Note di Matematica **26,** n. 1, 2006, 1–19.

Reflection simplicial maps and group cohomology

Stephan Klaus
Mathematisches Forschungsinstitut Oberwolfach Lorenzenhof,
77709 Oberwolfach-Walke
Germany klaus@mfo.de

Received: 10/9/2002; accepted: 18/3/2004.

Abstract. We define the reflection object of a simplicial object and the notion of a reflection simplicial map, which can be realized geometrically, too. Using the inversion in a group G, there is an interesting canonical reflection involution on the classifying space BG. We study its fixed point set and quotient space. As an application, we show that inversion invariant cohomology is isomorphic to the usual group cohomology if the coefficients are local away from 2. Then we construct reflection involutions on Eilenberg-MacLane spaces and simplicial pseudogroups which are local away from 2.

Keywords: reflection simplicial map, classifying space, group cohomology

MSC 2000 classification: 18G30, 20J06, 20N05, 55N20, 55P20, 55P45, 55S45, 55U10

1 Survey of results

First, we show that there is a unique non-trivial involutive automorphism τ of the simplicial category Δ which is induced by the flip $x \mapsto n-x$ of the object sets $[n] := \{0, 1, \ldots, n\}$. Using τ we define for a given simplicial object $X_\bullet$ its reflection simplicial object $X_\bullet^\tau$. In general, both objects are not isomorphic, but have homeomorphic geometric realizations. We define a reflection simplicial map as a simplicial map $X_\bullet \to Y_\bullet^\tau$.

In the third section, we show that group inversion defines an involutive reflection simplicial map γ on the classifying space $BG_\bullet$ of any group G. We show that its geometric realization Γ is homotopic to the identity. We consider inversion invariant group cohomology and show that it coincides with usual group cohomology if the coefficients are local away from 2. Then we show that the fixed point set BG^Γ is a homeomorphic deformation retraction of BG if G does not have 2-torsion. Furthermore, we show that the fundamental group of the quotient space BG/Γ is isomorphic to G if G does not have 2-torsion. In particular, for a finite group G of odd order, the space BG/Γ is a natural new construction for the classifying space of G which has a smaller number of cells than BG.

In the last section, we introduce reflection involution λ on Eilenberg-MacLane spaces $K(A,n)_\bullet$. Using our results on minimal Postnikov systems of H-spaces and simplicial loops [7], we prove that any connected minimal simplicial loop which is local away from 2 carries some reflection involution ρ generalizing λ. This raises the question if there is a similar result for inversion invariant generalized group cohomology as for usual group cohomology.

2 Reflection objects and reflection simplicial maps

Let Δ be the simplicial category, i.e. objects are the finite totally ordered sets $[n] := \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$ and morphisms are the monotonic maps [9]. We denote by Δ^n_m the set of monotonic maps $\phi : [m] \to [n]$. By the fundamental lemma for Δ, monotonic maps are generated by the standard maps $d^i \in \Delta^n_{n-1}$ and $s^i \in \Delta^n_{n+1}$ for $0 \le i \le n$. We recall that a simplicial object $X_\bullet$ in some category $\mathcal{C}$ is a contravariant functor $X_\bullet : \Delta \to \mathcal{C}$, in particular the standard maps act by morphisms $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$. Clearly, any covariant automorphism $\alpha : \Delta \to \Delta$ yields a simplicial object $X^\alpha_\bullet := X_\bullet \circ \alpha$.

1 Proposition. *There is exactly one non-trivial covariant automorphisms τ of the category Δ which is given by $\tau[n] = [n]$ and $\tau(\phi) = \tau_n \phi \tau_m$, where $\tau_m : [m] \to [m]$ denotes the antitonic bijection $\tau_m(k) := m - k$. In particular, $\tau^2 = \mathbb{I}$.*

Proof. As the objects $[n]$ are pairwise non-isomorphic in Δ, they have to be fixed by any automorphism τ. Now, τ permutes Δ^n_m and we will show that τ is fixed by its action on $\Delta^1_0 = \{d^0, d^1\}$: By the simplicial identities, the composition $s^i d^j$ cancels to the identity exactly in the cases $s^i d^i = s^i d^{i+1} = \mathbb{I}$. Thus for $\tau(d^i) = d^j$, it follows that either $\tau(d^{i+1}) = d^{j+1}$ (and $\tau(s^i) = s^j$) or $\tau(d^{i+1}) = d^{j-1}$ (and $\tau(s^i) = s^{j-1}$). Hence, either τ fixes the string $d^0, d^1, \ldots, d^n$, or reverses the order (and the same with $s^0, s^1, \ldots s^{n-1}$). The simplicial identities for $d^i d^j$ show that either τ fixes these strings in all dimensions n or reverses them all. With $\tau(\phi) := \tau_n \phi \tau_m$, we see that the second possibility is realized which by $\tau^2_k = \mathbb{I}$ is an involutive automorphism of Δ. $\boxed{QED}$

2 Definition. We call the non-trivial automorphism $\tau : \Delta \to \Delta$ the *reflection automorphism.* For a simplicial object $X_\bullet$, the simplicial object $X^\tau_\bullet$ is called its *reflection object.* As an object in $\mathcal{C}$, it is given by the sequence X_n, again, but the action of d_i and s_i on $X^\tau_n = X_n$ is given by d_{n-i} and s_{n-i} on X_n, respectively. Clearly, a simplicial morphism $f : X_\bullet \to Y_\bullet$ also is a simplicial morphism $f^\tau : X^\tau_\bullet \to Y^\tau_\bullet$. A *reflection simplicial morphism* between simplicial objects $X_\bullet$ and $Y_\bullet$ is a simplicial morphism $g : X_\bullet \to Y^\tau_\bullet$. Equivalently, g is a sequence of morphisms $g_n : X_n \to Y_n$ such that

$$g_{n-1} d_i = d_{n-i} g_n \quad \text{and} \quad g_{n+1} s_i = s_{n-i} g_n$$

for all i.

Clearly, there is an equality $(X_\bullet^\tau)^\tau = X_\bullet$ and $(f^\tau)^\tau = f$.

As an example, we consider the simplicial sphere $S_\bullet^n$ which is the simplicial set having exactly two non-degenerate simplicies $*$ and σ in dimensions 0 and n, respectively. Thus $S_k^n = \{ *, s_I\sigma \mid s_I \}$, where s_I is any composition in normal form of degeneration operators from dimension n to dimension $k \geq n$, and $*$ also denotes the degenerate base point in all dimensions. Then there is a reflection simplicial isomorphism $\lambda : S_\bullet^n \to (S_\bullet^n)^\tau$ which is given by $\lambda(*) = *$ and $\lambda(s_I\sigma) = \tau(s_I)\sigma$. In fact, λ is fixed because $(S_\bullet^n)^\tau$ again has exactly two non-degenerate simplicies $*$ and σ. It holds $\lambda\lambda^\tau = \mathbb{I} = \lambda^\tau\lambda$.

