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Enumeration of Nonsingular Buekenhout Unitals

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Abstract. The only known enumeration of Buekenhout unitals occurs in the Desarguesian plane $PG(2, q^2)$. In this paper we develop general techniques for enumerating the nonsingular Buekenhout unitals embedded in any two-dimensional translation plane, and apply these techniques to obtain such an enumeration in the regular nearfield planes, the odd-order Hall planes, and the flag-transitive affine planes. We also provide some computer data for small-order André planes of index two and give partial results toward an enumeration in this case.

Keywords: enumeration, Buekenhout unital, nonsingular quadric, translation plane

MSC 2000 classification: primary 51A45, secondary 51A40

1 Introduction

A *unital* is any 2–$(n^3 + 1, n + 1, 1)$ design, for some integer $n > 2$. We often refer to n as the *order* of the unital, even though this use of order is not the standard one used in design theory. The classical example is a Hermitian curve in the square-order Desarguesian plane $PG(2, q^2)$, where we take $n = q$ and the blocks become the intersections of the Hermitian curve with its "secant" lines. This example is often called the *classical unital.* In general, a unital with $n^3 + 1$ points which is contained in a projective plane of order n^2 is called an *embedded* unital in that plane. While most unitals are not embedded (see [23], [7]), the

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embedded ones are certainly of most interest to finite geometers. There is currently no example of a square-order projective plane which has been proven not to contain a unital, at least as far as we know. Moreover, unitals seem to play a key role in understanding the nature of square-order projective planes (see [11] for the connection with minimal blocking sets).

If we restrict to the family of translation planes which are two-dimensional over their kernels, and hence arise from line spreads of $\Sigma = PG(3,q)$, there are general techniques for constructing unitals embedded in such planes. These were developed by Buekenhout [12], and use the Bruck-Bose [9, 10] representation. Namely, let $\mathcal{S}$ be a spread in $\Sigma = PG(3,q)$, and embed Σ as a hyperplane at infinity in $\bar{\Sigma} = PG(4,q)$. Then the two-dimensional (affine) translation plane of order q^2 corresponding to $\mathcal{S}$ is the incidence structure whose points are the points of $\bar{\Sigma} \setminus \Sigma$, whose lines are the planes of $\bar{\Sigma}$ which meet the hyperplane Σ in a line of the spread $\mathcal{S}$, and whose incidence is inherited from $\bar{\Sigma}$. This affine plane is completed to a projective plane $\pi(\mathcal{S})$ by adding the spread lines of $\mathcal{S}$ as the points at infinity. The plane $\pi(\mathcal{S})$ is Desarguesian (that is, isomorphic to $PG(2,q^2)$) if and only if the spread $\mathcal{S}$ is regular (see [10]).

Buekenhout used this Bruck-Bose representation to present the following two constructions. If $\mathcal{S}$ is any spread of Σ and if $\mathcal{U}$ is an ovoidal cone of $\bar{\Sigma}$ (that is, the point cone over some 3-dimensional ovoid) that meets Σ in a line of $\mathcal{S}$, then $\mathcal{U}$ corresponds to a unital U in $\pi(\mathcal{S})$ which is tangent to the line at infinity. Similarly, if $\mathcal{U}$ is a nonsingular (parabolic) quadric in $\bar{\Sigma}$ that meets Σ in a regulus of the spread $\mathcal{S}$, then $\mathcal{U}$ corresponds to a unital U in $\pi(\mathcal{S})$ which meets the line at infinity in $q+1$ points. Of course, the second construction is valid only for those two-dimensional translation planes whose associated spread contains at least one regulus. Unitals embedded in any two-dimensional translation plane $\pi(\mathcal{S})$ which arise from Buekenhout's ovoidal cone construction will be called *ovoidal Buekenhout unitals*. If the ovoidal cone is an orthogonal cone (with an elliptic quadric as base), then the unital will be called an *orthogonal Buekenhout unital*. The unitals embedded in $\pi(\mathcal{S})$ which arise from Buekenhout's nonsingular quadric construction will be called *nonsingular Buekenhout unitals*.

Metz [21] used a counting argument to show that there are orthogonal cones which correspond to non-classical unitals in the Desarguesian plane $PG(2,q^2)$ for all $q > 2$. Barwick [5] also used a counting argument to show that in $PG(2,q^2)$ any unital arising from a nonsingular quadric is necessarily classical. Furthermore, the only known enumeration of Buekenhout unitals occurs in the Desarguesian plane $PG(2,q^2)$. In this plane any nonsingular Buekenhout unital is classical by Barwick's argument cited above, and the orthogonal Buekenhout-Metz unitals are completely enumerated in [2], [16]. In addition, the full stabilizers of these orthogonal Buekenhout-Metz unitals are computed in [2], [16].

To avoid the technical difficulties of dealing with quadrics in even characteristic and since most of the families discussed exist only in odd characteristic, we assume throughout this paper that q is an odd prime power. We discuss the enumeration of nonsingular Buekenhout unitals in several infinite families of two-dimensional translation planes, including the regular nearfield planes, the odd-order Hall planes, and the two-dimensional flag-transitive affine planes, thereby providing the first classification results for nonsingular Buekenhout unitals embedded in non-Desarguesian square-order planes. We also provide some computer data for nonsingular Buekenhout unitals embedded in small-order two-dimensional André planes of index two and give partial results toward an enumeration in this case. The enumeration of orthogonal Buekenhout unitals embedded in non-Desarguesian planes will be addressed in a future paper.

2 General Results

We begin with some notation and general results applicable to any translation plane arising from a spread of $PG(3,q)$. We let $(X_0, X_1, X_2, X_3, X_4)$ denote homogeneous coordinates for $\bar{\Sigma} = PG(4,q)$, where the hyperplane at infinity, $\Sigma \cong PG(3,q)$, has equation $X_0 = 0$ and contains the given spread $\mathcal{S}$. Points are represented by the row vectors containing their homogeneous coordinates. When using the Bruck-Bose representation in $\bar{\Sigma}$, we freely identify actions on $\pi(\mathcal{S})$ with the associated action on $\bar{\Sigma}$. We always denote a primitive element of $GF(q)$ by ω, and let A^T denote the transpose of A.

We now briefly recall how $\mathrm{Aut}(\pi(\mathcal{S}))$ is computationally obtained from $\mathrm{Aut}(\mathcal{S})$, where $\mathrm{Aut}(\mathcal{S})$ denotes the stabilizer of the spread $\mathcal{S}$ in the collineation group of Σ. As long as the spread $\mathcal{S}$ is not regular (and hence the plane $\pi(\mathcal{S})$ is not Desarguesian), the ideal line (that is, Σ) must be fixed. Hence in this case any automorphism of $\pi(\mathcal{S})$ is represented by a field automorphism applied to the $\bar{\Sigma}$-coordinates followed by right multiplication of a 5×5 "normalized" nonsingular matrix $\bar{M}$ over $GF(q)$, where the first column of $\bar{M}$ may be assumed to be $(1,0,0,0,0)^T$. Moreover, the 4×4 lower right submatrix M induces a collineation of Σ which stabilizes the spread $\mathcal{S}$. The four components of the first row of $\bar{M}$, other than the first component, are arbitrary elements of $GF(q)$, indicating the q^4 translations of $\pi(\mathcal{S})$. If these four components are all zero, then $\bar{M}$ represents an element in the *translation complement* of $\pi(\mathcal{S})$. In any case, one sees that $|\,\mathrm{Aut}(\pi(\mathcal{S}))| = q^4(q-1)|\,\mathrm{Aut}(\mathcal{S})|$.

We first consider the number of mutually inequivalent nonsingular Buekenhout unitals embedded in the translation plane $\pi(\mathcal{S})$ that might arise from a given regulus of $\mathcal{S}$. When we say $\mathcal{Q}_0$ is some quadratic form for a regulus $\mathcal{R}$, we mean that the isotropic vectors of $\mathcal{Q}_0$ are the points of the hyperbolic quadric $\mathcal{H}$ which has $\mathcal{R}$ as one class of ruling lines. All such forms are given by $k\mathcal{Q}_0$, as k varies over $GF(q)^*$.

