \epsilon^{\beta+b} = -1-\alpha a \\ (a;b) & \mid & [\infty;\beta] & \Longleftrightarrow & \epsilon^{\beta+b} = -a \\ (\infty;b) & \mid & [\alpha;\beta] & \Longleftrightarrow & \epsilon^{\beta+b} = -\alpha \\ (\infty;b) & \mid & [\infty;\beta] & \Longleftrightarrow & \epsilon^{\beta+b} = -1 \end{array}$$

Two points and two lines of $\mathcal{L}$ are parallel if and only if their first polar coordinates coincide. In fact, in two distinct points $(a:b:1)$ and $(\lambda a : \lambda b : 1)$ are joined by a line of $\mathcal{P}$ through the origin $(0:0:1)$; two lines $[\alpha:\beta:1]$ and $[\alpha':\beta':1]$ of $\mathcal{L}$ are parallel if and only if they meet in some point on $l \equiv [0:0:1]$, say $(x:y:0)$, and one obtains $\alpha x + \beta y = \alpha' x + \beta' y$, i.e. $(\alpha:\beta) = (\alpha':\beta')$. QED

Hence

$$\mathfrak{p}_a := \{(a;b) \,:\, b \in \mathbb{Z}_{q-1}\} \quad \text{and} \quad \mathfrak{l}_\alpha := \{[\alpha;\beta] \,:\, \beta \in \mathbb{Z}_{q-1}\}\,,$$

are hyperpencils of pairwise parallel points and lines, respectively. Choose $I := \mathbb{F}_q \cup \{\infty\}$ as convenient index set, as well as $(G,+) := (\mathbb{Z}_{q-1},+)$. Next we formulate and prove the following analogue of Proposition 8.

14 Proposition. *Denote by $K(\mathbb{Z}_{q-1})$ the complete graph on the vertex set $\mathbb{Z}_{q-1}$. Let r be a positive integer with $r \le \frac{q-2}{2}$. Let Π and Λ be two subgraphs of $K(\mathbb{Z}_{q-1})$, which are both r–regular, of order $q-1$, and of girth at least 5. Then Π and Λ are suitable if they have disjoint Cayley colours.*

Proof. First determine the pairs (a,α) for which $\mathfrak{p}_a \times \mathfrak{l}_\alpha$ contains only anti–flags. This happens if and only if $c_{a,\alpha} = 0$ and the equation for incidence has no solution. These pairs are $(a,\alpha) = (0,\infty)$ or $(\infty,0)$ or $(a,-a^{-1})$ with $a \in \mathbb{F}_q^*$. In the remaining cases, the rule characterizing incidence in terms of polar coordinates implies $\epsilon^{\beta+b} = \epsilon^\beta\,\epsilon^b = c_{a,\alpha}$, or, equivalently,

$$\beta = \iota(\epsilon^\beta) = \iota(c_{a,\alpha}\epsilon^{-b}) = \iota(c_{a,\alpha}) + \iota(\epsilon^{-b}) = \iota(c_{a,\alpha}) - b$$

. Hence $\mathfrak{p}_a \times \mathfrak{l}_\alpha$ gives rise to the following permutation:

$$\sigma_{a,\alpha} : \begin{cases} \mathbb{Z}_{q-1} & \longrightarrow \quad \mathbb{Z}_{q-1} \\ b & \longmapsto \quad \beta = \iota(c_{a,\alpha}) - b \end{cases}$$

This mapping leaves the Cayley colours of the edges of $K(\mathbb{Z}_{q-1})$ invariant since

$$v^{\sigma_{a,\alpha}} - w^{\sigma_{a,\alpha}} = \iota(c_{a,\alpha}) - v - (\iota(c_{a,\alpha}) - w) = -(v-w)$$

for all distinct $v,w \in K(\mathbb{Z}_{q-1})$. Thus "$\sigma_{a,\alpha}$–disjoint Cayley colours" again mean "disjoint Cayley colours." QED

15 Remark. Construction 2 furnishes k–regular graphs of girth 5, some of whose orders tie with or even improve the known upper bounds $rec(k,5)$:

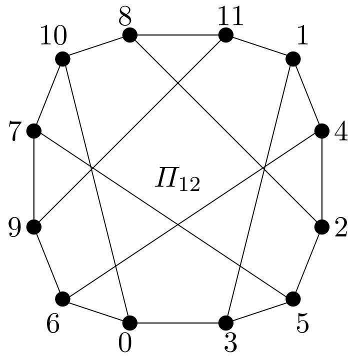
k	q	r	order of $\mathcal{L}(\Pi, \Lambda)$	smallest currently known order	first constructed by	reference(s)
9	7	2	**96**	**96**	Jørgensen	[14], Ex. 16
11	9	2	160	**156**	Jørgensen	[14]
13	11	2	**240**	**240**	Exoo	[10], Ex. 16
16	13	3	**336**	**336**	Jørgensen	[14], Ex. 17
18	16	2	510	**480**	Schwenk	[27]
20	17	3	**576**	**576**	Jørgensen	[14], Ex. 18
22	19	3	**720**	**720**	Jørgensen	[14], Ex. 18
27	23	4	**1056**	1200	new	Ex. 18
29	25	4	**1248**	1404	new	Ex. 18
31	27	4	**1456**	1624	new	Ex. 18

