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Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms

Un Kyu Kim

Department of Mathematics Education, College of Education, Sungkyunkwan University, Seoul 110-745, Korea
unkyukim@skku.edu

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Abstract. P. Alegre, D. Blair and A. Carriazo introduced and studied generalized Sasakian-space-forms. In this paper we classify conformally flat generalized Sasakian-space-forms under the assumption that the characteristic vector field is Killing. Also we classify locally symmetric generalized Sasakian-space-forms.

Keywords: generalized Sasakian-space-form, conformally flat, locally symmetric, Codazzi tensor, almost contact metric manifold

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1 Introduction

In Riemannian geometry many authors have studied curvature properties and to what extent they determined the manifold itself. Two important curvature properties are conformal flatness and local symmetry. For the conformal flatness, a recent development in contact geometry is due to K. Bang and D. Blair (see [2]). They proved that a conformally flat contact metric manifold such that the characteristic vector field is an eigenvector of the Ricci operator everywhere has constant curvature 0 or +1. For the local symmetry, S. Tanno (see [10]) showed that a locally symmetric K-contact manifold is of constant curvature 1 and is Sasakian. And D. Blair (see [4]) obtained a necessary and sufficient condition for the standard contact metric structure of the tangent sphere bundle of a Riemannian manifold to be locally symmetric.

Recently, P. Alegre, D. Blair and A. Carriazo (see [1]) introduced and studied generalized Sasakian-space-forms. These spaces are defined as follows: Given an almost contact metric manifold (M, ϕ, ξ, η, g), we say that M is a *generalized*

Sasakian-space-form if there exist three functions f_1, f_2 and f_3 on M such that

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} \\ + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. In such a case, we will write $M(f_1, f_2, f_3)$.

In this paper we study conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms, and obtain the following two theorems.

1 Theorem. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Then we have the following:*

(i) If $n > 1$, then M is conformally flat if and only if $f_2 = 0$.

(ii) If M is conformally flat and ξ is a Killing vector field, then it is flat, or of constant curvature, or locally the product $N^1 \times N^{2n}$, where N^1 is a 1-dimensional manifold and N^{2n} is a $2n$-dimensional almost Hermitian manifold of constant curvature. In any case, M is locally symmetric and has constant ϕ-sectional curvature.

2 Theorem. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional locally symmetric generalized Sasakian-space-form with the scalar curvature τ. Then we have the following:*

(i) $f_1 - f_3$ is a constant c.

(ii) If $\tau = -2n(2n+1)c$, then $M(f_1, f_2, f_3)$ is of a constant curvature $-c$.

(iii) If $\tau \neq -2n(2n+1)c$, then $M(f_1, f_2, f_3)$ is flat, or of constant curvature, or is locally the product $N^1 \times N^{2n}$, where N^1 is a 1-dimensional manifold and N^{2n} is a $2n$-dimensional almost Hermitian manifold of constant curvature or a complex space form.

We note that there are many almost contact metric manifolds with very different structures such that ξ is a Killing vector field (see [3]).

2 Preliminaries

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a (1,1) tensor field ϕ, a vector field ξ and a 1-form η such that $\eta(\xi) = 1, \phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) =$

$g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M (see [3]). $\phi\xi = 0$ and $\eta \circ \phi = 0$ are deducible from these conditions. We define the fundamental 2-form Φ on M by $\Phi(X,Y) = g(X, \phi Y)$. An almost contact metric manifold M is said to be a contact metric manifold if $d\eta(X,Y) = \Phi(X,Y)$. If ξ is a Killing vector field, then the contact metric manifold is said to be a K-contact manifold. The almost contact metric structure of M is said to be normal if $[\phi,\phi](X,Y) = -2d\eta(X,Y)\xi$, for any X,Y, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ. A normal contact metric manifold is called a Sasakian manifold. A normal almost contact metric manifold M with closed forms η and Φ is called a cosymplectic manifold. Cosymplectic manifolds are characterized by $\nabla_X\xi = 0$ and $(\nabla_X\phi)Y = 0$, for any vector fields X, Y on M. Given an almost contact metric manifold (M, ϕ, ξ, η, g), a ϕ-section of M at $p \in M$ is a plane section $\pi \subseteq T_pM$ spanned by a unit vector X_p orthogonal to ξ_p, and ϕX_p. The ϕ-sectional curvature of π is defined by $g(R(X,\phi X)\phi X, X)$. A cosymplectic-space-form, i.e., a cosymplectic manifold with constant ϕ-sectional curvature c, is a generalized Sasakian-space-form with $f_1 = f_2 = f_3 = \frac{c}{4}$ (see [7]). It is known that the ϕ-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ (see [1, Proposition 3.11]).