1 Theorem. *Let q be any odd prime power, and let $\mathcal{R}$ be some regulus contained in the spread $\mathcal{S}$. Let $\mathcal{Q}_0$ be a quadratic form for $\mathcal{R}$. Then the plane $\pi(\mathcal{S})$ has one or two inequivalent nonsingular Buekenhout unitals that meet the ideal line Σ in the points corresponding to $\mathcal{R}$, according to whether* $\mathrm{Aut}(\mathcal{S})$ *contains a collineation with an associated isometry that takes $\mathcal{Q}_0$ to $\omega\mathcal{Q}_0$.*

Proof. Consider a nonsingular Buekenhout unital of $\pi(\mathcal{S})$ that is represented by a nonsingular (parabolic) quadric $\mathcal{U}$ of $\bar{\Sigma}$ that meets Σ in the points covered by the regulus $\mathcal{R}$ of $\mathcal{S}$. If $\mathcal{Q}$ is a quadratic form for $\mathcal{U}$, then $\mathcal{Q}|_\Sigma$ must be a form for the hyperbolic quadric $\mathcal{H}$ of $\mathcal{R}$. Let $P = \Sigma^\perp$ be the pole of Σ with respect to the symmetric bilinear form associated with $\mathcal{Q}$, and consider the translation $\tau_{\bar{M}}$ induced by the matrix $\bar{M}$ which has first row $P = (1, t_1, t_2, t_3, t_4)$ and lower right 4×4 submatrix I. Thus $\tau_{\bar{M}}$ maps $P_0 = (1, 0, 0, 0, 0)$ to P. The image of $\mathcal{U}$ under $\tau_{\bar{M}}^{-1}$ will have $\Sigma^\perp = P_0$, and hence, after suitable scaling, will have as its equation

$$\mathcal{U}_t : X_0^2 - t\mathcal{Q}_0(X_1, X_2, X_3, X_4) = 0, \tag{1}$$

for some $t \in GF(q)^*$, where $\mathcal{Q}_0$ is a form for $\mathcal{H}$. Thus any nonsingular Buekenhout unital is equivalent to one with Equation (1), and any regulus $\mathcal{R}$ of $\mathcal{S}$ thus lifts to one or more inequivalent unitals.

Now, if we let ρ_k be the linear transformation whose associated matrix is $\mathrm{Diag}\{1, k, k, k, k\}$ for some $k \in GF(q)^*$, then ρ_k is in the homology subgroup associated with the kernel of $\pi(\mathcal{S})$. Straightforward computations show that ρ_k maps $\mathcal{U}_t$ to $\mathcal{U}_{\frac{t}{k^2}}$. Hence there are at most two inequivalent nonsingular Buekenhout unitals meeting the ideal line in the points corresponding to $\mathcal{R}$, namely those represented by the parabolic quadrics

$$\begin{aligned} \mathcal{U}_1 &: \quad X_0^2 - \mathcal{Q}_0(X_1, X_2, X_3, X_4) = 0, \\ \mathcal{U}_\omega &: \quad X_0^2 - \omega\mathcal{Q}_0(X_1, X_2, X_3, X_4) = 0. \end{aligned}$$

Moreover, to have a single equivalence class of nonsingular Buekenhout unitals arising from $\mathcal{R}$, it suffices to have an isometry taking $\mathcal{Q}_0$ to $\omega\mathcal{Q}_0$ which induces a collineation in $\mathrm{Aut}(\mathcal{S})$ that fixes $\mathcal{R}$. Conversely, if $\tau \in \mathrm{Aut}(\pi(\mathcal{S}))$ maps U_1 to U_ω, then it necessarily has an associated isometry of the underlying vector space taking $\mathcal{Q}_0$ to $\omega\mathcal{Q}_0$. $\boxed{QED}$

It should be noted that the order of the stabilizer in $\mathrm{Aut}(\pi(\mathcal{S}))$ of a nonsingular Buekenhout unital is naturally related to the order of the stabilizer in $\mathrm{Aut}(\mathcal{S})$ of the regulus $\mathcal{R}$ from which it arises.

2 Theorem. *Let q be any odd prime power, and consider a nonsingular Buekenhout unital of $\pi(\mathcal{S})$ which meets the ideal line in the points corresponding to some regulus $\mathcal{R}$ contained in $\mathcal{S}$. Then the stabilizer of this unital in* $\mathrm{Aut}(\pi(\mathcal{S}))$ *has either the same order or twice the order of the stabilizer of the regulus $\mathcal{R}$ in* $\mathrm{Aut}(\mathcal{S})$*, according to whether there are one or two equivalence classes of such unitals arising from $\mathcal{R}$. Moreover, some representative of the equivalence class of this unital has its stabilizer contained in the translation complement of $\pi(\mathcal{S})$.*

Proof. As discussed in the proof of Theorem 1, we may assume without loss of generality that our nonsingular Buekenhout unital is represented by a parabolic quadric in $\bar{\Sigma}$ of the form given by Equation (1). We define $\mathcal{P} = \{\mathcal{U}_t \mid t \in GF(q)^*\}$, and let G be the group of all elements of $\mathrm{Aut}(\pi(\mathcal{S}))$ which stabilize $\mathcal{P}$. Note that for every $\mathcal{U}_t$, we have $\Sigma^\perp = P_0$ as in the proof of Theorem 1, and hence every element of G fixes P_0. Thus G is a subgroup of the translation complement of $\mathrm{Aut}(\pi(\mathcal{S}))$.

Choose any $\mathcal{U} \in \mathcal{P}$. Straightforward matrix computations show that the stabilizer of $\mathcal{U}$ in $\mathrm{Aut}(\pi(\mathcal{S}))$ is indeed a subgroup of G. Now any $\tau \in \mathrm{Aut}(\mathcal{S})$ which fixes $\mathcal{R}$ has $q-1$ "liftings" to the translation complement of $\mathrm{Aut}(\pi(\mathcal{S}))$, according to the choice of the nonzero scalar multiple of the lower 4×4 matrix used in the representation. Moreover, any such lifting $\bar{\tau}$ belongs to G, and conversely any $\phi \in G$ must have $\phi|_\Sigma \in \mathrm{Aut}(\mathcal{S})$ with $\phi|_\Sigma$ fixing $\mathcal{R}$. Thus $|G| = (q-1)|Stab_{\mathrm{Aut}(\mathcal{S})}(\mathcal{R})|$. Since G acts transitively on $\mathcal{P}$ or has two orbits of size $\frac{q-1}{2}$ each, the result now follows from the Orbit-Stabilizer Theorem. $\boxed{QED}$

3 The Regular Spread of $PG(3, q)$

Before discussing the specific translation planes mentioned Section 1, we need a convenient representation for a regular spread of $PG(3, q)$ and its full automorphism group. To do this, it is best to start with the projective line $PG(1, q^2)$ over the finite field $GF(q^2)$. We let β denote a primitive element of $GF(q^2)$, so that $\omega = \beta^{q+1}$ is a primitive element of $GF(q)$. We choose $\{1, \epsilon = \beta^{\frac{q+1}{2}}\}$ for our ordered basis of $GF(q^2)$, treated as a vector space over its subfield $GF(q)$, noting that $\epsilon^2 = \omega$ and $\epsilon^q = -\epsilon$. We identify ordered pairs (s_1, s_2) from $GF(q)$ with the element $s = s_1 + s_2\epsilon$ in $GF(q^2)$ whenever it is useful, and observe that $s^{q+1} = s_1^2 - \omega s_2^2$.

The trivial (point) spread in $PG(1, q^2)$ consisting of all its q^2+1 points lifts to a Desarguesian (or regular) line spread of $PG(3, q)$ by viewing the underlying 2-dimensional vector space over $GF(q^2)$ for $PG(1, q^2)$ as a 4-dimensional vector space over $GF(q)$ (see [8], for instance). Namely, using the above ordered basis for $GF(q^2)$ over $GF(q)$, the projective point $\langle(1, s)\rangle = \langle(\epsilon, \epsilon s)\rangle$ of $PG(1, q^2)$ lifts to the line

$$\ell_s = \langle (1, 0, s_1, s_2), (0, 1, \omega s_2, s_1) \rangle$$

of $PG(3, q)$, and the projective point $\langle (0, 1) \rangle = \langle (0, \epsilon) \rangle$ of $PG(1, q^2)$ lifts to the line

$$\ell_\infty = \langle (0, 0, 1, 0), (0, 0, 0, 1) \rangle$$

of $PG(3, q)$. That is,

$$\mathcal{S}_0 = \{\ell_\infty\} \cup \{\ell_s \mid s \in GF(q^2)\}$$

is a regular line spread of $\Sigma = PG(3, q)$.