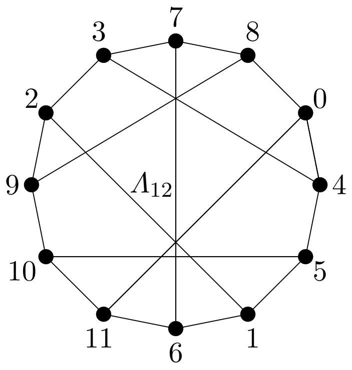
In the third column, r indicates the (highest) feasible degrees for suitable graphs Π and Λ of girth ≥ 5 on $q-1$ vertices. The handicap described in Remark 9 here affects the choice of r if q is an even prime power. Thus oddly regular graphs of girth at least 5 are eligible for all odd prime powers q. For $r = 3$ and $q = 11$ one might think of two copies of the Petersen graph for Π and Λ but any embedding of the first copy into $K_{10}(\mathbb{Z}_{10})$ already absorbs at least four Cayley colours out of five. Hence this idea is not feasible. For $r = 5$ and $q = 31$, an analogous idea would assign two $(5, 5)$–cages as Π and Λ, to be embedded into $K_{30}(\mathbb{Z}_{30})$ with disjoint Cayley colours. Its feasibility remains an open problem.

16 Example. For $r = 2$ and $q = 7, 11$ we choose Π and Λ to be the following $(q-1)$–cycles:

$q-1$	$E(\Pi)$	Cayley colours	$E(\Lambda)$	Cayley colours
6	$\{\{i, i+1\} : i \in \mathbb{Z}_6\}$	± 1	$\{\{0,3\},\{3,1\},\{1,5\},$ $\{5,2\},\{2,4\},\{4,0\}\}$	$\pm 2, \pm 3$
10	$\{\{i, i+1\} : i \in \mathbb{Z}_{10}\}$	± 1	$\{\{i, i+3\} : i \in \mathbb{Z}_{10}\}$	± 3

17 Example. A solution for $r = 3$ and $q = 13$ is due to Jørgensen [14], who pointed out that the two non–isomorphic cubic graphs of girth 5 and order 12 can be embedded into $K_{12}(\mathbb{Z}_{12})$ using disjoint Cayley colours, namely $\{\pm 2, \pm 3, 6\}$ and $\{\pm 1, \pm 4, \pm 5\}$ for Π_{12} and Λ_{12}, respectively:





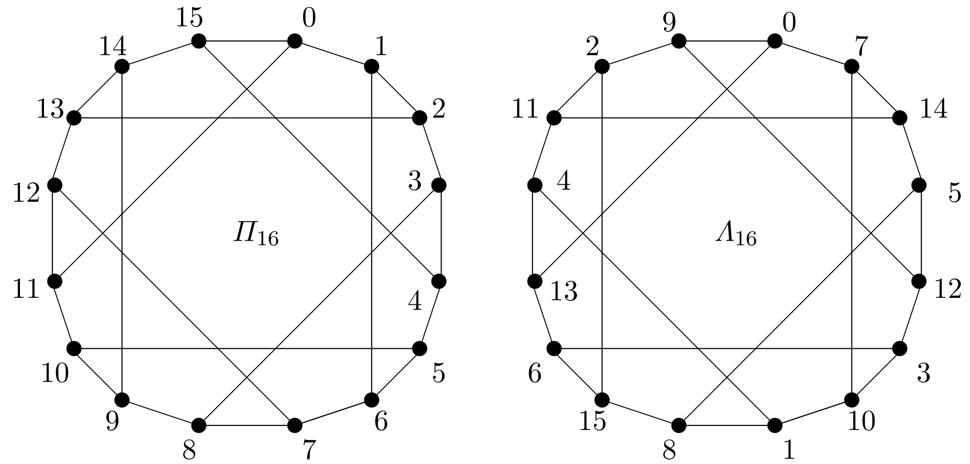
Recall that the *generalized Petersen graph* $P(\kappa, \mu)$ is defined as the cubic graph on 2κ vertices u_i, v_i with edges $\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+\mu}\}$, indices taken modulo κ, cf. e. g. [13]. Extent this notion and denote by $P(\kappa, \mu\,; \nu)$ the 4–regular graph obtained from $P(\kappa, \mu)$ by adding the edges $\{u_i, v_{i+\nu}\}$.

18 Example. Some suitable pairs of graphs Π and Λ with $r \geq 3$ and $q \geq 17$ are listed in the following table.

$q-1$	r	graph	edges (numbers taken mod $q-1$)	Cayley colours
16	3	Π_{16}	$\{i, i+1\}, \{2i, 2i-5\}$	$\pm1, \pm5$
		Λ_{16}	$\{i, i+7\}, \{2i, 2i-3\}$	$\pm3, \pm7$
18	3	Π_{18}	$\{i, i+1\}, \{2i, 2i+5\}$	$\pm1, \pm5$
		Λ_{18}	$\{i, i+9\}, \{2i, 2i+7\},$	$\pm7, \pm9$
			$\{4i, 4i+3\}, \{4i+2, 4i+5\}$	±3
22	4	Π_{22}	$\{2i, 2i+1\}, \{2i, 2i+2\},$	$\pm1, \pm2$
			$\{2i+1, 2i+6\}, \{2i+1, 2i+11\}$	$\pm5, \pm10$
		Λ_{22}	$\{2i, 2i+4\}, \{2i, 2i+7\}$	$\pm4, \pm7$
			$\{2i, 2i+9\}, \{2i+1, 2i+9\}$	$\pm8, \pm9$
24	4	Π_{24}	$\{2i, 2i+1\}, \{2i, 2i+2\}$	$\pm1, \pm2$
			$\{2i+1, 2i+6\}, \{2i+1, 2i+11\}$	$\pm5, \pm10$
		Λ_{24}	$\{3i, 3i+3\}$	±3
			$\{3i+1, 3i+8\}, \{3i+2, 3i+10\}$	$\pm7, \pm8$
			$\{3i, 3i\pm11\}, \{3i+2, 3i+13\}$	±11
26	4	Π_{26}	$\{i, i+1\}, \{2i, 2i+7\}, \{2i, 2i+11\}$	$\pm1, \pm7, \pm11$
		Λ_{26}	$\{i, i+5\}, \{2i, 2i+3\}, \{2i, 2i+9\}$	$\pm3, \pm5, \pm9$