As an example of a simplicial set $X_\bullet$ with $X_\bullet^\tau$ not being isomorphic to $X_\bullet$ as a simplicial set, we consider the horns $\Lambda_\bullet^{n,k}$ [9]. It is straight forward to see that $(\Lambda_\bullet^{n,k})^\tau$ is isomorphic to $\Lambda_\bullet^{n,n-k}$, which for $k \neq n-k$ is not isomorphic to $\Lambda_\bullet^{n,k}$.

We recall that the geometric realization of a simplicial set $X_\bullet$ is defined as the topological quotient space

$$|X_\bullet| = \coprod_{n\geq 0} X_n \times \Delta^n/(\phi^* x, t) \sim (x, \phi_* t),$$

where $\Delta^n \subset \mathbb{R}^{n+1}$ denotes the geometric standard n-simplex, $t = (t_0, t_1, \ldots, t_n) \in \Delta^n$ is a point, and ϕ^*, ϕ_* denote the action of morphisms ϕ in Δ. $|X_\bullet|$ is a CW-complex with n-cells corresponding to the non-degenerate n-simplicies in $X_\bullet$ [9].

3 Definition. Let the *reflection* $\tau_* : \Delta^n \to \Delta^n$ of the geometric n-simplex be given by the linear map $(t_0, t_1, \ldots, t_n) \to (t_n, t_{n-1}, \ldots, t_0)$ in $\mathbb{R}^{n+1}$.

4 Lemma. *Reflection induces a cellular homeomorphism* $\tau_* : |X_\bullet| \to |X_\bullet^\tau|$ *by* $(x, t) \mapsto (x, \tau_*(t))$. *For any simplicial map* $f : X_\bullet \to Y_\bullet$, *it holds* $|f^\tau|\tau_* = \tau_*|f| : |X_\bullet| \to |Y_\bullet^\tau|$.

Proof. By

$$\begin{aligned} |X_\bullet^\tau| &= \coprod_{n\geq 0} X_n \times \Delta^n/(\tau(\phi)^* x, t) \sim (x, \phi_* t) \\ &= \coprod_{n\geq 0} X_n \times \Delta^n/(\phi^* x, t) \sim (x, \tau(\phi)_* t) \end{aligned}$$

and by $\tau_*\phi_*\tau_* = \tau(\phi)_* : \Delta^n \to \Delta^m$, τ_* induces a continuous map. As $\tau_* \circ \tau_* = \mathbb{I}$ it follows that τ_* is a homeomorphism. Clearly, τ_* is cellular, and $|f^\tau|\tau_* = \tau_*|f|$ holds by definition. $\boxed{QED}$

5 Lemma. *Let $X_\bullet$ be a fibrant simplicial set. Then also $X_\bullet^\tau$ is fibrant and there exists a reflection homotopy equivalence $f : X_\bullet \to X_\bullet^\tau$. If $X_\bullet$ is fibrant and minimal, there exists a reflection simplicial isomorphism $f : X_\bullet \to X_\bullet^\tau$.*

Proof. As $X_\bullet$ satisfies the extension condition with respect to all maps from horns $\Lambda_\bullet^{n,k} \subset \Delta_\bullet^n$, the reflection set $X_\bullet^\tau$ satisfies the extension condition with respect to all maps from horns $\Lambda_\bullet^{n,n-k} \subset \Delta_\bullet^n$, hence also is fibrant. Thus the homeomorphism τ_* can be realized simplicially up to homotopy. Using the definition of minimality [9], it is straightforward to see that $X_\bullet^\tau$ is also minimal. The claim follows as any homotopy equivalence between minimal simplicial sets is an isomorphism. $\boxed{QED}$

We recall the definition of the normalized cochain complex $C^*(X_\bullet; A)$ of $X_\bullet$ with values in some abelian group A. In dimension n, it consists of the functions $c : X_n \to A$ which vanish on degenerate simplicies. The coboundary $d : C^n \to C^{n+1}$ is given by $dc(x) := \sum_{i=0}^{n+1} (-1)^i c(d_i x)$, where $x \in X_{n+1}$. The normalized cochain complex of $X_\bullet$ is canonically isomorphic to the cellular cochain complex of $|X_\bullet|$, where the n-cells of $|X_\bullet|$ carry the canonical orientation given by Δ^n.

6 Lemma. *The normalized cochain complex of $X_\bullet^\tau$ is given by that of $X_\bullet$ with multiplying the coboundary $d : C^n \to C^{n+1}$ by $(-1)^{n+1}$. The cellular homeomorphism τ_* induces an isomorphism of the cellular cochain complex of $|X_\bullet^\tau|$ with that of $|X_\bullet|$ which is given by multiplication with $\epsilon(n) := (-1)^{\frac{1}{2}n(n+1)}$ in dimension n.*

Proof. We can identify $C^n(X_\bullet^\tau; A) = C^n(X_\bullet; A)$ as groups. Let d' denote the coboundary of $C^n(X_\bullet^\tau; A)$, then $d' = \sum_{i=0}^{n+1} (-1)^i d_{n+1-i} = (-1)^{n+1} d$ because d_i acts by d_{n+1-i} on $X_{n+1}^\tau = X_{n+1}$. As the reflection $\tau_* : \Delta^n \to \Delta^n$ changes the orientation by $\epsilon(n) = \det(\tau_*)$, this also gives the induced isomorphism of the cellular homeomorphism τ_*. $\boxed{QED}$

3 Reflection inversion on the classifying space of a group

Let G be a monoid, then its classifying space $BG_\bullet$ is defined as the simplicial set [1], [9]

$$BG_n := G \times G \times \cdots \times G$$

(n factors). Here, the n-tuples are denoted by $[g_1, g_2, \ldots, g_n]$, with $BG_0 := \{[\,]\}$ the set consisting of the (empty) 0-tuple.