The general linear group $GL(2, q^2)$ of order $q^2(q^2 - 1)^2(q^2 + 1)$ acting on the underlying vector space for $PG(1, q^2)$ lifts to a projective subgroup of order $q^2(q^2 - 1)(q^2 + 1)(q + 1)$ stabilizing the regular spread $\mathcal{S}_0$; namely, the matrix $\begin{bmatrix} e & g \\ h & f \end{bmatrix} \in GL(2, q^2)$ naturally lifts to the linear collineation $\phi_{e,f,g,h}$ of $PG(3, q)$ stabilizing $\mathcal{S}_0$ that is induced by the matrix

$$M_{e,f,g,h} = \begin{bmatrix} e_1 & e_2 & g_1 & g_2 \\ \omega e_2 & e_1 & \omega g_2 & g_1 \\ h_1 & h_2 & f_1 & f_2 \\ \omega h_2 & h_1 & \omega f_2 & f_1 \end{bmatrix},$$

acting on row vectors by right multiplication, where $e, f, g, h \in GF(q^2)$ with $ef - gh \neq 0$. Of course, any nonzero $GF(q)$-scalar multiple of $M_{e,f,g,h}$ induces the same collineation $\phi_{e,f,g,h}$. Note that

$$\phi_{e,f,g,h} \colon \ell_s \mapsto \ell_{\frac{fs+g}{hs+e}}, \tag{2}$$

with the usual conventions on ∞.

The full (semilinear) stabilizer of $\mathcal{S}_0$ may be represented as follows. As always, we let (X_1, X_2, X_3, X_4) denote homogeneous coordinates for $\Sigma = PG(3, q)$. Let $q = p^n$, where p is an odd prime, and let σ be an automorphism of $GF(q^2)$ given by $x \mapsto x^r$, where $r = p^i$ for some $0 < i \leq 2n$. We let ψ_σ, or ψ_r for emphasis, denote the map which takes (X_1, X_2, X_3, X_4) to $(X_1^r, \omega^{\frac{r-1}{2}} X_2^r, X_3^r, \omega^{\frac{r-1}{2}} X_4^r)$. Using the notation introduced above, straightforward computations show that

$$\psi_\sigma \colon \ell_s \mapsto \ell_{s^\sigma}, \tag{3}$$

for any choice of σ. Also each ψ_σ fixes the line ℓ_∞, and thus each ψ_σ stabilizes the regular spread $\mathcal{S}_0$. While we would normally model a semilinear automorphism of $\mathcal{S}_0$ by first applying a field automorphism to each coordinate and then multiplying by a non-singular matrix, in this setting it is more convenient to first apply some ψ_σ and then apply some $\phi_{e,f,g,h}$ as defined above. Straightforward computations show that

$$\psi_\sigma \phi_{e,f,g,h} \psi_\sigma^{-1} = \phi_{e^{\sigma^{-1}}, f^{\sigma^{-1}}, g^{\sigma^{-1}}, h^{\sigma^{-1}}},$$

and thus we get a group stabilizing $\mathcal{S}_0$ of order $2nq^2(q^2-1)(q^2+1)(q+1)$. In fact, this is well known to be the order of $\mathrm{Aut}(\mathcal{S}_0)$ (see [8], for instance). It should be noted that $\psi_q\colon (X_1, X_2, X_3, X_4) \mapsto (X_1, -X_2, X_3, -X_4)$ is actually a linear collineation, so that the (full) projective subgroup of $\mathrm{Aut}(\mathcal{S}_0)$ has order $2q^2(q^2-1)(q^2+1)(q+1)$.

4 Regular Nearfield Planes

Two-dimensional regular nearfield planes are André translation planes, and hence can be constructed via the Bruck-Bose method by starting with a regular spread in $\Sigma = PG(3,q)$ and then reversing a certain “linear” set of disjoint reguli to obtain a regular nearfield spread of Σ. For a given odd prime power q, the regular nearfield spread is uniquely determined up to equivalence.

Let $\mathcal{S}_0 = \{\ell_\infty\} \cup \{\ell_s \mid s \in GF(q^2)\}$ be the regular spread of Σ discussed in the previous section. For each $t \in GF(q)^* = GF(q) \setminus \{0\}$, $\mathcal{R}_t = \{\ell_s \mid s^{q+1} = t\}$ is a regulus contained in $\mathcal{S}_0$. In fact, $\{\mathcal{R}_t \mid t \in GF(q)^*\}$ is a linear set of $q-1$ mutually disjoint reguli in $\mathcal{S}_0$ with carriers ℓ_0 and ℓ_∞ (see [8]). Straightforward computations show that $\mathcal{R}_t^{\mathrm{opp}} = \{m_s \mid s^{q+1} = t\}$ is the opposite regulus to $\mathcal{R}_t$, where

$$m_s = \langle (1, 0, s_1, s_2), (0, 1, -\omega s_2, -s_1) \rangle,$$

with the usual representation for s in terms of the basis $\{1, \epsilon\}$. Note that the line ℓ_s consists of the points whose homogeneous coordinates satisfy $y = sx$, while m_s contains those points with $y = sx^q$. It is also useful to note that $s \in \square_{q^2}$ if and only if $s^{q+1} = s_1^2 - \omega s_2^2 \in \square_q$, and similarly for non-squares, since $s^{\frac{q^2-1}{2}} = (s^{q+1})^{\frac{q-1}{2}}$ gives the quadratic character.

Replacing the lines of a regulus by the lines in its opposite regulus (covering the same hyperbolic quadric) is often called *reversing* the regulus. Reversing all the reguli in the above linear set produces the spread

$$\mathcal{S}_0' = \{\ell_0, \ell_\infty\} \cup \{m_s \mid s \in GF(q^2);\ s \neq 0\},$$

which is another regular spread of Σ sharing only the lines ℓ_0 and ℓ_∞ with $\mathcal{S}_0$.

The regular nearfield spread of Σ is obtained by reversing a linear subset of $\frac{q-1}{2}$ reguli $\mathcal{R}_t$ in the regular spread $\mathcal{S}_0$, where the subscripts t are all the nonzero squares (or, equivalently, all the non-squares) of $GF(q)$ (see Proposition II in [22]). We choose to reverse those reguli $\mathcal{R}_t$ where t is a nonzero square in $GF(q)$, thus obtaining the regular nearfield spread

$$\begin{aligned} \mathcal{S} &= \Big(\mathcal{S}_0 \setminus \cup_{t\in\Box_q}\mathcal{R}_t\Big) \cup \Big(\cup_{t\in\Box_q} \mathcal{R}_t^{\mathrm{opp}}\Big) \\ &= \{\ell_0,\ell_\infty\} \cup \{\ell_s \mid s^{q+1} \in \not\Box_q\} \cup \{m_s \mid s^{q+1} \in \Box_q\}. \end{aligned} \tag{4}$$

It is well known (see [1], for instance) that for $q \geq 5$

$$|\operatorname{Aut}(\mathcal{S})| = 4n(q^2-1)(q+1), \tag{5}$$

where $q = p^n$ for some odd prime p. It should be noted that for $q = 3$, the regular nearfield spread is equivalent to the Hall spread of $PG(3,3)$. The automorphism group of this spread has order 1920, and it acts transitively on the 10 lines and 10 reguli of the spread. The corresponding translation plane is called the *exceptional* nearfield plane. We thus assume from now on that $q \geq 5$.

We next develop a representation for the automorphisms of the regular nearfield spread $\mathcal{S}$, using the above model. This representation is very similar to the one given in [20]. Using the notation introduced in Section 3, straightforward computations show that

$$\psi_\sigma \colon m_s \mapsto m_{s^\sigma}, \tag{6}$$

for any field automorphism σ of $GF(q^2)$. Since any field automorphism σ fixes $\Box_{q^2}$ and $\not\Box_{q^2}$ as sets, it follows from Equations (3) and (6) that each ψ_σ is an automorphism of $\mathcal{S}$. Thus it suffices to determine those matrices which induce by right multiplication (on row vectors) a linear collineation of Σ fixing $\mathcal{S}$. We first deal with those actions which fix ℓ_∞.