$\Pi_{16} \cong \Lambda_{16} \cong \Gamma(8_3)$ is again the Möbius–Kantor graph (cf. the Figure below), while $\Pi_{18} \cong \Lambda_{18}$ is the Levi graph of the cyclic 9_3 configuration. In terms of generalized Petersen graphs, one has $\Pi_{22} \cong \Lambda_{22} \cong P(11, 5\,; 3)$ as well as

$\varPi_{24} \cong \varLambda_{24} \cong P(12, 5\,; 3)$. Finally, $\varPi_{26} \cong \varLambda_{26}$ is isomorphic to the Levi graph of the projective plane $PG(2, 3)$.



5 Elliptic Semiplanes of Type D

In this Section we discuss two constructions working in *Hughes planes* over the regular nearfields $N(2, 3)$ and $N(2, 5)$ of orders 9 and 25. Lacking an analogue of Propositions 8 and 14 for elliptic semiplanes of type D, we shall individually determine a subgraph $\varPi$ invariant under each permutation $\sigma_{a,\alpha}$ and look for a suitable subgraph $\varLambda$ in the complement of $\varPi$. These constructions will furnish k–regular graphs of girth 5, whose orders tie with or even improve the known upper bounds $rec(k, 5)$:

k	q^2	r	order of $\mathcal{D}(\varPi, \varLambda)$	smallest currently known order	first constructed by	reference(s)
11	9	2	**156**	**156**	Jørgensen	[14], Ex. 5.4
28	25	3	**1240**	1248	new	Ex. 5.5

The case $q^2 = 9$ will be preceded by a general construction of G–coordinates, where $G := N(2, q)\backslash\mathbb{F}_q$ is the subset of "imaginary" elements in the nearfield. In the case $q^2 = 25$, we adopt Room's somewhat different approach to obtain G–coordinates and use his incidence table WII(25), see [25, p. 301].

Let $\mathfrak{N} = N(2, q)$ be the regular nearfield of (odd) order q^2 (cf. e.g. [7, p. 34]): $\mathfrak{N}$ is obtained by taking the elements of the finite field $\mathbb{F}_{q^2}$, using the field addition, and defining a new multiplication in terms of the field multiplication:

$$x \cdot y = \begin{cases} x\,y & \text{if } y \text{ is a square in } \mathbb{F}_{q^2}\,; \\ x^q\,y & \text{otherwise.} \end{cases}$$

In $(\mathfrak{N}, +, \cdot)$, the non–zero elements make up a group under $\cdot$ and the right distributive law holds, i.e.

$$(a+b)\cdot c = a\cdot c + b\cdot c\,.$$

The centre and kernel of $\mathfrak{N}$ is the field $\mathbb{F}_q$. The automorphism group of $N(2,3)$ is the symmetric group S_3, which is sharply transitive on the elements not belonging to the kernel $\mathbb{F}_3$ of $N(2,3)$. If $q^2 \neq 9$, the automorphism group of $\mathfrak{N} = N(2,p^d)$ is cyclic of order dividing $2d$ (see e.g. [7, p. 229]).

The points of the *Hughes plane* $\mathcal{H}_{q^2}$ of order q^2 are the equivalence classes $(x : y : z)$ of 3–tuples in $\mathfrak{N}^3$ with $(x,y,z) \neq (0,0,0)$ under the equivalence relation

$$(x,y,z) \equiv (x',y',z') \quad \text{if and only if} \quad \begin{array}{l}(x',y',z') = (x\cdot t, y\cdot t, z\cdot t)\\ \text{for some } t \in \mathfrak{N} \text{ with } t \neq 0\,.\end{array}$$

The points $(x : y : z)$ with $x,y,z \in \mathbb{F}_q$ make up a Desarguesian Baer subplane $\mathcal{B}$ of order q and will be referred to as *central* points of $\mathcal{H}_{q^2}$. The set

$$l_\beta := \{(x : y : z) \;:\; x + \beta\cdot y + z = 0\}$$

is said to be a *special line* of $\mathcal{H}_{q^2}$ if $\beta = 1$ or $\beta \notin \mathbb{F}_q$. Choose a Singer matrix S for the Baer subplane. Then the set of all the lines of $\mathcal{H}_{q^2}$ is

$$\{l_\beta\, S^\alpha \;:\; \alpha \in \mathbb{Z}_{q^2+q+1}\,,\ \beta = 1 \text{ or } \beta \notin \mathbb{F}_q\}\,.$$