For a $(2n+1)$-dimensional almost contact metric manifold $(M, \phi, \xi, \eta, g), n \geq 1$, its Schouten tensor L is defined by

$$L = -\frac{1}{2n-1}Q + \frac{\tau}{4n(2n-1)}I, \tag{2}$$

where Q denotes the Ricci operator and τ is the scalar curvature, and the Weyl conformal curvature tensor is given by

$$\begin{aligned} W(X,Y)Z = R(X,Y)Z& \\ - [g(LX,Z)Y - g(Y,Z)LX& - g(LY,Z)X + g(X,Z)LY]. \end{aligned} \tag{3}$$

In dimensions > 3, that is $n > 1$, M is conformally flat if and only if $W = 0$ and in this case, L satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$, for any vector fields X, Y on M. In dimension 3, that is $n = 1$, $W = 0$ is automatically satisfied and M is conformally flat if and only if L satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$, for any vector fields X, Y on M.

A symmetric tensor field T of type (1,1) is a Codazzi tensor if it satisfies

$$(\nabla_X T)Y - (\nabla_Y T)X = 0.$$

For the later use we give the following lemma which was proved by Derdzinski.

3 Lemma ([5], [6]). *Let T be a Codazzi tensor on a Riemannian manifold M. Then we have the following: If T has more than one eigenvalue, then the* *eigenspaces for each eigenvalue ν form an integrable subbundle V_ν of constant multiplicity on open sets: If the multiplicity is greater than 1, then the integral submanifolds are umbilical submanifolds and each eigenfunction is constant along the integral submanifolds of its subbundle. Moreover, if ν is constant on M, then the integral submanifolds of V_ν are totally geodesic.*

3 Proof of Theorem 1

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Then the curvature tensor R of M is given by (1). From (1) we can easily see that

$$QX = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi, \tag{4}$$
$$\tau = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{5}$$

Moreover, we can see that

$$LX = [-\frac{1}{2}f_1 - \frac{3}{2(2n-1)}f_2]X + [\frac{3}{2n-1}f_2 + f_3]\eta(X)\xi. \tag{6}$$

Therefore, the Weyl conformal curvature tensor W can be written as

$$\begin{aligned} W(X,Y)Z = &-\frac{3}{2n-1}f_2[g(Y,Z)X - g(X,Z)Y] \\ &+ f_2[g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z] \\ &- \frac{3}{2n-1}f_2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \end{aligned} \tag{7}$$

Suppose that $M(f_1, f_2, f_3)$ is conformally flat and $n > 1$. Then we have $W = 0$.

If we put $X = \phi Y$ in (7), we find

$$\begin{aligned} f_2[(2-n)\{g(Y,Z)\phi Y - g(\phi Y,Z)Y - \eta(Y)\eta(Z)\phi Y + g(\phi Y,Z)\eta(Y)\xi\} \\ - (2n-1)\{g(Y,Y) - \eta(Y)\eta(Y)\}\phi Z] = 0. \end{aligned}$$

If we choose a unit vector U such that $g(U,\xi) = 0$ and put $Y = U$ in the above equation, then we have

$$f_2[(2-n)\{g(U,Z)\phi U - g(\phi U,Z)U\} - (2n-1)\phi Z] = 0.$$

Putting $Z = U$ gives $f_2(1-n) = 0$ and hence we get $f_2 = 0$.