For each $e = e_1 + e_2\epsilon$, $f = f_1 + f_2\epsilon \in GF(q^2)$, with $ef \neq 0$, we let $\phi_{e,f}$ denote the collineation of Σ induced by

$$M_{e,f} = \begin{bmatrix} e_1 & e_2\delta_{e,f} & 0 & 0 \\ \omega e_2 & e_1\delta_{e,f} & 0 & 0 \\ 0 & 0 & f_1 & f_2 \\ 0 & 0 & \omega f_2 & f_1 \end{bmatrix},$$

where $\delta_{e,f}$ is the quadratic character of ef in $GF(q^2)$; namely

$$\delta_{e,f} = \begin{cases} 1, & \text{if } ef \in \Box_{q^2}, \\ -1, & \text{if } ef \in \not\Box_{q^2}. \end{cases}$$

It should be noted that $\phi_{e,f}$ is not the same collineation as $\phi_{e,f,0,0}$ as previously defined. Indeed, it is precisely those $\phi_{e,f}$ with $ef \in \Box_{q^2}$ that are in the stabilizer of the regular spread $\mathcal{S}_0$ as discussed in Section 3. Straightforward computations show that the above matrices form a matrix group of order $(q^2-1)^2$, which naturally induces a collineation group of Σ whose order is $\frac{(q^2-1)^2}{q-1} = (q^2-1)(q+1)$ since any nonzero $GF(q)$-scalar multiple of $M_{e,f}$ induces the same collineation $\phi_{e,f}$ of Σ.

More direct computations show that if $ef \in \square_{q^2}$, then

$$\left.\begin{array}{lcl}\phi_{e,f} & : & \ell_s \mapsto \ell_{\frac{f}{e}s},\\ \phi_{e,f} & : & m_s \mapsto m_{\frac{f}{e^q}s}.\end{array}\right\} \tag{7}$$

Similarly, if $ef \in \not\square_{q^2}$, then

$$\left.\begin{array}{lcl}\phi_{e,f} & : & \ell_s \mapsto m_{\frac{f}{e}s},\\ \phi_{e,f} & : & m_s \mapsto \ell_{\frac{f}{e^q}s}.\end{array}\right\} \tag{8}$$

In particular, since the quadratic character of ef, $\frac{f}{e}$ and $\frac{f}{e^q}$ are all the same, we see that each $\phi_{e,f}$ leaves invariant the regular nearfield spread $\mathcal{S}$ defined in (4). Note that each $\phi_{e,f}$ fixes the lines ℓ_0 and ℓ_∞.

Moreover, one easily verifies that $\psi_\sigma \phi_{e,f} \psi_\sigma^{-1} = \phi_{e^{\sigma^{-1}}, f^{\sigma^{-1}}}$, and thus

$$K = \{\psi_\sigma \phi_{e,f} \mid \sigma \in \mathrm{Aut}(GF(q^2));\ e,f \in GF(q^2)^*\}$$

is a subgroup of $\mathrm{Aut}(\mathcal{S})$ which stabilizes ℓ_∞ (and ℓ_0). In fact, K is the semidirect product of $\{\phi_{e,f} \mid e,f \in GF(q^2)^*\}$ by $\{\psi_\sigma \mid \sigma \in \mathrm{Aut}(GF(q^2))\}$, and hence is an index-two subgroup in $\mathrm{Aut}(\mathcal{S})$ by Equation (5). To complete the description of $\mathrm{Aut}(\mathcal{S})$, we define ν to be the linear collineation of Σ induced by the mapping

$$(X_1, X_2, X_3, X_4) \mapsto (X_3, X_4, X_1, X_2).$$

One easily computes that

$$\left.\begin{array}{lcl}\nu & : & \ell_s \mapsto \ell_{\frac{1}{s}},\\ \nu & : & m_s \mapsto m_{\frac{1}{s^q}}\end{array}\right\} \tag{9}$$

for any $s \neq 0$, and observe that ν interchanges ℓ_0 and ℓ_∞. Hence the involution ν leaves invariant the regular nearfield spread $\mathcal{S}$ defined in (4). Moreover, one easily checks that $\nu\psi_\sigma\phi_{e,f}\nu = \psi_\sigma\phi_{f,e}$ when $ef \in \square_{q^2}$, and $\nu\psi_\sigma\phi_{e,f}\nu = \psi_{q\sigma}\phi_{f^q,e^q}$ when $ef \in \not\square_{q^2}$. Letting J be the cyclic subgroup generated by the involution ν, we have thus shown that $G = K \rtimes J$ is the full automorphism group of $\mathcal{S}$.

Next we obtain $\mathrm{Aut}(\pi(\mathcal{S}))$ from $\mathrm{Aut}(\mathcal{S})$ using the Bruck-Bose representation in $\bar{\Sigma}$ as described in Section 2. Thus we know that $|\,\mathrm{Aut}(\pi(\mathcal{S}))| =$

$4nq^4(q^2-1)^2$. To account for field automorphisms, we use the lifting $\bar{\psi}_r$ of ψ_r which takes $(X_0, X_1, X_2, X_3, X_4)$ to $(X_0^r, X_1^r, \omega^{\frac{r-1}{2}} X_2^r, X_3^r, \omega^{\frac{r-1}{2}} X_4^r)$. Defining linear collineations $\bar{\phi}_{e,f}$ and $\bar{\nu}$ in a similar fashion (bordering the associated matrices with a 1 in the $(1,1)$-position and 0's elsewhere in the first row and column), we see that every element of the translation complement of $\pi(\mathcal{S})$ may be written uniquely as $\bar{\psi}_r\bar{\phi}_{e,f}$ or as $\bar{\psi}_r\bar{\phi}_{e,f}\bar{\nu}$, for some $e, f \in GF(q^2)^*$ and some integer $r = p^i$ with $0 < i \leq 2n$. In particular, this translation complement has order $4n(q^2-1)^2$ since $\bar{\phi}_{e,f} \neq \bar{\phi}_{e',f'}$ whenever $(e,f) \neq (e',f')$.

The previously described group actions show that $\mathrm{Aut}(\mathcal{S})$ has two orbits on the lines of $\mathcal{S}$, namely $\{\ell_0, \ell_\infty\}$ and $\mathcal{S} \setminus \{\ell_0, \ell_\infty\}$. More importantly, for our purposes we need to discuss the action of $\mathrm{Aut}(\mathcal{S})$ on the reguli of $\mathcal{S}$. While this action seems to be well-known, we could not find any suitable reference in print and hence we briefly discuss here the results that we will need.

3 Theorem. *Let $q \geq 5$ be an odd prime power. Then the regular nearfield spread $\mathcal{S}$ of Σ has precisely $2q$ reguli.*

Proof. We assume $\mathcal{S}$ is defined as in Equation (4), and thus $\mathcal{S}$ contains at least $q-1$ reguli, namely $\{\mathcal{R}_t \mid t \in \not\square_q\} \cup \{\mathcal{R}_t^{\mathrm{opp}} \mid t \in \square_q\}$. These reguli partition the lines of $\mathcal{S} \setminus \{\ell_0, \ell_\infty\}$. Also note that every line of $\mathcal{S}$ belongs to either $\mathcal{S}_0$ or $\mathcal{S}_0'$, both of which are regular spreads. Moreover, the carriers $\{\ell_0, \ell_\infty\}$ belong to both $\mathcal{S}_0$ and $\mathcal{S}_0'$.

Now let $\mathcal{R}$ denote an arbitrary regulus contained in $\mathcal{S}$. Then $\mathcal{R}$ contains zero, one, or two lines from the carriers $\{\ell_0, \ell_\infty\}$. Suppose first that $\mathcal{R}$ contains both of these lines, and let $\ell \in \mathcal{R} \setminus \{\ell_0, \ell_\infty\}$. If $\ell \in \mathcal{S}_0$, then $\mathcal{R} \subset \mathcal{S}_0$ since $\mathcal{S}_0$ is a regular spread. Similarly, if $\ell \in \mathcal{S}_0'$, then $\mathcal{R} \subset \mathcal{S}_0'$. Thus $\mathcal{R}$ consists of $\{\ell_0, \ell_\infty\}$ and either two lines from each regulus in $\{\mathcal{R}_t \mid t \in \not\square_q\}$ or two lines from each regulus in $\{\mathcal{R}_t^{\mathrm{opp}} \mid t \in \square_q\}$. From the well-known correspondence between the lines and reguli of a regular spread in Σ and the points and circles of a Miquelian inversive plane (see [8], for instance), and using the associated computational techniques (as discussed in [17], for instance), one immediately sees that there are $\frac{1}{2}(q+1)$ reguli of $\mathcal{S}_0$ through $\{\ell_0, \ell_\infty\}$ which contain two lines from each of $\{\mathcal{R}_t \mid t \in \not\square_q\}$. And there are $\frac{1}{2}(q+1)$ reguli of $\mathcal{S}_0'$ through $\{\ell_0, \ell_\infty\}$ which contain two lines from each of $\{\mathcal{R}_t^{\mathrm{opp}} \mid t \in \square_q\}$. Thus we get a total of $q+1$ choices for a regulus $\mathcal{R}$ in $\mathcal{S}$ that contains both ℓ_0 and ℓ_∞.