Incidence is defined by set theoretic inclusion. Thus the lines are uniquely determined by the coordinates $[\alpha;\beta]$ with $\alpha \in \mathbb{Z}_{q^2+q+1}$ and $\beta = 1$ or $\beta \notin \mathbb{F}_q$. The *central* lines (belonging to the Baer subplane) are those with $\beta = 1$. The non–central points incident with the special line l_1 are the points $(b : 1 : -1-b)$ with $b \notin \mathbb{F}_q$. The orbits

$$\{(b : 1 : -1-b)\, S^a \;:\; a \in \mathbb{Z}_{q2+q+1}\}$$

of these points partition the set of non–central points into $q^2 - q$ subsets of size q^2+q+1 each. The central points make up one further orbit, namely

$$\{(1 : 1 : -2)\, S^a \;:\; a \in \mathbb{Z}_{q^2+q+1}\}\,.$$

Hence each point is uniquely determined by the coordinates $(a;b)$ with $a \in \mathbb{Z}_{q^2+q+1}$ and $b = 1$ or $b \notin \mathbb{F}_q$.

19 Lemma. *Incidence is given by the following rule:*

$$(a;b) \,|\, [\alpha;\beta] \iff (b,1,-1-b)\, S^{a-\alpha} \equiv \begin{cases} (x,1,-x-\beta) \;\; \textit{for some} \;\; x \notin \mathbb{F}_q \\ \quad \textit{or} \\ (1,0,-1) \end{cases}$$

Proof. $(a;b) \,|\, [\alpha;\beta]$ holds if and only if $(b,1,-1-b)\, S^a \in l_\beta\, S^\alpha$ or, equivalently, $(b,1,-1-b)\, S^{a-\alpha} =: (\xi,\eta,\zeta) \in l_\beta$. By definition, this holds if and only if $\xi + \beta \cdot \eta + \zeta = 0$. Now distinguish two cases: if $\eta \neq 0$, we conclude

$$(b,1,-1-b)\, S^{a-\alpha} = (\xi,\eta,-\xi-\beta\cdot\eta) \equiv (\xi\cdot\eta',1,-\xi\cdot\eta'-\beta)$$

where $\eta\cdot\eta' = 1$ and the statement follows if we put $x := \xi\cdot\eta'$; if $\eta = 0$, one has

$$(b,1,-1-b)\, S^{a-\alpha} = (\xi,0,-\xi) \equiv (1,0,-1)\,.$$

QED

Let $\mathcal{H}^D_{q^2}$ be the elliptic semiplane of type D obtained from $\mathcal{H}_{q^2}$ by deleting the central Baer subplane $\mathcal{B}$. In terms of the above coordinates, this means just to exclude the options $b = 1$ and $\beta = 1$. Hence the points and lines of $\mathcal{H}^D_{q^2}$ have $\mathfrak{N}\backslash\mathbb{F}_q$–coordinates $(a;b)$ and $[\alpha;\beta]$ with $a,\alpha \in \mathbb{Z}_{q^2+q+1}$ and $b,\beta \in \mathfrak{N}\backslash\mathbb{F}_q$, incidence being given by

$$(a;b) \,|\, [\alpha;\beta] \iff (b,1,-1-b)\, S^{a-\alpha} \equiv (x,1,-x-\beta) \;\text{ for some }\; x \notin \mathbb{F}_q\,.$$

Two points and two lines are parallel if and only if their first $\mathfrak{N}\backslash\mathbb{F}_q$–coordinates coincide: in $\mathcal{H}_{q^2}$, two distinct points $(a;b),(a;b')$ are joined by the central line $l_1\, S^a$ with coordinates $[a;1]$ and two distinct lines $[\alpha;\beta],[\alpha;\beta']$ meet in the central point $(1:0:-1)\, S^\alpha \equiv (1:1:-2)\, S^{\alpha+\gamma}$ with coordinates $(\alpha+\gamma;1)$ for some $\gamma \in \mathbb{Z}_{q^2+q+1}$.

Hence

$$\mathfrak{p}_a := \{(a;b) \,:\, b \in \mathfrak{N}\backslash\mathbb{F}_q\} \quad\text{and}\quad \mathfrak{l}_\alpha := \{[\alpha;\beta]) \,:\, \beta \in \mathfrak{N}\backslash\mathbb{F}_q\}$$

are pencils of pairwise parallel points and lines, respectively.

20 Lemma. *Let b range in $\mathfrak{N}\backslash\mathbb{F}_q$. The m flags (if any) $(a;b) \,|\, [\alpha;\beta]$ belonging to $\mathfrak{p}_a \times \mathfrak{l}_\alpha$ give rise to the permutation*

$$\sigma_{a,\alpha} : \begin{cases} \mathfrak{N}\backslash\mathbb{F}_q & \longrightarrow & \mathfrak{N}\backslash\mathbb{F}_q \\ \quad b & \longmapsto & \quad x \end{cases}$$

where x is defined by $(b,1,-1-b)\, S^{a-\alpha} \equiv: (x,1,-x-\beta)$.