Conversely, if $f_2 = 0$, then from (7) we have $W(X,Y)Z = 0$ and hence $M(f_1, f_2, f_3)$ is conformally flat. Therefore, when $n > 1$, $M(f_1, f_2, f_3)$ is conformally flat if and only if $f_2 = 0$. Thus the first part (i) of the Theorem 1 is proved.

For the proof of the second part (ii), we assume that $M(f_1, f_2, f_3)$ is conformally flat and ξ is Killing. Then the Schouten tensor L of the manifold is a Codazzi tensor, that is,

$$(\nabla_X L)Y - (\nabla_Y L)X = 0 \tag{8}$$

for any vector fields X, Y on M. Also, if $n > 1$, then we have $f_2 = 0$ by the first part (i) and hence from (6) the Schouten tensor is given by

$$LX = -\frac{1}{2} f_1 X + f_3 \eta(X)\xi. \tag{9}$$

If $n = 1$, then $W = 0$ is automatically satisfied. From (3) we get

$$R(X,Y)Z = g(LX,Z)Y - g(Y,Z)LX - g(LY,Z)X + g(X,Z)LY$$

for any vector fields X, Y, Z. In the 3-dimensional manifold $M(f_1, f_2, f_3)$ the Schouten tensor is given by

$$LX = -\frac{1}{2}(f_1 + 3f_2)X + (3f_2 + f_3)\eta(X)\xi.$$

From these two equations we obtain

$$\begin{aligned} R(X,Y)Z &= f_1^*[g(Y,Z)X - g(X,Z)Y] \\ &+ f_3^*[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \end{aligned} \tag{10}$$

where f_1^* and f_3^* are given by

$$f_1^* = f_1 + 3f_2, \quad f_3^* = 3f_2 + f_3.$$

Hence the Schouten tensor is written as

$$LX = -\frac{1}{2} f_1^* X + f_3^* \eta(X)\xi.$$

Consequently, we also use the same symbols f_1 and f_3 instead of f_1^* and f_3^* for the 3-dimensional manifold $M(f_1, f_2, f_3)$. Equation (9) gives

$$L\xi = (f_3 - \frac{1}{2} f_1)\xi. \tag{11}$$

If X is a vector orthogonal to ξ, then we get

$$LX = -\frac{1}{2}f_1 X. \tag{12}$$

Let $\xi, E_1, \ldots, E_{2n}$ be local orthonormal vector fields on $M(f_1, f_2, f_3)$. Then from (8), (9) and (12) we get

$$(\nabla_{E_i}L)E_j - (\nabla_{E_j}L)E_i = -\frac{1}{2}(E_i f_1)E_j + \frac{1}{2}(E_j f_1)E_i + f_3\eta(\nabla_{E_j}E_i - \nabla_{E_i}E_j)\xi = 0.$$

Taking the inner product with E_j, we have $E_i f_1 = 0$ $(i = 1, 2, \ldots, 2n)$. Since $\nabla_{E_i}\xi$ is orthogonal to ξ,

$$(\nabla_{E_j}L)\xi + L\nabla_{E_j}\xi = (E_j f_3)\xi + (f_3 - \frac{1}{2}f_1)\nabla_{E_j}\xi$$

which gives

$$(\nabla_{E_j}L)\xi = (E_j f_3)\xi + f_3\nabla_{E_j}\xi.$$

Since ξ is Killing,

$$(\nabla_\xi L)E_j + L\nabla_\xi E_j = -\frac{1}{2}(\xi f_1)E_j - \frac{1}{2}f_1\nabla_\xi E_j,$$

$$\begin{aligned} L\nabla_\xi E_j &= -\frac{1}{2}f_1\nabla_\xi E_j + f_3 g(\xi, \nabla_\xi E_j)\xi \\ &= -\frac{1}{2}f_1\nabla_\xi E_j - f_3 g(\nabla_\xi \xi, E_j)\xi = -\frac{1}{2}f_1\nabla_\xi E_j, \end{aligned}$$

and we have

$$(E_j f_3)\xi + f_3\nabla_{E_j}\xi = -\frac{1}{2}(\xi f_1)E_j. \tag{13}$$

Taking the inner product with E_j implies $\xi f_1 = 0$. Thus f_1 is constant on M.