Now suppose that $\mathcal{R}$ contains exactly one of ℓ_0, ℓ_∞. The remaining $q \geq 4$ lines of $\mathcal{R}$ are partitioned by $\mathcal{S}_0 \setminus \{\ell_0, \ell_\infty\}$ and $\mathcal{S}_0' \setminus \{\ell_0, \ell_\infty\}$. Thus $\mathcal{R}$ must be a regulus of $\mathcal{S}_0$ or $\mathcal{S}_0'$, since both are regular spreads, and so without loss of generality we may assume that $\mathcal{R} \subset \mathcal{S}_0$. Then the remaining q lines of $\mathcal{R}$ must include at least three lines from one of $\{\mathcal{R}_t \mid t \in \not\square_q\}$, which implies that $\mathcal{R}$ must be one of these reguli. But this contradicts the fact that $\mathcal{R}$ contains one of $\{\ell_0, \ell_\infty\}$, and thus this case cannot occur.

Finally, suppose that $\mathcal{R}$ contains neither ℓ_0 nor ℓ_∞. The same reasoning as above shows that $\mathcal{R}$ must be one of $\{\mathcal{R}_t \mid t \in \not\square_q\} \cup \{\mathcal{R}_t^{\mathrm{opp}} \mid t \in \square_q\}$. As these $q-1$ reguli are indeed reguli of $\mathcal{S}$, we have a total of $2q$ choices for $\mathcal{R}$ and the result follows. $\boxed{QED}$

It is interesting to note that $q+1$ of the reguli in $\mathcal{S}$ contain $\{\ell_0, \ell_\infty\}$, and the remaining $q-1$ reguli in $\mathcal{S}$ are disjoint from $\{\ell_0,\ \ell_\infty\}$. Moreover, q of the reguli in $\mathcal{S}$ are reguli of the regular spread $\mathcal{S}_0$, and the remaining q reguli in $\mathcal{S}$ are reguli of the regular spread $\mathcal{S}_0'$.

4 Theorem. *Let $q \geq 5$ be an odd prime power, and let $\mathcal{S}$ be a regular nearfield spread of Σ. Then* $\mathrm{Aut}(\mathcal{S})$ *has precisely two orbits on the $2q$ reguli contained in $\mathcal{S}$.*

Proof. We again assume that $\mathcal{S}$ is defined as in (4), and we let $\mathcal{O}_1 = \{\mathcal{R}_t \mid t \in \not\square_q\} \cup \{\mathcal{R}_t^{\mathrm{opp}} \mid t \in \square_q\}$. Since $\mathrm{Aut}(\mathcal{S})$ leaves invariant $\{\ell_0,\ \ell_\infty\}$ and maps reguli of $\mathcal{S}$ to reguli of $\mathcal{S}$, the proof of Theorem 3 implies that $\mathrm{Aut}(\mathcal{S})$ naturally acts on the $q-1$ reguli in $\mathcal{O}_1$. Choose $\mathcal{R}_1^{\mathrm{opp}} = \{m_s \mid s^{q+1} = 1\} \in \mathcal{O}_1$, and consider the stabilizer of this regulus in $G = \mathrm{Aut}(\mathcal{S})$.

From our description of $G = K \rtimes J$ and using Equations (3), (6), (7), (8), and (9), we see that the stabilizer of $\mathcal{R}_1^{\mathrm{opp}}$ is $K_1 \rtimes J$, where

$$K_1 = \{\psi_\sigma \phi_{e,f} \mid \sigma \in \mathrm{Aut}(GF(q^2);\ e, f \in GF(q^2)^*;\ (f/e)^{q+1} = 1\}.$$

Now there are $q+1$ choices for f, given any $e \in GF(q^2)^*$, so that $(f/e)^{q+1} = 1$. And for any such pair (e, f), the $q-1$ pairs (ke, kf), as k varies over $GF(q)^*$, all satisfy this condition. Hence elementary counting shows that the order of the stabilizer of $\mathcal{R}_1^{\mathrm{opp}}$ is $2n(q+1)^2(2) = 4n(q+1)^2$. An application of the Orbit-Stabilizer Theorem now shows that $\mathcal{O}_1$ is a single orbit under $\mathrm{Aut}(\mathcal{S})$.

Next we define $\mathcal{O}_2$ to be the remaining $q+1$ reguli contained in $\mathcal{S}$, all of which contain $\{\ell_0, \ell_\infty\}$ by the proof of Theorem 3. If $\mathcal{R} = \{m_s \mid s \in GF(q)^*\} \cup \{\ell_0, \ell_\infty\}$, then $\mathcal{R}$ is a subset of $q+1$ lines contained in $\mathcal{S}$ from Equation (4), and straightforward computations show that $\mathcal{R}$ is in fact a regulus of $\mathcal{S}$. Namely the hyperbolic quadric covered by the lines in $\mathcal{R}$ has equation $X_1X_4 + X_2X_3 = 0$. In particular, $\mathcal{R}$ is a regulus in $\mathcal{O}_2$. Again $G = \mathrm{Aut}(\mathcal{S})$ acts naturally on the reguli in $\mathcal{O}_2$. From our previous discussion of $G = K \rtimes J$ and using the same group action equations as above, we see that the stabilizer of $\mathcal{R}$ is $K_2 \rtimes J$, where

$$K_2 = \{\psi_\sigma \phi_{e,f} \mid \sigma \in \mathrm{Aut}(GF(q^2);\ e, f \in GF(q^2)^*;\ \frac{f}{e^q} \in GF(q)^*\}.$$

There are exactly $(q^2-1)(q-1)$ pairs (e, f) satisfying the condition $\frac{f}{e^q} \in GF(q)^*$, as we may choose any nonzero e and set $f = te^q$ for each $t \in GF(q)^*$. Moreover, this set includes all $GF(q)$-homogeneous pairs. Hence the stabilizer of $\mathcal{R}$ has order $2n(q^2-1)(2) = 4n(q^2-1)$, and an application of the Orbit-Stabilizer Theorem shows that $\mathcal{O}_2$ is a single orbit under $\mathrm{Aut}(\mathcal{S})$, proving the result. $\boxed{QED}$

We set aside for future use the following stabilizer results contained in the above proof.

5 Corollary. *Let $q = p^n \geq 5$ be an odd prime power, and let $\mathcal{R}_1^{\mathrm{opp}}$ and $\mathcal{R}$ be the inequivalent reguli in the regular nearfield $\mathcal{S}$ described in the proof of Theorem 4. Then*

(1) the full stabilizer of $\mathcal{R}_1^{\mathrm{opp}}$ in $\mathrm{Aut}(\mathcal{S})$ has order $4n(q+1)^2$,

(2) the full stabilizer of $\mathcal{R}$ in $\mathrm{Aut}(\mathcal{S})$ has order $4n(q^2-1)$.

Finally, we determine all mutually inequivalent nonsingular Buekenhout unitals embedded in the regular nearfield plane $\pi(\mathcal{S})$, and compute their stabilizer subgroups.

6 Theorem. *Let $q \geq 5$ be an odd prime power, and let $\mathcal{S}$ be a regular nearfield spread of Σ. Then the regular nearfield plane $\pi(\mathcal{S})$ has precisely two inequivalent nonsingular Buekenhout unitals embedded in it.*

Proof. We need only consider one regulus from each of the two orbits $\mathcal{O}_1, \mathcal{O}_2$ as given in Theorem 4, whose notation we use here. In particular, for orbit $\mathcal{O}_1$ we again choose $\mathcal{R}_1^{\mathrm{opp}}$ as its representative. Straightforward computations show that the hyperbolic quadric $\mathcal{H}_1$ covered by the lines of $\mathcal{R}_1^{\mathrm{opp}}$ has equation

$$\mathcal{H}_1 \colon X_1^2 - wX_2^2 - X_3^2 + wX_4^2 = 0.$$

By Theorem 1 there is a single equivalence class of nonsingular Buekenhout unitals arising from $\mathcal{H}_1$ if there exists a semilinear transformation associated with some $\tau \in \mathrm{Aut}(\mathcal{S})$ that takes $\mathcal{Q}_1(X_1,X_2,X_3,X_4) = X_1^2 - \omega X_2^2 - X_3^2 + \omega X_4^2$ to $\omega\mathcal{Q}_1$ More straightforward computations show that the matrix $M_{\epsilon,\epsilon}N$ corresponding to $\phi_{\epsilon,\epsilon}\nu$ provides such a transformation. We let U_1 denote a unital of $\pi(\mathcal{S})$ from this uniquely determined equivalence class.