PROOF. The image of b under $\sigma_{a,\alpha}$ is well defined by normalizing the second coordinate of $(b, 1, -1-b)\, S^{a-\alpha}$ to be 1 again. Note that the second coordinate would be zero only if $(b, 1, -1-b)\, S^{a-\alpha}$ were a central point. QED

To go ahead, we need more concreteness concerning the Singer matrix S:

21 Example. Let $q^2 = 9$. We use additive notations and, for typographic reason, prefer to write 2 instead of -1. So $\mathbb{F}_3 = \{0, 1, 2\}$ and $\mathbb{F}_9$ can be written as $\mathbb{F}_3[\omega]/(\omega^2 + 2\omega + 2)$. The elements of $\mathbb{F}_9$ are represented by residues $a + b\omega$ with $a, b \in \mathbb{F}_3$. We shall write ab instead of $a + b\omega$. In the following multiplication table of $\mathfrak{N} = N(2, 3)$, the elements are ordered lexicographically:

·	01	02	10	11	12	20	21	22
01	20	10	01	12	22	02	11	21
02	10	20	02	21	11	01	22	12
10	01	02	10	11	12	20	21	22
11	21	12	11	20	01	22	02	10
12	11	22	12	02	20	21	10	01
20	02	01	20	22	21	10	12	11
21	22	11	21	01	10	12	20	02
22	12	21	22	10	02	11	01	20

The central elements in $\mathfrak{Q}$ are $00, 10, 20$. Let the matrix

$$S = \begin{pmatrix} 20 & 00 & 10 \\ 10 & 00 & 00 \\ 00 & 10 & 00 \end{pmatrix}$$

act on row vectors of coordinates. S induces a Singer cycle in the Baer subplane. Order the elements of $\mathfrak{N} \backslash \mathbb{F}_3$ lexicographically. Together with the canonic order in $\mathbb{Z}_{13}$, this induces a lexicographic order for all points $(a; b)$ and lines $[\alpha; \beta]$ of $\mathcal{H}_9^D$, to which all the following incidence matrices refer. A calculation shows that, for $i \neq 2, 3, 5, 11$, the flags in $\mathfrak{p}_i \times \mathfrak{l}_0$ give rise to four distinct incidence matrices, and consequently, to four permutations:

$\mathfrak{p}_0 \times \mathfrak{l}_0$, $\mathfrak{p}_9 \times \mathfrak{l}_0$, $\mathfrak{p}_{10} \times \mathfrak{l}_0$	$\begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$	$\sigma_{0,0} = \sigma_{9,0} = \sigma_{10,0} = id$
$\mathfrak{p}_1 \times \mathfrak{l}_0$, $\mathfrak{p}_7 \times \mathfrak{l}_0$	$\begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$	$\sigma_{1,0} = \sigma_{7,0} = (01\ \ 02)\,(11\ \ 22)\,(12\ \ 21)$
$\mathfrak{p}_4 \times \mathfrak{l}_0$, $\mathfrak{p}_{12} \times \mathfrak{l}_0$	$\begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$	$\sigma_{4,0} = \sigma_{12,0} = (01\ \ 22)\,(02\ \ 21)\,(11\ \ 12)$
$\mathfrak{p}_6 \times \mathfrak{l}_0$, $\mathfrak{p}_8 \times \mathfrak{l}_0$	$\begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{pmatrix}$	$\sigma_{6,0} = \sigma_{8,0} = (01\ \ 12)\,(11\ \ 02)(21\ \ 22)$

The point–line–pairs in

$$\mathfrak{p}_2 \times \mathfrak{l}_0\,,\; \mathfrak{p}_3 \times \mathfrak{l}_0\,,\; \mathfrak{p}_5 \times \mathfrak{l}_0\,,\; \mathfrak{p}_{11} \times \mathfrak{l}_0\,,$$

are all anti–flags. The 6–cycle

$$\Pi \quad : \quad 01\,,\; 22\,,\; 11\,,\; 02\,,\; 21\,,\; 12\,,\; 01$$

turns out to be invariant under all four permutations. The Cayley colours of this 6–cycle belong to $\{\pm 12,\, \pm 11\}$. Choose

$$\Lambda \quad : \quad 01\,,\; 11\,,\; 21\,,\; 22\,,\; 12\,,\; 02\,,\; 01$$

as an edge–disjoint 6–cycle. Since the edges of Λ have Cayley colours lying in $\{\pm 10,\, \pm 01\}$, the Cayley colours of Π and Λ are $\sigma_{a,\alpha}$-disjoint for all $a, \alpha \in \mathbb{Z}_{13}$. Applying Construction 2, we obtain an 11–regular graph $\mathcal{D}(\Pi, \Lambda)$ of girth 5 on 156 vertices.

22 Remark. $\mathcal{D}(\Pi, \Lambda)$ is isomorphic to the graph constructed by Jørgensen [14, Example 12]. The vertex set of his graph is $\mathbb{Z}_{13} \times S_3 \times \{1, 2\}$. He distinguishes edges of types I, $II.1$, and $II.2$, which correspond to edges in $\Gamma(\mathcal{H}_9^D)$, Π, and Λ, respectively. The edges of type I are defined in terms of a $(13, 6, 9, 1)$ relative difference set with forbidden subgroup $\{0\} \times S_3$ in the group $\mathbb{Z}_{13} \times S_3$, pointed out by Pott [22, Example 1.1.10.6]. The permutations $\sigma_{1,0}$, $\sigma_{4,0}$, and $\sigma_{6,0}$ generate a subgroup of S_6 acting on the elements in $\mathfrak{N} \backslash \mathbb{F}_3$. This subgroup turns out to be isomorphic to S_3. With these data, an isomorphism between the two constructions of $\Gamma(\mathcal{H}_9^D)$ can easily be established if the elements in $\mathfrak{N} \backslash \mathbb{F}_3$ are ordered in the following way: $01, 11, 21, 22, 02, 12$.