Taking the inner product with ξ gives $E_j f_3 = 0$ $(j = 1, 2, \ldots, 2n)$ and $f_3\nabla_{E_j}\xi = 0$ $(j = 1, 2, \ldots, 2n)$. Combining this with $\nabla_\xi\xi = 0$ gives

$$f_3\nabla_X\xi = 0, \tag{14}$$

for any vector field X.

From (14) we get

$$(Yf_3)\nabla_X\xi + f_3\nabla_Y\nabla_X\xi = 0.$$

This equation and (14) give

$$(Xf_3)\nabla_Y\xi - (Yf_3)\nabla_X\xi + f_3[\nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi] = 0.$$

Multiplying this equation with f_3 and using (14) give

$$f_3^2 R(X,Y)\xi = 0.$$

This equation and (1) imply

$$f_3^2(f_1 - f_3)[\eta(Y)X - \eta(X)Y] = 0,$$

from which we obtain $f_3(f_1 - f_3) = 0$.

Consider the case $f_1 = 0$. In this case we have $f_3 = 0$ on M and hence M is flat.

Next consider the case $f_1 \neq 0$. Differentiating $f_3(f_1 - f_3) = 0$ with ξ gives $(f_1 - 2f_3)(\xi f_3) = 0$. If $f_3(p) = 0$ at a point $p \in M$, then $\xi f_3 = 0$ at p. If $f_3(p) \neq 0$, then $f_3 = f_1$ in an open neighborhood $\mathcal{O}$ of p. Hence we get $\xi f_3 = 0$ in $\mathcal{O}$. Thus we have $\xi f_3 = 0$ on M. Since $E_j f_3 = 0$ $(j = 1, 2, \ldots, 2n)$ f_3 is constant on M. Hence we have

(a) If $f_3 = 0$, then M is of constant curvature f_1,

(b) If $f_3 \neq 0$, then we have $f_1 = f_3$ and $\nabla_X\xi = 0$ for any vector X on M. Hence the Schouten tensor L has two distinct constant eigenvalues $\frac{1}{2}f_1$ with multiplicity 1 and $-\frac{1}{2}f_1$ with multiplicity $2n$. Therefore, we have the decomposition $\mathfrak{D} \oplus [\xi]$, where $\mathfrak{D}$ is the distribution defined by $\eta = 0$ and $[\xi]$ is the distribution spanned by the vector ξ. By Lemma 3, $\mathfrak{D}$ is integrable. Hence, M is locally the product of an integral submanifold N^1 of $[\xi]$ and an integral submanifold N^{2n} of $\mathfrak{D}$. Since the eigenvalue is constant on M, N^{2n} is a totally geodesic submanifold of M by Lemma 3. If we denote the restriction of ϕ in $\mathfrak{D}$ by J, then

$$J^2X = \phi^2X = -X + \eta(X)\xi = -X$$

for any $X \in \mathfrak{D}$. Hence J defines an almost complex structure on N^{2n}.

Also, $g'(JX, JY) = g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) = g'(X,Y)$ for any $X, Y \in \mathfrak{D}$, where g' is the induced metric on N^{2n} from g. Hence (N^{2n}, J, g') is an almost Hermitian manifold. Since N^{2n} is a totally geodesic hypersurface of M, the equation of Gauss is given by

$$R(X,Y)Z = R'(X,Y)Z$$

for any vector fields X, Y and Z tangent to N^{2n}, where R' is the curvature tensor of N^{2n}. Thus, we get

$$R'(X,Y)Z = f_1[g'(Y,Z)X - g'(X,Z)Y],$$

and hence N^{2n} is a space of constant curvature f_1. In any case, from the above arguments we can easily see that $M(f_1, f_2, f_3)$ is locally symmetric. Since f_1 and f_3 are constants, we can see that M is of constant ϕ-sectional curvature. This completes the proof of the Theorem 1.