Similarly, for the orbit $\mathcal{O}_2$ we choose $\mathcal{R} = \{\ell_0, \ell_\infty\} \cup \{m_s \mid s \in GF(q)^*\}$ as its representative. The hyperbolic quadric $\mathcal{H}_2$ covered by the lines of $\mathcal{R}$ has equation

$$\mathcal{H}_2 \colon X_1X_4 + X_2X_3 = 0.$$

As in the above case, we need only show that there exists a semilinear transformation associated with some τ in $\mathrm{Aut}(\mathcal{S})$ that takes $\mathcal{Q}_2(X_1,X_2,X_3,X_4) = X_1X_4 + X_2X_3$ to $\omega\mathcal{Q}_2$ to show that the "lifted" nonsingular Buekenhout unital is unique up to equivalence. The matrix $M_{\epsilon,\epsilon}$ corresponding to $\tau = \phi_{\epsilon,\epsilon}$ provides such a transformation, and we let U_2 denote a unital from this uniquely determined equivalence class.

As the unitals U_1 and U_2 are clearly inequivalent, we see that $\pi(\mathcal{S})$ contains precisely two inequivalent nonsingular Buekenhout unitals. $\boxed{QED}$

7 Theorem. *Let $q = p^n \geq 5$ be an odd prime power, and let U_1 and U_2 be the inequivalent nonsingular Buekenhout unitals embedded in the regular nearfield plane $\pi(\mathcal{S})$, as described in the proof of Theorem 6. Then*

(1) the full stabilizer of U_1 in $\mathrm{Aut}(\pi(\mathcal{S}))$ has order $4n(q+1)^2$,

(2) the full stabilizer of U_2 in $\mathrm{Aut}(\pi(\mathcal{S}))$ has order $4n(q^2-1)$.

Proof. The result follows immediately from Theorem 2 and Corollary 5. $\boxed{QED}$

5 Hall planes

To construct a Hall spread, unique up to equivalence, we start with a regular spread in $\Sigma = PG(3,q)$ and reverse a regulus. Using the notation of the previous sections, we work with the regular spread $\mathcal{S}_0 = \{\ell_s \mid s \in GF(q^2) \cup \{\infty\}\}$ of Σ and reverse the regulus $\mathcal{R}_1 = \{\ell_s \mid s^{q+1} = 1\}$ to obtain the Hall spread $\mathcal{S}$. That is,

$$\begin{aligned} \mathcal{S} &= \Big(\mathcal{S}_0 \setminus \mathcal{R}_1\Big) \cup \Big(\mathcal{R}_1^{\mathrm{opp}}\Big) \\ &= \{\ell_\infty\} \cup \{\ell_s \mid s^{q+1} \neq 1\} \cup \{m_s \mid s^{q+1} = 1\}. \end{aligned}$$

We now develop a representation for the automorphisms of $\mathcal{S}$, just as we did for the regular nearfield spread.

Since $q > 3$, all collineations of $\mathrm{Aut}(\mathcal{S})$ are inherited from collineations of $\mathrm{Aut}(\mathcal{S}_0)$, where $\mathcal{S}_0$ is the regular spread used in the construction of $\mathcal{S}$ (see [19]). Hence the stabilizer of $\mathcal{R}_1$ in $\mathrm{Aut}(\mathcal{S}_0)$ is precisely $\mathrm{Aut}(\mathcal{S})$, where $\mathcal{S}$ is the Hall spread defined above. From Equation (3) we see that every $\psi_\sigma \in \mathrm{Aut}(\mathcal{S})$, and thus we concentrate on determining which $\phi_{e,f,g,h}$ stabilize $\mathcal{R}_1$. From Equation (2) we see that $\phi_{e,f,g,h} \in \mathrm{Aut}(\mathcal{S})$ if and only if $(fs+g)^{q+1} = (hs+e)^{q+1}$ whenever $s^{q+1} = 1$. This is true if and only if

$$(f^q g - h^q e)s^q + (g^q f - e^q h)s + f^{q+1} + g^{q+1} - h^{q+1} - e^{q+1} = 0$$

for all $s \in GF(q^2)$ such that $s^{q+1} = 1$. Treating the above equation as a polynomial in s of degree q with $q+1$ roots, we see that this polynomial must be the zero polynomial, and we obtain the necessary and sufficient conditions that $f^q g = h^q e$ and $e^{q+1} - f^{q+1} - g^{q+1} + h^{q+1} = 0$. Multiplying the second equation by ef and then using the first equation (and its companion $fg^q = he^q$) to substitute, we get $(ef - gh)e^{q+1} = (ef - gh)f^{q+1}$ and therefore $e^{q+1} = f^{q+1}$, which further implies that $g^{q+1} = h^{q+1}$. Thus $\phi_{e,f,g,h} \in \mathrm{Aut}(\mathcal{S})$ if and only if

$$e^{q+1} = f^{q+1} \quad \neq \quad g^{q+1} = h^{q+1}, \tag{10}$$

$$f^q g \quad = \quad h^q e. \tag{11}$$

Note that the condition $f^{q+1} \neq g^{q+1}$ in the above system is equivalent to the non-singularity condition $ef - gh \neq 0$ for $\phi_{e,f,g,h}$.

It is now straightforward counting (partitioning into cases $e = 0$ and $e \neq 0$) to show that the number of choices for (e, f, g, h) is

$$(q^2 - 1)(q + 1) + (q^2 - 1)(q + 1)(q^2 - (q + 1)) = q(q^2 - 1)^2,$$

and thus $|\,\mathrm{Aut}(\mathcal{S})| = 2nq(q^2 - 1)^2/(q - 1) = 2nq(q^2 - 1)(q + 1)$ for any odd prime power $q > 3$ (see [19]).

We next determine the orbit structure on reguli contained in the Hall spread $\mathcal{S}$.

8 Theorem. *Let $q > 3$ be any prime power. Then the Hall spread $\mathcal{S}$ of Σ has precisely $\frac{1}{2}q(q - 1)(q - 2) + 1$ reguli.*

Proof. Since $q \geq 4$, any regulus in $\mathcal{S}$ not equal to $\mathcal{R}_1^{\mathrm{opp}}$ contains at most two lines of $\mathcal{R}_1^{\mathrm{opp}}$ and thus must contain at least three lines of $\mathcal{S}_0$. Hence such a regulus is contained in the regular spread $\mathcal{S}_0$, and is necessarily disjoint from $\mathcal{R}_1^{\mathrm{opp}}$. From the previously mentioned one-to-one correspondence between the lines and reguli of $\mathcal{S}_0$ and the points and circles of the Miquelian inversive plane $M(q)$, it follows that there are precisely $\frac{1}{2}q(q - 1)(q - 2)$ reguli disjoint from a given regulus (see [14]). The result now follows immediately. $\boxed{QED}$

To determine the orbit structure on the reguli contained in the Hall spread, we find it useful to introduce the concept of self-reciprocal polynomials. Since this is a rather abrupt shift in content, we briefly review this notion and some related elementary facts. The *reciprocal polynomial* f^* of a polynomial f of degree $d > 0$ is given by $f^*(x) = x^d f(\frac{1}{x})$, and f is said to be *self-reciprocal* if $f = f^*$. For any $t \in GF(q)^*$, where $q = p^n$ for some (odd) prime p, either the (monic) minimal polynomial f of t over $GF(p)$ is self-reciprocal, or the least degree self-reciprocal monic polynomial in $GF(p)[x]$ which has t as a root is an associate of $f(x)f^*(x)$, namely $\frac{1}{f(0)}f(x)f^*(x)$. In any case, the monic least degree self-reciprocal polynomial which has t as a root is referred to as the *minimal self-reciprocal monic polynomial* of t over $GF(p)$, or *min-srmp* for short. Note that when $t = -1$, the min-srmp is $x + 1$ and this is the only min-srmp of degree one. When $t = 1$, the min-srmp is $x^2 - 2x + 1 = (x - 1)^2$, and this is the only min-srmp with a repeated root. Moreover, the binary relation on

$GF(q) \setminus \{0,1\}$ defined by $t_1 \sim t_2$ if and only if t_1 and t_2 have the same min-srmp is an equivalence relation with equivalence class sizes given by the degrees of the various min-srmp. The importance of min-srmp for our purposes lies in the following lemma.