Face and degeneracy maps are defined by

$$\begin{aligned}
d_0[g_1,\ldots,g_n] &:= [g_2,\ldots,g_n],\\
d_i[g_1,\ldots,g_n] &:= [g_1,\ldots,g_{i-1},g_ig_{i+1},g_{i+2},\ldots,g_n] \quad \text{for } 0<i<n,\\
d_n[g_1,\ldots,g_n] &:= [g_1,\ldots,g_{n-1}],\\
s_j[g_1,\ldots,g_n] &:= [g_1,\ldots,g_j,1,g_{j+1},\ldots,g_n] \quad \text{for } 0\le j\le n
\end{aligned}$$

(with $s_0[\,] = [1]$ and $d_0[g] = d_1[g] = [\,]$ for all g). A homomorphism of monoids $f : G \to H$ induces a simplicial map $Bf : BG_\bullet \to BH_\bullet$ by $Bf[g_1,\ldots,g_n] := [f(g_1),\ldots,f(g_n)]$. The following rigidity result is well-known:

7 Lemma. *Let G, H be monoids, then B gives a bijection of the set of homomorphisms $G \to H$ to the set of simplicial maps $\alpha : BG_\bullet \to BH_\bullet$,*

$$\mathrm{Hom}(G,H) \cong \mathrm{map}_\Delta(BG_\bullet, BH_\bullet), \qquad f \mapsto Bf.$$

The inverse is given by $\alpha \mapsto (\alpha_1 : G \to H)$.

Proof. Let $g_1, g_2 \in G$ and $h_i := \alpha_1[g_i] \in H$. We denote $\alpha_2[g_1,g_2] =: [h,h'] \in BH_2 = H \times H$. Application of d_0 shows $h' = h_1$, and application of d_2 shows $h = h_0$. Application of d_1 to $\alpha_2[g_1,g_2] = [h_1,h_2]$ shows that $\alpha_1[g_1g_2] = [h_1h_2]$, i.e. α_1 is a homomorphism. Clearly $Bf_1 = f$. Furthermore, induction on n shows that $\alpha_n[g_1,\ldots,g_n] = [h_1,\ldots,h_n]$ with $h_i = \alpha_1[g_i]$ by applying d_0 and d_n. $\boxed{QED}$

For groups G and H, the simplicial sets $BG_\bullet$ and $BH_\bullet$ are fibrant and it holds for the set of homotopy classes of maps that

$$\mathrm{Hom}(G,H) = \mathrm{map}_\Delta(BG_\bullet, BH_\bullet) = [BG_\bullet, BH_\bullet].$$

In particular, for a non-abelian group G it is not possible to realize the inversion $\gamma : G \to G$, $\gamma(x) := x^{-1}$ by a simplicial map $BG_\bullet \to BG_\bullet$ as γ is not a homomorphism. But we can construct a reflection simplicial map from γ:

8 Definition. For any group G, define a map

$$\gamma : BG_\bullet \longrightarrow BG_\bullet^\tau$$
$$\gamma[g_1,g_2,\ldots,g_n] := [g_n^{-1}, g_{n-1}^{-1},\ldots,g_1^{-1}].$$

It is straightforward to check that γ is a reflection simplicial map, which we call *reflection inversion* on $BG_\bullet$. It holds $\gamma\gamma^\tau = \mathbb{I} = \gamma^\tau\gamma$ and we define the *geometric reflection inversion* by the composition

$$\Gamma := \tau_* \circ |\gamma| : BG \longrightarrow BG.$$

Hence, Γ is the cellular homeomorphism given by

$$\Gamma([g_1,\ldots,g_n],(t_0,\ldots,t_n)) = ([g_n^{-1},\ldots,g_1^{-1}],(t_n,\ldots,t_0)),$$

and it holds $\Gamma^2 = \mathbb{I}$.

9 Theorem. *For any group G, geometric reflection inversion Γ is homotopic to the identity of BG. An explicit homotopy is given by*

$$h : I \times BG \longrightarrow BG,$$

$$\begin{aligned} &h_s([g_1, \ldots, g_n], (t_0, \ldots, t_n)) := \\ &\qquad ([g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}], (s't_0, s't_1, \ldots, s't_{n-1}, t_n, st_{n-1}, \ldots, st_1, st_0)) \end{aligned}$$

with $s' := 1 - s$.

PROOF. Γ maps the 1-cell $[g]$ to the 1-cell $[g^{-1}]$, but reverses the orientation. Thus it induces the identity on the fundamental group $\pi_1(BG) = G$ and by $\mathrm{Hom}(G, G) = [BG, BG]$, it follows $\Gamma \simeq \mathbb{I}$. Now we will show that h_s is well-defined for all s, with $h_0 = \mathbb{I}$ and $h_1 = \Gamma$. The gluing maps of the cells in BG are given by the identification $(\phi^* x, t) \sim (x, \phi_* t)$ which is generated by the relations

$$\begin{aligned} (d_0): \quad & ([g_2, \ldots, g_n], (t_0, \ldots, t_{n-1})) \\ & \sim ([g_1, \ldots, g_n], (0, t_0, \ldots, t_{n-1})) \\ (d_i): \quad & ([g_1, \ldots, g_i g_{i+1}, \ldots, g_n], (t_0, \ldots, t_{n-1})) \\ & \sim ([g_1, \ldots, g_n], (t_0, \ldots, 0, \ldots, t_{n-1})) \\ (d_n): \quad & ([g_1, \ldots, g_{n-1}], (t_0, \ldots, t_{n-1})) \\ & \sim ([g_1, \ldots, g_n], (t_0, \ldots, t_{n-1}, 0)) \\ (s_j): \quad & ([g_1, \ldots, 1, \ldots, g_n], (t_0, \ldots, t_{n+1})) \\ & \sim ([g_1, \ldots, g_n], (t_0, \ldots, t_j + t_{j+1}, \ldots, t_{n+1})) \end{aligned}$$

with $0 < i < n$ and $0 \leq j \leq n$. In particular, it follows $h_0 = \mathbb{I}$ by n-fold application of $d_{\max}$ and $h_1 = \Gamma$ by n-fold application of d_0. We also have to check that the relations (d_j), (s_j) are respected by h_s for $0 < s < 1$. For h_s respecting (d_0),

$$\begin{aligned} &([g_2, \ldots, g_n, g_n^{-1}, \ldots, g_2^{-1}], (s't_0, \ldots, s't_{n-2}, t_{n-1}, st_{n-2}, \ldots, st_0)) \\ &\qquad \sim ([g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}], (0, s't_0, \ldots, s't_{n-2}, t_{n-1}, st_{n-2}, \ldots, st_0, 0)) \end{aligned}$$

holds by application of relations (d_0) and (d_{2n}). For h_s respecting (d_i),

$$\begin{aligned} &([g_1, \ldots, g_i g_{i+1}, \ldots, g_n, g_n^{-1}, \ldots, g_{i+1}^{-1} g_i^{-1}, \ldots, g_1^{-1}], \\ &\qquad\qquad (s't_0, \ldots, s't_{n-2}, t_{n-1}, st_{n-2}, \ldots, st_0)) \\ &\qquad\qquad\qquad \sim ([g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}], \\ &\qquad\qquad\qquad\quad (s't_0, \ldots, 0, \ldots, s't_{n-2}, t_{n-1}, st_{n-2}, \ldots, 0, \ldots, st_0)) \end{aligned}$$