4 Remark. In the Theorem 1 the condition "ξ is a Killing vector field" cannot be removed. For example, given (N, J, g) with constant curvature c, say, a 6-dimensional sphere with nearly Kaehler structure (see [7]), the warped product $M = \mathbb{R} \times_f N$, where $f > 0$ is a nonconstant function on $\mathbb{R}$, can be endowed with an almost contact metric structure (ϕ, ξ, η, g_f). Moreover, $M = \mathbb{R} \times_f N$ is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with functions

$$f_1 = \frac{c - f'^2}{f^2}, \quad f_2 = 0, \quad f_3 = \frac{c - f'^2}{f^2} + \frac{f''}{f}$$

(see [1, Theorem 4.8]). Since $f_2 = 0$, M is conformally flat. Also, we can see that

$$g_f(\nabla_X \xi, X) = g_f(\frac{\xi f}{f} X, X),$$

where X is a nonzero vector field on N (see [8]).

But, it is easy to find a function f on $\mathbb{R}$ such that $\xi f = \frac{\partial}{\partial t} f = f' \neq 0$ and f_1 is not constant. In such a manifold $M = \mathbb{R} \times_f N$ the characteristic vector is not a Killing vector field and the conclusion (ii) of the Theorem 1 is not satisfied.

4 Proof of Theorem 2

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional locally symmetric generalized Sasakian-space-form. Then the curvature tensor is given by (1). From (1) we can see that

$$QX = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi, \tag{15}$$

$$\tau = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{16}$$

From (1) we get

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}. \tag{17}$$

From (17) we get $R(X, \xi)\xi = (f_1 - f_3)\{X - \eta(X)\xi\}$. Since M is locally symmetric, this equation gives

$$\begin{aligned} R(\nabla_Z X, \xi)\xi + R(X, \nabla_Z \xi)\xi + R(X, \xi)\nabla_Z \xi &= Z(f_1 - f_3)[X - \eta(X)\xi] \\ &+ (f_1 - f_3)[\nabla_Z X - (\nabla_Z \eta)(X)\xi - \eta(\nabla_Z X)\xi - \eta(X)\nabla_Z \xi]. \end{aligned}$$

From the equation (17) and $R(X,\xi)\nabla_Z\xi = -(f_1 - f_3)g(X,\nabla_Z\xi)\xi$ we get

$$Z(f_1 - f_3)[X - \eta(X)\xi] = 0,$$

which shows that $f_1 - f_3$ is a constant. Hence, we can put $f_3 - f_1 = c$ for a constant c. Thus, using (15) and (16) the Ricci operator Q and the scalar curvature τ are given by

$$QX = [(2n-1)f_1 + 3f_2 - c]X - [3f_2 + (2n-1)f_1 + (2n-1)c]\eta(X)\xi, \tag{18}$$
$$\tau = 2n[(2n-1)f_1 + 3f_2 - 2c]. \tag{19}$$

Since Q is parallel and τ is constant, (18) gives

$$[3f_2 + (2n-1)f_1 + (2n-1)c]\nabla_Y\xi = 0, \tag{20}$$

where $3f_2 + (2n-1)f_1 + (2n-1)c$ is a constant.

On the other hand, since R is parallel we have from (1)

$$\begin{aligned}(Vf_1)[g(Y,Z)X - g(X,Z)Y] + (Vf_2)[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z]\\ + f_2[g(X,(\nabla_V\phi)Z)\phi Y + g(X,\phi Z)(\nabla_V\phi)Y - g(Y,(\nabla_V\phi)Z)\phi X\\ - g(Y,\phi Z)(\nabla_V\phi)X + 2g(X,(\nabla_V\phi)Y)\phi Z + 2g(X,\phi Y)(\nabla_V\phi)Z]\\ + (Vf_1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]\\ + (f_1 + c)[(\nabla_V\eta)(X)\eta(Z)Y + \eta(X)(\nabla_V\eta)(Z)Y - (\nabla_V\eta)(Y)\eta(Z)X\\ - \eta(Y)(\nabla_V\eta)(Z)X + g(X,Z)(\nabla_V\eta)(Y)\xi + g(X,Z)\eta(Y)\nabla_V\xi\\ - g(Y,Z)(\nabla_V\eta)(X)\xi - g(Y,Z)\eta(X)\nabla_V\xi] = 0.\end{aligned} \tag{21}$$