9 Lemma. *Let $q = p^n \geq 5$ be an odd prime power, and consider the previously defined reguli $\mathcal{R}_{t_1}$ and $\mathcal{R}_{t_2}$ contained in the Hall spread $\mathcal{S}$, for some $t_1, t_2 \in GF(q) \setminus \{0,1\}$. Then $\mathcal{R}_{t_1}$ and $\mathcal{R}_{t_2}$ are equivalent under $G = \mathrm{Aut}(\mathcal{S})$ if and only if t_1 and t_2 have the same min-srmp. Moreover, the stabilizer of $\mathcal{R}_{t_1}$ in G has order $\frac{4n}{d}(q+1)^2$, where d is the degree of the min-srmp of t_1 over $GF(p)$.*

PROOF. Recall that each element of $\mathrm{Aut}(\mathcal{S})$ may be written as $\psi_\sigma \phi_{e,f,g,h}$, subject to the conditions in Equation (10) and Equation (11). Since ψ_σ maps the regulus $\mathcal{R}_{t_1}$ to $\mathcal{R}_{t_1^\sigma}$ by Equation (3), we may concentrate on those $\phi_{e,f,g,h}$ which map $\mathcal{R}_{t_1^\sigma}$ to $\mathcal{R}_{t_2}$. Computations similar to those given above show that such a mapping exists if and only if

$$(f^q g - t_2 h^q e)s^q + (g^q f - t_2 e^q h)s + t_1^\sigma f^{q+1} + g^{q+1} - t_1^\sigma t_2 h^{q+1} - t_2 e^{q+1} = 0$$

for all $s \in GF(q^2)$ with $s^{q+1} = t_1^\sigma$. The usual polynomial argument shows that this is equivalent to the above polynomial in s being the zero polynomial, and hence we obtain the necessary and sufficient conditions

$$\begin{aligned} (1 - t_2)h^q e &= 0, \\ (t_1^\sigma - t_2)f^{q+1} + (1 - t_1^\sigma t_2)g^{q+1} &= 0. \end{aligned}$$

Since $t_2 \neq 1$, we necessarily have $h = 0$ or $e = 0$. Conditions (10) further imply that either $g = h = 0$ or $e = f = 0$, but not both, and in either case Condition (11) is satisfied. Moreover, in the first case we have $t_2 = t_1^\sigma$, and in the second case we have $t_2 = \frac{1}{t_1^\sigma}$. As the roots of the min-srmp of t_1 are precisely these elements t_2, as σ varies over all automorphisms of $GF(q^2)$, the first assertion in the lemma follows. Note that each of these roots is obtained twice as σ varies over $\mathrm{Aut}(GF(q^2))$ (and once each as σ varies over $\mathrm{Aut}(GF(q))$).

To prove the second assertion, we observe that $H = \{\psi_\sigma \phi_{e,f,g,h} \mid g = h = 0 \text{ or } e = f = 0\}$ is a subgroup of $\mathrm{Aut}(\mathcal{S})$ of order $4n(q+1)^2$ which contains the stabilizer of $\mathcal{R}_{t_1}$ and acts transitively on the collection $\{\mathcal{R}_t \mid t \sim t_1\}$. The stated order for the stabilizer of $\mathcal{R}_{t_1}$ then follows from the Orbit-Stabilizer Theorem. $\boxed{QED}$

Next observe that $\frac{x^{q-1}-1}{x-1}$ is a self-reciprocal monic polynomial in $GF(p)[x]$ which factors as the product of the distinct min-srmp for $t \in GF(q) \setminus \{0,1\}$ (each such min-srmp is either a minimal polynomial or the product of two distinct minimal polynomials). Let N denote the number of factors in this factorization, and write

$$\frac{x^{q-1}-1}{x-1} = f_1(x)f_2(x)\cdots f_N(x).$$

Let d_i denote the degree of $f_i(x)$, and let t_i be one of its roots. Clearly $\sum_1^N d_i = q-2$.

10 Theorem. *Let $q \geq 5$ be an odd prime power, let $\mathcal{S}$ be the Hall spread of Σ, and let $G = \mathrm{Aut}(\mathcal{S})$. Then G has precisely $1+N$ orbits on the reguli contained in $\mathcal{S}$. These orbits are represented by the regulus $\mathcal{R}_1^{\mathrm{opp}}$ and the reguli $\mathcal{R}_{t_i}, i = 1, 2, \ldots, N$.*

Proof. Clearly, $\{\mathcal{R}_1^{\mathrm{opp}}\}$ is an orbit of size one under the action of G. When the full stabilizer $G = \mathrm{Aut}(\mathcal{S})$ is used in place of H from Lemma 9, we see that the orbit of $\mathcal{R}_{t_i}$ has size $d_i\frac{q(q-1)}{2}$ since $[G : H] = \frac{q(q-1)}{2}$. Summing these sizes, we see that the orbits of the various $\mathcal{R}_t$ account for

$$\sum_1^N d_i \frac{q(q-1)}{2} = \frac{q(q-1)}{2}\sum_1^N d_i = \frac{q(q-1)(q-2)}{2}$$

reguli, and the result follows from Theorem 8. $\boxed{QED}$

It is an easy computation to determine the number N for a specific value of $q = p^n$, however a general formula would require the use of Möbius inversion (which is needed to count the number of irreducible polynomials of a given degree). When $n = 1$, so q is prime, it is easily seen that $N = \frac{1}{2}(q-1)$. If n is odd, then since $x+1$ is the only self-reciprocal polynomial of odd degree, it follows that N is $\frac{1}{2}(1+N_0)$, where N_0 is the number of irreducible factors in the complete factorization of $\frac{x^{q-1}-1}{x-1}$ over $GF(p)$. As another example, one can verify that $N = \frac{(p-1)(p+3)}{4}$ for $q = p^2$ (where there are $\frac{p-1}{2}$ self-reciprocal polynomials of degree 2). In any case N is a polynomial in p whose leading term is always $\frac{q}{2n}$.

11 Theorem. *Let $q \geq 5$ be an odd prime power, and let $\mathcal{S}$ be the Hall spread of $\Sigma = PG(3,q)$. Then the Hall plane $\pi(\mathcal{S})$ has precisely $1+N$ mutually inequivalent nonsingular Buekenhout unitals, where N is the number of min-srmp factors of $\frac{x^{q-1}-1}{x-1}$ in $GF(p)[x]$. If q is prime, then $1+N = \frac{q+1}{2}$ is the number of inequivalent nonsingular Buekenhout unitals in $\pi(\mathcal{S})$.*

Proof. By Theorem 10, it suffices to consider those nonsingular quadrics of $\bar{\Sigma}$ that meet Σ in one of the hyperbolic quadrics $\mathcal{H}_t$, covered by the lines of the regulus $\mathcal{R}_t$, for some nonzero t (note we have used the fact that $\mathcal{R}_1$ and $\mathcal{R}_1^{\mathrm{opp}}$ cover the same points). Moreover, by Theorem 1 we know that for a given

t, we only need consider the following two parabolic quadrics in $\bar{\Sigma}$:

$$\begin{aligned} \mathcal{U}_t &: X_0^2 - tX_1^2 + t\omega X_2^2 + X_3^2 - \omega X_4^2 = 0 \\ \mathcal{U}_t' &: X_0^2 - \omega\big[tX_1^2 - t\omega X_2^2 - X_3^2 + \omega X_4^2\big] = 0. \end{aligned}$$

Since the linear collineation $\phi_{\beta,\beta,0,0} \in \mathrm{Aut}(\mathcal{S})$ has an associated isometry that maps $\mathcal{Q}_0(X_1, X_2, X_3, X_4) = tX_1^2 - t\omega X_2^2 - X_3^2 + \omega X_4^2$ to $\omega\mathcal{Q}_0$, by Theorem 1 there is a single equivalence class of nonsingular Buekenhout unitals arising from $\mathcal{H}_t$ for any given value of t, and the result follows. $\boxed{QED}$

We let U_t denote the unital embedded in the Hall plane $\pi(\mathcal{S})$ corresponding to the nonsingular quadric $\mathcal{U}_t$ of $\bar{\Sigma}$ described above.