holds by application of (d_i) and (d_{2n-i}). For h_s respecting (d_n),

$$\begin{aligned}([g_1,\dots,g_{n-1},g_{n-1}^{-1},\dots,g_1^{-1}],(s't_0,\dots,s't_{n-2},t_{n-1},st_{n-2},\dots,st_0))\\ \sim([g_1,\dots,g_n,g_n^{-1},\dots,g_1^{-1}],(s't_0,\dots,s't_{n-1},0,st_{n-1},\dots,st_0))\end{aligned}$$

holds by application of (d_n) and then (s_n) as $s't_{n-1}+st_{n-1}=t_{n-1}$. For h_s respecting (s_j),

$$\begin{aligned}([g_1,\dots,1,\dots,g_n,g_n^{-1},\dots,1,\dots,g_1^{-1}],(s't_0,\dots,s't_n,t_{n+1},st_n,\dots,st_0))\\ \sim([g_1,\dots,g_n,g_n^{-1},\dots,g_1^{-1}],\\ (s't_0,\dots,s'(t_j+t_{j+1}),\dots,s't_n,t_{n+1},st_n,\dots,s(t_j+t_{j+1}),\dots,st_0))\end{aligned}$$

holds by application of (s_j) and (s_{2n-j}). $\boxed{QED}$

We remark that the homotopy h is not a cellular map as $h_{\frac{1}{2}}$ maps an n-cell into a $2n$-cell.

In the cellular cochain complex $C^*(BG;A)$ of BG, geometric reflection inversion Γ induces the chain homomorphism

$$\Gamma^*(c)[g_1,\dots,g_n]=\epsilon(n)c[g_n^{-1},\dots,g_1^{-1}]$$

which is chain homotopic to the identity. Here, $\epsilon(n)=(-1)^{\frac{1}{2}n(n+1)}$ was defined as the orientation character of τ_*.

10 Definition. Define the invariant cochain complex $C^*_\Gamma(BG;A)$ with respect to the reflection inversion Γ as the subcomplex

$$C^*_\Gamma(BG;A)=\{\,c\in C^*(BG;A)\mid\Gamma^*c=c\,\}$$

and let $C^*_{\bar\Gamma}(BG;A)=\{\,c\in C^*(BG;A)\mid\Gamma^*c=-c\,\}$ be the subcomplex of anti-invariant cochains. The homology of these subcomplexes is denoted by $H^*_\Gamma(BG;A)$ and $H^*_{\bar\Gamma}(BG;A)$ and is called *inversion (anti-)invariant cohomology.*

11 Theorem. *Let the coefficient group A be local away from 2 (i.e., multiplication by 2 is an isomorphism). Then the inclusion $C^*_\Gamma(BG;A)\subset C^*(BG;A)$ is a chain equivalence. The inversion invariant cohomology is isomorphic to the usual group cohomology*

$$H^*_\Gamma(BG;A)\cong H^*(BG;A),$$

and the inversion anti-invariant cohomology $H^*_{\bar\Gamma}(BG;A)$ vanishes.

Proof. As $\Gamma^* \simeq \mathbb{I}$, the chain map $c \mapsto \frac{1}{2}(c + \Gamma^* c)$ gives a deformation retraction of $C^*(BG;A)$ to the subcomplex $C^*_\Gamma(BG;A)$. As $C^*_{\bar{\Gamma}}(BG;A)$ can be identified with the factor complex, the long exact sequence shows that its homology vanishes. $\boxed{QED}$

We recall that the product structure in cohomology can be defined explicitly by the Alexander-Whitney product [9]. For normalized m- and n-cochains c, c' on $BG_\bullet$ with values in a commutative ring A, it is given by

$$(c \cup c')[g_1, g_2, \ldots, g_{m+n}] := c[g_1, \ldots, g_m] \cdot c'[g_{m+1}, \ldots, g_{m+n}].$$

By the coboundary formula

$$d(c \cup c') = dc \cup c' + (-1)^m c \cup dc',$$

the product of cocycles is a cocycle, which gives the cup product on cohomology. We recall that this gives a graded commutative product on cohomology level (not on cochain level):

$$x \cup y = (-1)^{mn} y \cup x$$

for two cohomology classes x, y in dimensions m and n.

For two Γ-invariant cocycles c, c' in dimensions m and n, we have

$$(c \cup c')[g_1, \ldots, g_{m+n}] = c[g_1, \ldots, g_m] c'[g_{m+1}, \ldots, g_{m+n}]$$
$$= \epsilon(m)\epsilon(n) c[g_m^{-1}, \ldots, g_1^{-1}] c'[g_{m+n}^{-1}, \ldots, g_{m+1}^{-1}]$$
$$= \epsilon(m)\epsilon(n)(c' \cup c)[g_{m+n}^{-1}, \ldots, g_1^{-1}].$$

By graded commutativity, this is in the same cohomology class as

$$\epsilon(m)\epsilon(n)(-1)^{mn}(c \cup c')[g_{m+n}^{-1}, \ldots, g_1^{-1}].$$

As $\epsilon(m)\epsilon(n)(-1)^{mn} = \epsilon(m+n)$, it follows that $c \cup c'$ and $\Gamma^*(c \cup c')$ are in the same cohomology class. Thus we have showed:

12 Proposition. *Let the coefficients A be a commutative ring which is local away from 2. The bilinear Γ-invariant pairing on $C^*_\Gamma(BG;A)$*

$$c \circ c' := \frac{1}{2}(c \cup c' + \Gamma^*(c \cup c'))$$

defines a product structure in $H^*_\Gamma(BG;A)$ which coincides with the cup product in group cohomology under the isomorphism induced by the inclusion $C^*_\Gamma(BG;A) \subset C^*(BG;A)$.