Room's approach [25] to construct incidence tables for Hughes planes of order q^2 differs slightly from the construction described above. First, the rôle of the special line is played by the set

$$l_\beta := \{(x : y : z) \;:\; y + \beta \cdot z = 0\}\,.$$

Secondly, Room by–passes the notion of a regular nearfield $\mathfrak{N}(2, q)$. Instead, he constructs $\mathcal{H}_{q^2}$ from the Desarguesian plane $PG(2, q^2)$ by "transferring points from some line l to the conjugate line l^*" (cf. [25, p. 136]. But when defining these new incidences, Room's case distinction (i) and (ii) [25, p. 297] can easily be unified using multiplication in $\mathfrak{N}(2, q)$ rather than in $\mathbb{F}_{q^2}$.

23 Example. Let $q^2 = 25$. Let $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ and consider $\mathbb{F}_{25} = \mathbb{F}_5[\omega]/(\omega^2 + 3)$. The elements of $\mathbb{F}_{25}$ are represented by residues $a + b\omega$ with $a, b \in \mathbb{F}_5$. Again we write ab instead of $a + b\omega$. From Room [25, Section 5], we take over the order for the elements in $G := \mathfrak{N}(2, 5) \backslash \mathbb{F}_5$, namely

$$01, 11, 42, 44, 32, 24, 13, 23, 34, 02, 03, 31, 22, 12, 21, 33, 41, 43, 14, 04.$$

In [25], these elements are respectively denoted by

$$1, 2, \ldots, 10, -10, -9, \ldots, -1.$$

We greatly appreciate, and willingly rely on, the calculations for $\mathcal{H}_{25}$ reported in [25]. Room's points $W_{j,r}$ and lines $w_{i,0}$ have G–coordinates $(r; j)$ and $[0; i]$, respectively. Equivalently, $(a; b)$ and $[\alpha; \beta]$ represent the point $W_{b,a}$ and the line $w_{\beta,\alpha}$.

Two points and lines are parallel if their first G–coordinates coincide.

With these data, the permutations $\sigma_{a,0}$ can be extracted from Room's incidence table WII(25) [25, p. 301]:

$$\begin{aligned}
\sigma_{1,0} &= (11\ 41)(12\ 42)(13\ 43)(14\ 44)(21\ 31)(22\ 32)(23\ 33)(24\ 34)\\
\sigma_{2,0} &= (01\ 03)(02\ 04)(11\ 44)(12\ 21)(13\ 24)(14\ 41)(31\ 42)(34\ 43)\\
\sigma_{3,0} &= (01\ 11)(02\ 12)(03\ 13)(04\ 14)(21\ 41)(22\ 42)(23\ 43)(24\ 44)\\
\sigma_{4,0} &= (01\ 33)(02\ 22)(03\ 23)(04\ 32)(12\ 13)(21\ 43)(24\ 42)(31\ 34)(41\ 44)\\
\sigma_{5,0} &= (01\ 12)(02\ 03)(04\ 13)(11\ 21)(14\ 24)(22\ 31)(23\ 34)(32\ 33)(41\ 44)\\
\sigma_{6,0} &= (01\ 33)(02\ 43)(03\ 42)(04\ 32)(12\ 44)(13\ 41)(21\ 22)(23\ 24)\\
\sigma_{7,0} &= (01\ 02)(03\ 04)(12\ 34)(13\ 31)(21\ 43)(22\ 33)(23\ 32)(24\ 42)\\
\sigma_{8,0} &= (01\ 41)(02\ 42)(03\ 43)(04\ 44)(11\ 31)(12\ 32)(13\ 33)(14\ 34)\\
\sigma_{9,0} &= (01\ 23)(02\ 32)(03\ 33)(04\ 22)(11\ 14)(12\ 34)(13\ 31)(21\ 24)(42\ 43)\\
\sigma_{10,0} &= (01\ 24)(02\ 41)(03\ 44)(04\ 21)(11\ 13)(12\ 14)(22\ 31)(23\ 34)\\
\sigma_{11,0} &= (02\ 03)(11\ 33)(12\ 42)(13\ 43)(14\ 32)(21\ 24)(22\ 44)(23\ 41)(31\ 34)\\
\sigma_{12,0} &= (01\ 31)(02\ 32)(03\ 33)(04\ 34)(11\ 21)(12\ 22)(13\ 23)(14\ 24)\\
\sigma_{14,0} &= (01\ 04)(02\ 24)(03\ 21)(11\ 14)(12\ 44)(13\ 41)(22\ 42)(23\ 43)(32\ 33)\\
\sigma_{15,0} &= (02\ 24)(03\ 21)(11\ 33)(12\ 23)(13\ 22)(14\ 32)(41\ 42)(43\ 44)\\
\sigma_{16,0} &= (01\ 23)(02\ 13)(03\ 12)(04\ 22)(11\ 43)(14\ 42)(31\ 32)(33\ 34)\\
\sigma_{17,0} &= (01\ 04)(11\ 22)(12\ 13)(14\ 23)(21\ 31)(24\ 34)(32\ 41)(33\ 44)(42\ 43)\\
\sigma_{18,0} &= (01\ 21)(02\ 22)(03\ 23)(04\ 24)(31\ 41)(32\ 42)(33\ 43)(34\ 44)\\
\sigma_{20,0} &= (01\ 41)(02\ 11)(03\ 14)(04\ 44)(12\ 13)(21\ 24)(31\ 42)(32\ 33)(34\ 43)\\
\sigma_{21,0} &= (01\ 11)(02\ 41)(03\ 44)(04\ 14)(12\ 21)(13\ 24)(22\ 23)(31\ 34)(42\ 43)\\
\sigma_{22,0} &= (01\ 04)(02\ 34)(03\ 31)(11\ 43)(12\ 32)(13\ 33)(14\ 42)(22\ 23)(41\ 44)\\
\sigma_{24,0} &= (01\ 42)(02\ 03)(04\ 43)(11\ 14)(21\ 32)(22\ 23)(24\ 33)(31\ 41)(34\ 44)\\
\sigma_{25,0} &= (01\ 42)(04\ 43)(11\ 22)(14\ 23)(21\ 44)(24\ 41)(31\ 33)(32\ 34)\\
\sigma_{26,0} &= (02\ 34)(03\ 31)(11\ 12)(13\ 14)(22\ 44)(23\ 41)(32\ 43)(33\ 42)\\
\sigma_{27,0} &= (01\ 12)(04\ 13)(11\ 34)(14\ 31)(21\ 23)(22\ 24)(32\ 41)(33\ 44)\\
\sigma_{29,0} &= (01\ 34)(02\ 11)(03\ 14)(04\ 31)(21\ 32)(24\ 33)(41\ 43)(42\ 44)
\end{aligned}$$