12 Theorem. *Let $q = p^n \geq 5$ be an odd prime power, and consider nonsingular Buekenhout unitals U_t embedded in the Hall plane $\pi(\mathcal{S})$. For $t \neq 1$, let d be the degree of the min-srmp of t over $GF(p)$. Then*

(1) the full stabilizer of U_1 in $\mathrm{Aut}(\pi(\mathcal{S}))$ has order $2nq(q^2-1)(q+1)$,

(2) the full stabilizer of U_t in $\mathrm{Aut}(\pi(\mathcal{S}))$, for $t \neq 1$, has order $\frac{4n}{d}(q+1)^2$.

Proof. The result follows immediately from Theorem 2, the proof of Theorem 11, Lemma 9, and the fact that $\mathrm{Aut}(\mathcal{S})$ leaves invariant $\mathcal{R}_1^{\mathrm{opp}}$. $\boxed{QED}$

13 Remark. Note that $d = 1$ when $t = -1$, and $d = 2$ when $t \in GF(p) \setminus \{0, 1, -1\}$. Hence, if $q = p$ is prime, then there is one such stabilizer of order $2q(q^2-1)(q+1)$, one of order $4(q+1)^2$, and $\frac{1}{2}(q-3)$ of order $2(q+1)^2$.

6 Flag-transitive planes

An affine plane is called *flag-transitive* if the automorphism group of the affine plane acts transitively on point-line incident pairs. By a result of Wagner [25] every finite flag-transitive affine plane is necessarily a translation plane and hence arises from some spread. Using the translation group which acts transitively on the affine points, one sees that a finite affine plane is flag-transitive if and only if the associated spread admits an automorphism group which acts transitively on it.

In [3] the two-dimensional flag-transitive affine planes of odd order q^2 are classified modulo a certain gcd condition involving q; namely, provided $\gcd(\frac{1}{2}(q^2+1), n) = 1$ where $q = p^n$ with p an odd prime. In particular, if $q = p$ is an odd prime, this gcd condition is certainly satisfied and the enumeration shows that there are precisely $\frac{1}{2}(p-1)$ mutually non-isomorphic two-dimensional flag-transitive affine planes of order p^2. A formula involving the Euler totient function can be given in the general setting (see [3]), again subject to the above gcd condition. Moreover, all the associated spreads in $PG(3,q)$ are shown in [4] to be regulus-free, and thus the projective completions of these affine planes have no embedded nonsingular Buekenhout unitals. The only known two-dimensional flag-transitive affine planes of even order are the Lüneburg-Tits planes order q^2, where $q \geq 8$ is an odd power of 2. This plane is uniquely determined, up to isomorphism, for each such value of q. Once again the associated spread in $PG(3,q)$ is known to be regulus-free (see [8]). Hence we have the following result.

14 Theorem. *The projective completions of all known two-dimensional flag-transitive affine planes have no embedded nonsingular Buekenhout unitals.*

7 Index-Two André Planes

The two-dimensional André planes of *index two* are translation planes of order q^2 which can be constructed via the Bruck-Bose method by starting with a regular spread in $\Sigma = PG(3,q)$ and reversing a (linear) pair of disjoint reguli. It is determined in [8] that there are precisely $\frac{1}{2}(p-1)$ such planes of order p^2 when $q = p \geq 5$ is prime. In fact, using the information developed above for the Hall plane, it is straightforward to show that the number of such planes of order q^2, for any odd prime power q, is given by the integer N in the statement of Theorem 10. Moreover, if $\mathcal{R}_1$ and $\mathcal{R}_{t_0}$ are reversed to obtain the index two André spread, it is not difficult to show, at least for $q \geq 7$, that the stabilizer of such an André spread has order $\frac{8n}{d_0}(q+1)^2$, where d_0 is given as in Lemma 9. Before making additional observations, we provide some computer data about the embedded nonsingular Buekenhout unitals in these planes for small values of q. All data was obtained using the software package Magma [13].

For $q \geq 7$, every regulus in such an André spread is inherited from the ambient regular spread (the second line of the table, where $q = 5$ and $t_0 = -1$, has only 7 of the 10 reguli inherited). The proof that the corresponding hyperbolic quadric lifts uniquely to a Buekenhout nonsingular unital again follows from Theorem 1 exactly as in the proof of Theorem 11 since the required mapping is in the automorphism group. Thus the number of mutually inequivalent Buekenhout nonsingular unitals in such an André plane is the same as the number of regulus orbits in the associated spread. Counting the number of inherited reguli in the Andrè spread requires a general representation for the reguli of the regular spread $\mathcal{S}_0$ or an equivalent use of counting in the Miquelian inversive plane $M(q)$; however, it is clear that this number is either $(q-1) + \frac{1}{4}(q+1)(q-3)(q-5) + \frac{1}{2}(q+1)$ or $(q-1) + \frac{1}{4}(q+1)(q-3)(q-5)$, according to whether t_0 is a square or a nonsquare. The $q-1$ reguli from the linear set will always be a union of orbits, each of length m for some divisor m

of $4n$. The extra $\frac{1}{2}(q+1)$ reguli that occur when t_0 is a square are always a single orbit. The remaining $\frac{1}{4}(q+1)(q-3)(q-5)$ reguli are all in orbits whose length is $m(q+1)$ for some divisor m of $4n$. Thus we do not expect there to be a simple explicit expression for the number of inequivalent such unitals in an index two André plane of odd order q^2. The stabilizer orders of the table are as expected given the comments on the lengths of the orbits of reguli.

q	$\lvert \mathrm{Aut}(\mathcal{S})\rvert$	regs in $\mathcal{S}$	orbits	unitals	stabilizer orders
5	$2^5 3^2$	4	1	1	$2^3 3^2$
5	$2^6 3^2$	10	2	2	$2^4 3^2, 2^5 3$
7	2^8	26	7	7	$2^8, 2^8, 2^7, 2^7, 2^6, 2^5, 2^5$
7	2^8	22	4	4	$2^7, 2^7, 2^7, 2^4$
7	2^9	22	3	3	$2^8, 2^7, 2^5$
9	$2^6 5^2$	73	6	6	$2^5 5^2, 2^5 5^2, 2^4 5^2,$ $2^6 5, 2^4 5, 2^3 5$
9	$2^5 5^2$	73	8	8	$2^4 5^2, 2^4 5^2, 2^4 5^2, 2^4 5^2,$ $2^5 5, 2^3 5, 2^3 5, 2^3 5$
9	$2^4 5^2$	68	7	7	$2^3 5^2, 2^3 5^2, 2^3 5^2,$ $2^3 5^2, 2^2 5, 2^2 5, 2^2 5$

Table 1. Nonsingular Buekenhout Unitals in Index-Two André Planes

8 Conclusion

The techniques developed in this paper may be used with any translation plane, provided one has a convenient description of the full automorphism group of the spread $\mathcal{S}$. A determination of distinct representatives for the orbits on reguli of $\mathcal{S}$ will then allow one to enumerate the nonsingular Buekenhout unitals embedded in $\pi(\mathcal{S})$ and compute their stabilizers in $\mathrm{Aut}(\pi(\mathcal{S}))$. It should also be noted that this approach gives an alternative (short) proof of Barwick's result that the only nonsingular Buekenhout unital embedded in the Desarguesian plane is the classical unital.

While Witt's Theorem could have been used to help sort out the equivalences among these nonsingular Buekenhout unitals, our approach seemed to be more insightful with respect to the stabilizers of such unitals. In a future paper we will use a similar technique to enumerate the orthogonal Buekenhout unitals embedded in various families of two-dimensional translation planes.

A number of authors have dealt with Buekenhout unitals embedded in the Hall plane, although no attempt at a complete enumeration had previously been made. For example, Barwick [6] and Rinaldi [24] studied nonsingular Buekenhout unitals in the Hall plane that are inherited from classical unitals in the Desarguesian plane, while Dover [15] studied unitals in the Hall plane that are inherited from non-classical unitals in the Desarguesian plane. In fact, Dover's approach lead to the construction of unitals embedded in the Hall plane which do not arise from either of Buekenhout's constructions, the only such known examples in any two-dimensional translation plane. Grüning [18] studied a class of nonsingular Buekenhout unitals in the Hall plane that turned out to be self-dual, and hence are embedded in both the Hall plane and the dual Hall plane.

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