Now we consider the fixed point sets of γ and Γ. We have $\gamma(a) = a$ for $a \in G \times G \times \cdots \times G$ if and only if

$$a = [g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}]$$

in even dimensions $2n$, and

$$a = [g_1, \ldots, g_n, g, g_n^{-1}, \ldots, g_1^{-1}]$$

with $g^2 = 1$ in odd dimensions $2n+1$. The fixed point set of the reflection τ_* on Δ^m is given by the set of points

$$(t_0, t_1, \ldots, t_{n-1}, t_n, t_{n-1}, \ldots, t_1, t_0)$$

if $m = 2n$, and by

$$(t_0, t_1, \ldots, t_n, t_n, \ldots, t_1, t_0)$$

if $m = 2n + 1$. In both cases, the fixed point set is linearly homeomorphic to Δ^n, where the linear homeomorphism is given by

$$(s_0, s_1, \ldots, s_{n-1}, s_n) \mapsto \frac{1}{2}(s_0, s_1, \ldots, s_{n-1}, 2s_n, s_{n-1}, \ldots, s_1, s_0)$$

for $m = 2n$ and by

$$(s_0, s_1, \ldots, s_n) \mapsto \frac{1}{2}(s_0, s_1, \ldots, s_n, s_n, \ldots, s_1, s_0)$$

for $m = 2n + 1$. Thus the geometric fixed point set BG^Γ is the union of points

$$([g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}], \frac{1}{2}(s_0, \ldots, s_{n-1}, 2s_n, s_{n-1}, \ldots, s_0))$$

and

$$([g_1, \ldots, g_n, g, g_n^{-1}, \ldots, g_1^{-1}], \frac{1}{2}(s_0, \ldots, s_n, s_n, \ldots, s_0))$$

with $(s_0, \ldots, s_n) \in \Delta^n$ and $g^2 = 1$. Here, all g_i and g have to be non-trivial because otherwise the tuple in $BG_\bullet$ is degenerate.

13 Theorem. *Let the group G be without 2-torsion. Then the fixed point set BG^Γ is homeomorphic to BG. In fact, the homotopy h above gives a homeomorphism $h_{\frac{1}{2}}$ from BG onto BG^Γ, which also shows that BG^Γ is a deformation retract of BG.*

Proof. As the condition $g^2 = 1$ is only satisfied by $g = 1$ and thus leads to a degenerate simplex, the fixed point set is located in cells of even dimension $m = 2n$:

$$BG^\Gamma = \{ ([g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}], \frac{1}{2}(s_0, \ldots, s_{n-1}, 2s_n, s_{n-1}, \ldots, s_0)) \}.$$

This set of points is equal to the image of $h_{\frac{1}{2}}$ which is an injective map. As each n-cell of BG is mapped to a linear sub-n-cell within a $2n$-cell of BG, $h_{\frac{1}{2}}$ is a homeomorphism onto its image. $\boxed{QED}$

In general, one can prove that BG^Γ consists of the disjoint union of one homeomorphic copy of BG for each g with $g^2 = 1$.

Last, we consider the quotient set BG/Γ of Γ. As Γ is a cellular involution, the projection $p : BG \to BG/\Gamma$ and the transfer map $tr : H^*(BG; A) \to H^*(BG/\Gamma; A)$ in singular cohomology are related by [2]

$$tr \circ p^* = 2, \qquad p^* \circ tr = 1 + \Gamma* = 2.$$

In particular, p^* and tr are isomorphisms of singular cohomology if the coefficients A are local away from 2.

In order to give the quotient a CW structure, we have to subdivide the cells of BG that contain fixed points. The finer cell decomposition of BG has to have the property that Γ either fixes a cell pointwise or maps it homeomorphically to another cell, in order to get an induced cell decomposition of BG/Γ. We call this property *strong invariance*.

To this end we first look for such a cell decomposition for the action of the reflection involution τ_* on Δ^m. Let sign : $\mathbb{R} \to \{ -, 0, + \}$ be the sign function, then we define in Δ^{2n} and Δ^{2n-1}

$$\delta_{a_0 \ldots a_{n-1}} := \{ (t_0, \ldots, t_{2n}) \in \Delta^{2n} \mid \operatorname{sign}(t_i - t_{2n-i}) = a_i \text{ for } 0 \leq i \leq n-1 \},$$

$$\delta'_{a_0 \ldots a_{n-1}} := \{ (t_0, \ldots, t_{2n-1}) \in \Delta^{2n-1} \mid \operatorname{sign}(t_i - t_{2n-1-i}) = a_i \text{ for } 0 \leq i \leq n-1 \},$$

where $a_i \in \{ -, 0, + \}$. This gives a cell decomposition of Δ^m into 3^n cells with $\dim(\delta_{a_0 \ldots a_{n-1}}) = 2n - d$ and $\dim(\delta'_{a_0 \ldots a_{n-1}}) = 2n - 1 - d$, where d denotes the number of a_i with $a_i = 0$. The fixed point set in Δ^m is given by the case where all $a_i = 0$. The map τ_* maps the cells in the following way:

$$\tau_*(\delta_{a_0 \ldots a_{n-1}}) = \delta_{-a_0 \ldots -a_{n-1}}$$

and the same for δ'. Thus τ_* permutes all cells in pairs with the exception of the fixed point cell. It follows that Δ^m/τ_*, $m = 2n$ ($m = 2n - 1$, respectively) has a cell decomposition into $1 + \frac{1}{2}(3^n - 1)$ cells, starting with the fixed point cell in dimension n ($n-1$, respectively) and ending with 2^{n-1} cells in the top dimension m.

Now we accordingly subdivide the m-cells of BG that contain fixed points. The new CW-structure of BG is strongly invariant under Γ and hence induces a CW-structure on BG/Γ.

14 Theorem. *If G contains no 2-torsion, then $p : BG \to BG/\Gamma$ induces an isomorphism of the fundamental groups.*

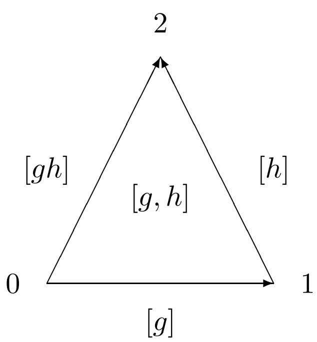
Proof. As $g^2 = 1$ only has the trivial solution, the fixed points are contained in the cells $[g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}]$ of even dimension $2n$. The refined cell decomposition of these cells contributes with cells of dimension n, $n+1, \ldots$, $2n$ to BG/Γ. Hence the 2-skeleton of BG/Γ, which we need to know in order to compute π_1, can be read of the refined 4-skeleton of BG which we denote by Sk^4. Sk^4 has exactly one 0-cell $[\,]$.

The 1-cells of Sk^4 are the 1-cells $[g]$, $g \neq 1$ of BG and the fixed point sets $[g, g^{-1}]^\Gamma$ of the 2-cells $[g, g^{-1}]$, $g \neq 1$. We denote the 1-cell $[g, g^{-1}]^\Gamma$ by $\delta_0(g)$.