The point–line–pairs in

$$\mathfrak{p}_0 \times \mathfrak{l}_0\,,\ \mathfrak{p}_{13} \times \mathfrak{l}_0\,,\ \mathfrak{p}_{19} \times \mathfrak{l}_0\,,\ \mathfrak{p}_{23} \times \mathfrak{l}_0\,,\ \mathfrak{p}_{28} \times \mathfrak{l}_0\,,\ \mathfrak{p}_{30} \times \mathfrak{l}_0\,,$$

are all anti–flags. In Table WII(25), we encounter six blank entries in each row and column. Extract a $(0, 1)$–matrix, say M, from WII(25) by writing 1 for each blank entry and 0 otherwise. Being symmetric, we can interpret M as the adjacency matrix of a 6–regular graph Φ with vertices in G. With the help of the software *Groups and Graphs* [16], we check that each $\sigma_{a,0}$ is an automorphism of Φ. In particular, we can partition the edge set of Φ into two subsets indicated by entries $\mathbf{1}$ and $\mathit{1}$, respectively:

$$M = \begin{pmatrix}
o & o & \mathbf{0} & \mathbf{1} & \mathit{1} & o & \mathit{1} & \mathbf{0} & o & \mathbf{0} & o & o & \mathbf{1} & o & o & o & o & \mathit{1} & \mathbf{1} & o \\
o & o & \mathit{1} & o & \mathbf{1} & \mathbf{1} & o & \mathit{1} & o & \mathbf{0} & \mathit{1} & o & o & \mathbf{0} & o & \mathbf{0} & o & o & o & \mathbf{1} \\
\mathbf{0} & \mathit{1} & o & o & o & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathit{1} & o & o & o & o & o & \mathbf{1} & o & \mathbf{0} & o & o & \mathit{1} \\
\mathbf{1} & o & o & o & \mathit{1} & \mathbf{0} & \mathbf{1} & \mathit{1} & o & \mathit{1} & o & \mathbf{1} & o & \mathbf{0} & o & o & o & o & \mathbf{0} & o \\
\mathit{1} & \mathbf{1} & o & \mathit{1} & o & \mathit{1} & \mathbf{1} & o & o & o & \mathbf{1} & \mathbf{0} & o & o & o & o & \mathbf{0} & o & \mathbf{0} & o \\
o & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathit{1} & o & o & o & o & o & \mathit{1} & \mathbf{1} & o & \mathit{1} & o & o & o & \mathbf{1} & o & \mathbf{0} \\
\mathit{1} & o & \mathbf{1} & \mathbf{1} & \mathbf{1} & o & o & o & \mathit{1} & o & o & o & o & o & \mathit{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & o & o \\
\mathbf{0} & \mathit{1} & \mathbf{1} & \mathit{1} & o & o & o & o & o & \mathbf{1} & \mathbf{0} & \mathit{1} & o & o & o & o & o & \mathbf{0} & o & \mathbf{1} \\
o & o & \mathit{1} & o & o & o & \mathit{1} & o & o & \mathbf{0} & \mathbf{1} & o & \mathit{1} & o & \mathbf{1} & o & \mathbf{1} & o & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & o & \mathit{1} & o & o & o & \mathbf{1} & \mathbf{0} & o & o & \mathbf{1} & o & o & \mathit{1} & \mathbf{1} & o & o & \mathit{1} & o \\
o & o & \mathit{1} & o & \mathbf{1} & \mathit{1} & o & \mathbf{0} & \mathbf{1} & o & o & \mathbf{0} & \mathbf{1} & o & o & \mathbf{0} & \mathit{1} & o & o & o \\
o & o & o & \mathbf{1} & \mathbf{0} & \mathbf{1} & o & \mathit{1} & o & \mathbf{1} & \mathbf{0} & o & \mathbf{0} & \mathit{1} & o & o & o & \mathit{1} & o & o \\
\mathbf{1} & o & o & o & o & o & o & o & \mathit{1} & o & \mathbf{1} & \mathbf{0} & o & o & \mathbf{0} & o & \mathit{1} & \mathbf{1} & \mathit{1} & \mathbf{0} \\
o & \mathbf{0} & o & \mathbf{0} & o & \mathit{1} & o & o & o & o & o & \mathit{1} & o & o & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & o & \mathit{1} \\
o & o & \mathbf{1} & o & o & o & \mathit{1} & o & \mathbf{1} & \mathit{1} & o & o & \mathbf{0} & \mathbf{0} & o & \mathit{1} & o & \mathbf{0} & \mathbf{1} & o \\
o & \mathbf{0} & o & o & o & o & \mathbf{0} & o & o & \mathbf{1} & \mathbf{0} & o & o & \mathbf{1} & \mathit{1} & o & \mathit{1} & o & \mathbf{1} & \mathit{1} \\
o & o & \mathbf{0} & o & \mathbf{0} & o & \mathbf{0} & o & \mathbf{1} & o & \mathit{1} & o & \mathit{1} & \mathbf{1} & o & \mathit{1} & o & o & o & \mathbf{1} \\
\mathit{1} & o & o & o & o & \mathbf{1} & \mathbf{0} & \mathbf{0} & o & o & o & \mathit{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & o & o & o & \mathit{1} & o \\
\mathbf{1} & o & o & \mathbf{0} & \mathbf{0} & o & o & o & \mathbf{0} & \mathit{1} & o & o & \mathit{1} & o & \mathbf{1} & \mathbf{1} & o & \mathit{1} & o & o \\
o & \mathbf{1} & \mathit{1} & o & o & \mathbf{0} & o & \mathbf{1} & \mathbf{0} & o & o & o & \mathbf{0} & \mathit{1} & o & \mathit{1} & \mathbf{1} & o & o & o
\end{pmatrix}$$