If we put $Z = \xi$ in (21), we get

$$\begin{aligned}f_2[g(X,(\nabla_V\phi)\xi)\phi Y - g(Y,(\nabla_V\phi)\xi)\phi X + 2g(X,\phi Y)(\nabla_V\phi)\xi]\\ + (f_1 + c)[(\nabla_V\eta)(X)Y - (\nabla_V\eta)(Y)X + \eta(X)(\nabla_V\eta)(Y)\xi\\ - \eta(Y)(\nabla_V\eta)(X)\xi] = 0.\end{aligned}$$

Applying ϕ and taking account of $(\nabla_V\phi)\xi = -\phi\nabla_V\xi$ we have

$$\begin{aligned}(f_1 + c)[g(\nabla_V\xi, X)\phi Y - g(\nabla_V\xi, Y)\phi X] + f_2[g(\nabla_V\xi,\phi X)(-Y + \eta(Y)\xi)\\ - g(\nabla_V\xi,\phi Y)(-X + \eta(X)\xi) + 2g(X,\phi Y)\nabla_V\xi] = 0.\end{aligned}$$

If we put $Y = \phi X$ in the above equation and we take a unit vector X orthogonal to ξ, then we have

$$(f_1 + f_2 + c)[g(\nabla_V\xi, X)X + g(\nabla_V\xi,\phi X)\phi X] + 2f_2\nabla_V\xi = 0.$$

Take a local ϕ-basis $\xi, e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n$. Then we have

$$\nabla_V \xi = g(\nabla_V \xi, e_1)e_1 + g(\nabla_V \xi, \phi e_1)\phi e_1 + \cdots + g(\nabla_V \xi, e_n)e_n + g(\nabla_V \xi, \phi e_n)\phi e_n.$$

Combining the last two equations we get

$$[f_1 + (2n+1)f_2 + c]\nabla_V \xi = 0. \tag{22}$$

Case 1. $\tau = -2n(2n+1)c$

In this case we have $3f_2 + (2n-1)f_1 + (2n-1)c = 0$. Hence the Schouten tensor L is given by $L = \frac{c}{2}I$. Therefore, we have

$$W(X,Y)Z = R(X,Y)Z + c[g(Y,Z)X - g(X,Z)Y]. \tag{23}$$

If $n = 1$, then W is automatically zero and hence

$$R(X,Y)Z = -c[g(Y,Z)X - g(X,Z)Y].$$

This shows that M is of constant curvature $-c$.

Now consider the case $n > 1$. If $f_1(p) + (2n+1)f_2(p) = -c$ at some $p \in M$, then we have $f_2(p) = 0$ since $(2n-1)f_1(p) + 3f_2(p) = (1-2n)c$. Suppose that there exists a point $p \in M$ such that $f_2(p) \neq 0$. Then we have $f_1(p) + (2n+1)f_2(p) \neq -c$ and hence (22) implies $\nabla_X \xi = 0$ in an open neighborhood $\mathcal{O}$ of p. We now give equation (21) by using components. Since $\nabla_X \xi = 0$ in $\mathcal{O}$ the equation can be written as