The 2-cells of Sk^4 are the 2-cells $[g, h]$ of BG, where g, h, and $gh \neq 1$, the subdivided 2-cells of $[g, g^{-1}]$, $g \neq 1$ which we denote by $\delta_+(g)$ and $\delta_-(g)$, and the fixed point sets $[g, h, h^{-1}, g^{-1}]^\Gamma$ of the 4-cells $[g, h, h^{-1}, g^{-1}]$, $g, h \neq 1$, which we denote by $\delta_{00}(g, h)$.

The 2-cells are glued to the 1-cells in the following way. The boundary of $x := [g, h]$ is the closed path $\partial x := d_2 x d_0 x d_1 x^{-1}$:

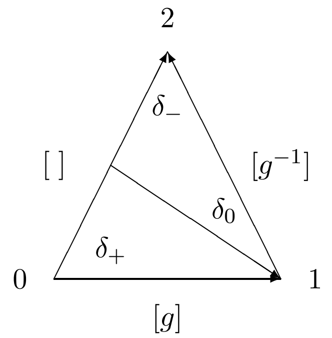
$$\partial[g, h] = [g][h][gh]^{-1}.$$



In particular, $[g, g^{-1}]$ has boundary $[g][g^{-1}]$ as $[1]$ is the constant path in $[\,]$. Furthermore, $[g, g^{-1}]$ is divided into two 2-cells $\delta_+(g)$ and $\delta_-(g)$ that have the 1-cell $\delta_0(g) = [g, g^{-1}]^\Gamma$ as common boundary. Thus we have

$$\partial(\delta_+(g)) = [g]\delta_0(g)^{-1},$$

$$\partial(\delta_-(g)) = \delta_0(g)[g^{-1}].$$



The fixed point set in $[g, h, h^{-1}, g^{-1}]$ consists of the points

$$([g, h, h^{-1}, g^{-1}], (t_0, t_1, t_2, t_1, t_0))$$

and thus has d_0-boundary $[h, h^{-1}]^\Gamma$ (set $t_0 = 0$), d_1-boundary $[gh, h^{-1}g^{-1}]^\Gamma$ (set $t_1 = 0$), and d_2-boundary $[g, g^{-1}]^\Gamma$ (set $t_2 = 0$). Hence, the boundary of the 2-cell $\delta_{00}(g, h) = [g, h, h^{-1}, g^{-1}]^\Gamma$ is the closed path

$$\partial(\delta_{00}(g, h)) = \delta_0(g)\delta_0(h)\delta_0(gh)^{-1}.$$

In particular, we have changed the canonical representation

$$\pi_1 BG = \left\langle [g] \mid [g][h][gh]^{-1} \right\rangle = G$$

with $g, h \in G - \{1\}$ to the 'refined' representation

$$\pi_1 Sk^4 = \left\langle [g], \delta_0(g) \mid [g][h][gh]^{-1}, [g]\delta_0(g)^{-1}, \delta_0(g)[g^{-1}], \delta_0(g)\delta_0(h)\delta_0(gh)^{-1} \right\rangle$$

(with $gh \neq 1$) which gives G, again, as the second relation replaces $\delta_0(g)$ by $[g]$.

Now we consider the induced cell structure on BG/Γ, in particular its 2-skeleton, which we read from Sk^4/Γ. As each non-trivial element $g \in G$ is different from its inverse g^{-1}, we can choose a transversal set P for the equivalence relation $g \sim g^{-1}$ on $G - \{1\}$. In particular, G is a disjoint union

$$G = \{1\} \sqcup P \sqcup P^{-1}.$$

Each pair of 1-cells $[g]$, $[g^{-1}]$ is identified by Γ with orientation being reversed (because of $\epsilon(1) = -1$). This contributes one 1-cell to BG/Γ for each $g \in P$, which we denote by $< g >$. We put on $< g >$ the orientation given by $[g]$. If $g \in P^{-1}$, we define $< g > := < g^{-1} >^{-1}$ (reverse orientation). The 1-cells $\delta_0(g) = [g, g^{-1}]^\Gamma$ are pointwise fixed and thus also give 1-cells in BG/Γ.

Each 2-cell $[g,h]$ with $g,h,gh \neq 1$ is identified by Γ with $[h^{-1},g^{-1}]$, orientation being reversed again (because of $\epsilon(2)=-1$). We denote these 2-cells by $< g,h >$. Their boundary is given by the closed path

$$\partial < g,h > = < g >< h >< gh >^{-1} .$$

We remark that we cannot restrict to $g,h \in P$, here, because of

$$(G - \{ 1 \}) \times (G - \{ 1 \}) = P \times P \sqcup P \times N \sqcup N \times P \sqcup N \times N.$$

The involution Γ interchanges $P \times P$ and $N \times N$, but maps $P \times N$ and $N \times P$ to themselves. Furthermore, Γ maps the 2-cell $\delta_+(g)$ to the 2-cell $\delta_-(g)$ and vice versa (orientation reversing). We denote the identified 2-cell by $\delta_\pm(g)$ and get

$$\partial(\delta_\pm(g)) = < g > \delta_0(g).$$

The 2-cell $\partial(\delta_{00}(g,h))$ consists of fixed points and hence also gives a 2-cell in BG/Γ with

$$\partial(\delta_{00}(g,h)) = \delta_0(g)\delta_0(h)\delta_0(gh)^{-1}.$$

Thus we have

$$\begin{aligned}\pi_1(BG/\Gamma) = \big\langle < g >, \delta_0(g) \mid \; & < g >< g^{-1} >, < g >< h >< gh >^{-1}, \\ & < g > \delta_0(g), \delta_0(g)\delta_0(h)\delta_0(gh)^{-1} \big\rangle\end{aligned}$$

with $g,h,gh \neq 1$, which proves our claim. $\boxed{QED}$

4 Reflection involutions on Eilenberg-MacLane spaces and simplicial loops

We recall [5] that the minimal Eilenberg-MacLane complex $K(A,n)_\bullet$ for an abelian group A can be defined as

$$K(A,n)_\bullet = A[S^n_\bullet]/A[0_\bullet],$$

i.e. the free reduced A-'module' generated by the simplicial n-sphere. Thus the k-simplices are the elements of the finite direct sum $\oplus_{s_I} A \cdot s_I \sigma$ where summation is over all degeneracy operators s_I in normal form from dimension n to dimension k (in particular, 0 for $k < n$). Geometric realization gives a topological Eilenberg-MacLane space $K(A,n) := |K(A,n)_\bullet|$ which is a CW-complex with cellular abelian group structure.