The entries $\mathbf{1}$ and $\mathit{1}$ represent two (edge–disjoint) bipartite cubic graphs of girth 6, say Π and Π', which are Levi graphs of 10_3 configurations. Using Kantor's classification [15], one has $\Pi \cong \Gamma(10_3B) \cong P(10, 3)$ and $\Pi' \cong \Gamma(10_3A)$. In the complement of Φ, let $\Lambda \cong P(10, 4)$ be the cubic graph of girth 5 whose adjacency matrix is obtained from M by substituting 1 for the entries $\mathbf{0}$ and 0 for all the other entries ($\mathbf{1}$, $\mathit{1}$, or o), respectively. Then Π and Λ are $\sigma_{a,\alpha}$–disjoint for all $a, \alpha \in \mathbb{Z}_{31}$. The Cayley colours of Π and Π' belong to

$$\{\pm 12, \pm 13, \pm 21, \pm 24, \} ,$$

whereas those of Λ lie in

$$\{\pm 01, \pm 14, \pm 20, \pm 22, \pm 23\} .$$

Applying Construction 2, we obtain a 28–regular graph $\mathcal{D}(\Pi, \Lambda)$ of girth 5 on 1240 vertices, which has eight vertices less than the instance in Jørgensen's series [14, Theorem 17].

6 Appendix: Deletion of Parallel Classes

We survey a well known deletion technique, eligible for all elliptic semiplanes $\mathcal{S}$ of types C and L. Fix a permutation $\pi \in S_\mu$, acting on I. If $\mathcal{S}$ is of type L, we additionally assume that $\mathfrak{p}_i \times \mathfrak{l}_{i^\pi}$ consists only of anti–flags for all $i \in I$; this holds true if we put $i^\pi := i'$ in the Proof of Lemma 3. For a positive integer $\lambda < \mu$, choose a λ–subset $J \subseteq I$. Let $\mathcal{S}^{(\lambda)}$ be the configuration obtained from $\mathcal{S}$ by deleting, for each $j \in J$, the m points and m lines belonging to $\mathfrak{p}_j$ and $\mathfrak{l}_j$, respectively. Then $\mathcal{S}^{(\lambda)}$ is a configuration of type $(m(\mu-\lambda))_{n-\lambda}$ and its Levi graph $\Gamma(\mathcal{S}^{(\lambda)})$ is an $(n-\lambda)$–regular bipartite graph of girth ≥ 6 and order $2m(\mu-\lambda)$. Construction 2 still works and for any suitable pair Π, Λ and one has the following

24 Theorem. *The amalgam $\mathcal{S}^{(\lambda)}(\Pi,\Lambda)$ is an $(n+r-\lambda)$–regular simple graph of girth* 5 *and order $2m(\mu-\lambda)$.* QED

25 Remark. This deletion technique successfully applies in the following cases, yielding graphs whose orders tie with those of the (currently known) smallest girth 5 graphs of the same degree:

ell. spl. type	cfg. type	parameters n,r,k,m,μ	λ	degree $k-\lambda$	order $2m(\mu-\lambda)$	known graph of same order and degree	ref.
C	25_5	5,2,7,5,5	1	6	**40**	(5,6)–cage	([9])
			2	5	**30**	(5,5)–cage: $\lvert \text{Aut} \rvert = 20$	([9])
L	168_{13}	13,3,16,12,14	1	15	**312**	Jørgensen	[14]
			2	14	**288**	Jørgensen	[14]
C	256_{16}	16,3,19,16,16	1	18	**480**	Schwenk	[27]
			2	17	**448**	Schwenk	[27]

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