$$\begin{aligned}
&(\nabla_l f_1)(g_{ji}g_{kh} - g_{ki}g_{jh}) + (\nabla_l f_2)(\phi_{ik}\phi_{jh} - \phi_{ij}\phi_{kh} + 2\phi_{jk}\phi_{ih}) \\
&+ f_2[(\nabla_l \phi_{ik})\phi_{jh} + (\nabla_l \phi_{jh})\phi_{ik} - (\nabla_l \phi_{ij})\phi_{kh} - \phi_{ij}\nabla_l \phi_{kh} + 2(\nabla_l \phi_{jk})\phi_{ih} + 2\phi_{jk}\nabla_l \phi_{ih}] \\
&+ (\nabla_l f_1)(\eta_k\eta_i g_{jh} - \eta_j\eta_i g_{kh} + g_{ki}\eta_j\eta_k - g_{ji}\eta_k\eta_h) = 0,
\end{aligned} \tag{24}$$

where ∇_j denotes the operator of covariant differentiation. Applying ϕ^{kh} gives

$$[\nabla_l f_1 + (2n+1)(\nabla_l f_2)]\phi_{ji} + f_2(2n+2)\nabla_l \phi_{ji} = 0 \tag{25}$$

in $\mathcal{O}$. Applying ϕ^{ji} again gives

$$\nabla_l f_1 + (2n+1)\nabla_l f_2 = 0, \quad \nabla_l \phi_{ji} = 0 \tag{26}$$

in $\mathcal{O}$. Therefore, the open submanifold $\mathcal{O}$ of M is a cosymplectic manifold. Since $(2n-1)\nabla_l f_1 + 3\nabla_l f_2 = 0$ from the condition $3f_2 + (2n-1)f_1 + (2n-1)c = 0$ we have, by the help of (26), $\nabla_l f_2 = 0$ and $\nabla_l f_1 = 0$. So f_1 and f_2 are constants in $\mathcal{O}$. Since $M(f_1, f_2, f_3)$ has ϕ-sectional curvature $f_1 + 3f_2$ and since f_1 and f_2 are constants in $\mathcal{O}$, the open submanifold $\mathcal{O}$ of M is a cosymplectic space form.

So, we have $f_1 = f_2 = f_3 = f_1 + c$. Therefore we have $c = 0$. But, in our case we have $(2n+2)f_2 = (2n-1)f_1 + 3f_2 = (1-2n)c = 0$. Hence we get $f_2 = 0$ in $\mathcal{O}$. This is a contradiction. Thus, there does not exist a point p in M such that $f_2(p) \neq 0$ and hence $f_2 = 0$ in M. By (i) of Theorem 1, M is conformally flat. This and (23) imply that M is of constant curvature $-c$. This completes the proof of (ii) of the Theorem 2.

Case 2. $\tau \neq -2n(2n+1)c$.

In this case we have $3f_2 + (2n-1)f_1 + (2n-1)c \neq 0$. From (20) we get $\nabla_X \xi = 0$ on M.

First, we consider the case $n = 1$. The Schouten tensor is given by

$$LX = -QX + \frac{\tau}{4}X.$$

Since Q is parallel and τ is constant, L is parallel. Therefore we have $(\nabla_X L)Y - (\nabla_Y L)X = 0$ and hence M is conformally flat. Since $\nabla_X \xi = 0$, the characteristic vector field ξ is a Killing vector field. Hence, we have the same conclusion as (ii) of Theorem 1.

Second, we consider the case $n > 1$. Since τ is constant we have

$$(2n-1)X(f_1) + 3X(f_2) = 0.$$

Since $\nabla_X \xi = 0$, we also have (25) and hence

$$X(f_1) + (2n+1)X(f_2) = 0, \quad f_2(\nabla_X \phi)Y = 0.$$

These equations imply that f_1 and f_2 are constant on M. If $f_2 \neq 0$, then we have $(\nabla_X \phi)Y = 0$, for any vector fields X, Y on M. Hence M is a cosymplectic manifold and since f_1 and f_2 are constants M is a cosymplectic space form. Hence we have $f_1 = f_2 = f_3 = f_1 + c$ and $c = 0$. Equations (15) and (16) give the Schouten tensor