As we already defined a reflection isomorphism $\lambda : S^n_\bullet \to (S^n_\bullet)^\tau$, linear extension gives a reflection isomorphism

$$\lambda : K(A,n)_\bullet \longrightarrow K(A,n)^\tau_\bullet,$$

$$\lambda\left(\sum_{s_I} a_I \cdot s_I \sigma\right) = \sum_{s_I} a_I \cdot \tau(s_I)\sigma.$$

Clearly, $\lambda(x+y) = \lambda(x) + \lambda(y)$ and $\lambda\lambda^\tau = \mathbb{I} = \lambda^\tau\lambda$. We remark that $\gamma = -\lambda$ in the special case $K(A,1)_\bullet = BA_\bullet$.

15 Definition. Define the *geometric reflection involution* on $K(A,n)$ as

$$\Lambda := \tau_* \circ |\lambda| : K(A,n) \longrightarrow K(A,n).$$

Clearly, Λ is a cellular isomorphism of CW abelian groups and $\Lambda^2 = \mathbb{I}$.

16 Lemma. *The homotopy class of geometric reflection involution is given by*

$$\Lambda \simeq \epsilon(n)\mathbb{I}.$$

Proof. There holds the following well-known rigidity results for maps between simplicial Eilenberg-MacLane spaces of equal dimension [9]

$$\mathrm{map}_\Delta(K(A,n)_\bullet, K(B,n)_\bullet) \cong [K(A,n)_\bullet, K(B,n)_\bullet] \cong \mathrm{Hom}(A,B)$$

where the isomorphism is given by the induced map in $\pi_n(-)$. Hence also

$$[K(A,n), K(B,n)] = \mathrm{Hom}(A,B)$$

in the topological category. Now, λ is the identity on $K(A,n)_n = A \cdot \sigma$, but Λ changes the orientation of each n-cell in $K(A,n)$ which is associated to a non-trivial $a \in A \cdot \sigma$, by the factor $\epsilon(n)$. As these n-cells are spheres and represent the corresponding element $a \in \pi_n(K(A,n)) = A$, the induced map of Λ in π_n is given by $a \mapsto \epsilon(n)a$. $\boxed{QED}$

Now we can recover our isomorphism result on inversion invariant cohomology in the following way.

In the simplicial category, there holds the following representability for normalized cocycles:

$$Z^n(X_\bullet; A) \cong \mathrm{map}_\Delta(X_\bullet, K(A,n)_\bullet)$$

where the isomorphism is given by pull back of the universal cocycle [9]

$$z \in Z^n(K(A,n)_\bullet; A).$$

Thus we may represent a cocycle in $Z^n(BG_\bullet; A)$ by its map

$$c : BG_\bullet \longrightarrow K(A; n)_\bullet$$

and for A local away from 2 we may form the new cocycle

$$c' := \frac{1}{2}(c + \epsilon(n)\lambda^\tau c^\tau \gamma)$$

which is equivariant in the sense that $\lambda c' = \epsilon(n)(c')^\tau \gamma$. By $|\epsilon(n)\lambda^\tau c^\tau \gamma| = \epsilon(n)\Lambda|c|\Gamma \simeq |c|$, it follows that $c' \simeq \frac{1}{2}(c + c) = c$, hence every cocycle c has an equivariant representative c'.

Now we consider the problem how this result can be generalized to other cohomology theories. To this end we recall the following simplicial representability result concerning generalized cohomology theories which we proved in [7]:

17 Theorem. *Let $E^*(-)$ be a generalized cohomology theory given by a topological spectrum E. Then, on the category of simplicial sets, $E^*(| - |)$ can be represented as the homology of a loop-valued cochain functor $C^*_E(-)$. If the coefficients $\pi_*(E)$ are local away from 2, then the cochain functor can be chosen with values in commutative loops.*

We recall that a *loop* is a group without associativity, i.e. a set L with a distinguished element $1 \in L$ and a binary operation $* : L \times L \to L$, such that 1 is a two-sided unit, and the equation $x * y := z$ in L has to be uniquely solvable for any given x and z (or y and z) [3], [4]. Of course, L is called *commutative* if $x * y = y * x$ for all x, y.

We showed the above result by representing $E^*(-)$ for simplicial sets $X_\bullet$ as $E^*(|X_\bullet|) = Z^*_E(X_\bullet)/B^*_E(X_\bullet)$ with loop-valued cocycles Z_E and coboundaries B_E in a loop-valued chain complex given by

$$C^k_E(X_\bullet) = \mathrm{map}_\Delta(X_\bullet, PL^{k+1}_\bullet).$$

Here, $L^n_\bullet$, $n \in \mathbb{Z}$, are simplicial loops which we constructed from $E^*(-)$, and P denotes the simplicial path space functor. Moreover, the cocycles are given by

$$Z^k_E(X_\bullet) = \mathrm{map}_\Delta(X_\bullet, \Omega L^{k+1}_\bullet)$$

with Ω denoting the simplicial loop space functor, and there are isomorphisms of simplicial loops $\Omega L^{k+1}_\bullet = L^k_\bullet$. Hence $L^n_\bullet$ is a minimal fibrant simplicial set which generalizes the simplicial Eilenberg-MacLane complex $K(A, n)_\bullet$.

Now, we consider the problem if there is a reflection involution on the simplicial loop $L^n_\bullet$ generalizing λ on $K(A, n)_\bullet$. The assumption that $L_\bullet$ is local away from 2 will enable us to prove existence of a similar strong symmetry as in [7], where we proved existence of a strict commutative loop space structure, strengthening a theorem of Irye and Kono [6].

18 Theorem. *Let $(L_\bullet, *)$ be a simplicial loop which is connected, minimal and local away from 2 (i.e. all homotopy groups $\pi_k(L_\bullet)$ are local away from 2). Then there exists a reflection isomorphism*

$$\rho : L_\bullet \to L_\bullet^\tau$$

such that $\rho\rho^\tau = \mathbb{I} = \rho^\tau\rho$. The geometric realization $\tau_*|\rho|$ is homotopic to the identity.

We remark that we do not assume that $*$ has to be commutative. Before we give the proof of this theorem, we need the following lemma concerning central loop extensions (see [3], [4], [7] for the notion of a *central extension* of loops). For a simplicial abelian group $Z_\bullet$, we denote by $BZ_\bullet$ its classifying 'space' (simplicial set). $BZ_\bullet$ is a simplicial abelian group, again, and there is a canonical isomorphism $Z_\bullet = \Omega BZ_\bullet$ (see [9]).

19 Lemma. *Let*

$$Z_\bullet \hookrightarrow \tilde{M}_\bullet \xrightarrow{p} M_\bullet$$

be a central extension of simplicial loops. We denote the loop multiplication in $Z_\bullet$ by $+$, and in $\tilde{M}_\bullet$, $M_\bullet$ by $*$.