$$LX = -\frac{n+1}{2n-1}f_2 X + \frac{2(n+1)}{2n-1}f_2 \eta(X)\xi. \tag{27}$$

This equation shows that

$$L\xi = \frac{n+1}{2n-1}f_2 \xi, \tag{28}$$

$$LX = -\frac{n+1}{2n-1}f_2 X, \tag{29}$$

for any vector field X orthogonal to ξ. Hence L has distinct two constant eigenvalues $-(n+1)f_2/(2n-1)$ and $(n+1)f_2/(2n-1)$ with multiplicity $2n$ and 1, respectively. The equation (27) and $\nabla_X \xi = 0$ give $(\nabla_Y L)X = 0$. Therefore, L

is a Codazzi tensor. Consequently, we have the decomposition $[\xi] \oplus \mathfrak{D}$, where $[\xi]$ is the distribution spanned by the vector ξ and $\mathfrak{D}$ is the distribution defined by $\eta = 0$. By the Lemma 3 $\mathfrak{D}$ is integrable. Hence, M is locally the product of an integral submanifold N^1 of $[\xi]$ and an integral submanifold N^{2n} of $\mathfrak{D}$. Since eigenvalue $-(n+1)f_2/(2n-1)$ is constant on M, N^{2n} is a totally geodesic submanifold of M. The restriction J of ϕ in $\mathfrak{D}$ defines an almost complex structure on N^{2n}. Hence (N^{2n}, J, g') is an almost Hermitian manifold, where g' is the induced metric on N^{2n}. Since N^{2n} is a totally geodesic hypersurface of M, the Gauss formula is given by $\nabla_X Y = {}'\nabla_X Y$ for any vector fields X, Y on N^{2n}. Since

$$0 = (\nabla_X \phi)Y = \nabla_X(\phi Y) - \phi(\nabla_X Y) = {}'\nabla_X(JY) - J({}'\nabla_X Y) = ({}'\nabla_X J)Y,$$

for any vector fields X, Y on N^{2n}, it is a Kaehler manifold. Since N^{2n} is a totally geodesic hypersurface with ξ as the normal vector field, the curvature tensor R' of N^{2n} is given by

$$\begin{aligned} R'(X,Y)Z = f_2[&g'(Y,Z)X - g'(X,Z)Y \\ &+ g'(X,JZ)JY - g'(X,JY)JZ + 2g'(X,JY)JZ]. \end{aligned}$$

Therefore N^{2n} is a complex space form.

If $f_2 = 0$, then by (i) of Theorem 1 M is conformally flat. Since $\nabla_X \xi = 0$ on M ξ is a Killing vector field. Hence we have the same conclusion as (ii) of Theorem 1. This completes the proof of (iii) of the Theorem 2.

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References

[1] P. Alegre, D. Blair, A. Carriazo: *Generalized Sasakian-space-forms*, to appear.

[2] K. Bang, D. Blair: *On conformally flat contact metric manifolds*, to appear.

[3] D. Blair: Riemannian Geometry of Contact and Sympletic manifolds, Birkhäuser, Boston 2002.

[4] D. Blair: *When is the tangent sphere bundle locally symmetric?* Geometry and Topology, World Scientific, Singapore, 15–30.

[5] A. Derdzinski: *Some remarks on the local structure of Codazzi tensors*, Lecture Notes in Math. vol. 838, Springer, Berlin 1981, 251–255.

[6] A. Derdzinski: *Classification of Certain Compact Riemannian Manifolds with Harmonic Curvature and non-parallel Ricci Tensor*, Math. Z., **172**, (1980), 273–280.

[7] G. Ludden: *Submanifolds of cosymplectic manifolds*, J. Differential Geometry, **4**, (1970), 237–244.

[8] B. O'Neill: Semi-Riemannian Geometry with Applications to Relativity, Pure and Applied Mathematics 103, Academic Press, New York 1983.

[9] S. Tanno: *Constancy of holomorphic sectional curvature in almost Hermitian manifolds*, Kōdai Math. Sem. Rep., **25**, (1973), 191–201.

[10] S. Tanno: *Locally symmetric K-contact Riemannian manifolds*, Proc. Japan Acad., **43**, 581–583.