- *There exist simplicial maps $k : M_\bullet \to BZ_\bullet$ (the* classifying map*) and $\tilde{k} : \tilde{M}_\bullet \to PBZ_\bullet$ such that the following holds: Let $Z_\bullet \hookrightarrow E(k)_\bullet \xrightarrow{pr_1} M_\bullet$ with $E(k)_\bullet := \{\, (x,v) \in M_\bullet \times PBZ_\bullet \mid k(x) = dv \,\}$ be the pullback of the universal principal fibre bundle $Z_\bullet \to PBZ_\bullet \xrightarrow{d} BZ_\bullet$ by k. Then the central extension $Z_\bullet \to \tilde{M}_\bullet \to M_\bullet$ is as a principal fibre bundle isomorphic to $Z_\bullet \to E(k)_\bullet \to M_\bullet$ with isomorphism given by $(p, \tilde{k}) : \tilde{M}_\bullet \to E(k)_\bullet$. It holds $d\tilde{k} = kp$ and $\tilde{k}(a * x) = a + \tilde{k}(x)$ for $a \in Z_\bullet$ and $x \in L_\bullet$.*

- *The classifying map k is determined up to homotopy. The bundle map $\tilde{k}$ is for fixed k determined up to bundle automorphism.*

- *Let $x, y \in M_\bullet$ and define $l(x,y) := \tilde{k}(\tilde{x} * \tilde{y}) - \tilde{k}(\tilde{x}) - \tilde{k}(\tilde{y})$ where $p(\tilde{x}) = x$ and $p(\tilde{y}) = y$. Then l is a well-defined simplicial map $l : M_\bullet \times M_\bullet \to PBZ_\bullet$ and it holds $dl(x,y) = k(x * y) - k(x) - k(y)$.*

- *The definition $(x,v) * (y,w) := (x * y, v + w + l(x,y))$ gives a well-defined simplicial loop structure $* : E(k)_\bullet \times E(k)_\bullet \to E(k)_\bullet$ on $E(k)_\bullet$. The principal fibre bundle $Z_\bullet \to E(k)_\bullet \to M_\bullet$ is a simplicial central loop extension and $(p, \tilde{k}) : \tilde{M}_\bullet \to E(k)_\bullet$ is a simplicial isomorphism of central loop extensions.*

Proof. We showed in [7] that a simplicial loop extension is a fibration. As the extension is central, $Z_\bullet$ is a simplicial abelian group and we have a free and transitive action of the fibre $Z_\bullet$ on $\tilde{M}_\bullet$. Hence the extension is a simplicial principal fibre bundle [9]. Classification by homotopy classes of maps from the basis $M_\bullet$ to the classifying space $BZ_\bullet$ is a standard fact [9]. (We remark that the universal principle fibre bundle $Z_\bullet \to EZ_\bullet \to BZ_\bullet$ can be canonically identified with the bundle $\Omega BZ_\bullet \to PBZ_\bullet \to BZ_\bullet$.) This proves the first two statements. The map $(\tilde{x}, \tilde{y}) \mapsto \tilde{k}(\tilde{x} * \tilde{y}) - \tilde{k}(\tilde{x}) - \tilde{k}(\tilde{y})$ from $\tilde{M}_\bullet \times \tilde{M}_\bullet$ to $PBZ_\bullet$ is invariant with respect to the replacement $(\tilde{x}, \tilde{y}) \mapsto (a * \tilde{x}, b * \tilde{y})$ for any $a, b \in Z_\bullet$. Here we use that $Z_\bullet$ is central: $(a*\tilde{x})*(b*\tilde{y}) = (a+b)*(\tilde{x}*\tilde{y})$. This proves that l is well-defined, and the formula for dl follows with $d\tilde{k} = kp$. The last statement follows from $(p, \tilde{k})(\tilde{x} * \tilde{y}) = (x * y, v + w + l(x, y))$ where we have set $(p, \tilde{k})(\tilde{x}) =: (x, v)$ and $(p, \tilde{k})(\tilde{y}) =: (y, w)$. $\boxed{QED}$

Now we give the proof of the above theorem.

Proof. In [7] we have proved that the canonical Postnikov decomposition of a connected minimal simplicial loop $L_\bullet$ consists of simplicial central loop extensions

$$K(\pi_n(L_\bullet), n)_\bullet \hookrightarrow L_\bullet^{(n)} \longrightarrow L_\bullet^{(n-1)}.$$

Now, our proof works by inductive construction of the reflection isomorphism $\rho^{(n)} : L_\bullet^{(n)} \to (L_\bullet^{(n)})^\tau$ from $\rho^{(n-1)} : L_\bullet^{(n-1)} \to (L_\bullet^{(n-1)})^\tau$. Moreover, we will show that there is a commutative diagram

$$\begin{array}{ccccc}
K(\pi_n, n)_\bullet^\tau & \hookrightarrow & (L_\bullet^{(n)})^\tau & \longrightarrow & (L_\bullet^{(n-1)})^\tau \\
\Big\uparrow \epsilon(n)\lambda & & \Big\uparrow \rho^{(n)} & & \Big\uparrow \rho^{(n-1)} \\
K(\pi_n, n)_\bullet & \hookrightarrow & L_\bullet^{(n)} & \longrightarrow & L_\bullet^{(n-1)}
\end{array}$$

Then the limit $n \to \infty$ gives a well-defined reflection isomorphism $\rho : L_\bullet \to L_\bullet^\tau$ as $L^{(n)}$ and $\rho^{(n)}$ coincide with $L^{(n-1)}$ and $\rho^{(n-1)}$ below dimension n.

In order to carry out the inductive construction, we use the preceding lemma with $Z_\bullet := K(\pi_n, n)_\bullet$, $\tilde{M}_\bullet := L_\bullet^{(n)}$ and $M_\bullet := L_\bullet^{(n-1)}$. We assume that $\rho : M_\bullet \to M_\bullet^\tau$ is already constructed and satisfies $\tau_*|\rho| \simeq \mathbb{I}$ and $\rho\rho^\tau = \mathbb{I} = \rho^\tau\rho$. We take a classifying map $k : M_\bullet \to BZ_\bullet$ and form

$$k_0 := \frac{1}{2}(k + \epsilon(n)\lambda^\tau k^\tau \rho)$$

which is a map $k_0 : M_\bullet \to BZ_\bullet$. As the realizations of $\epsilon(n)\lambda$ and ρ are homotopic to the identity, $k_0 \simeq k$ also serves as a classifying map. Taking a bundle map

$\tilde{k}_0$ over k_0 gives us an isomorphism $(p, \tilde{k}_0) : \tilde{M}_\bullet \to E(k_0)_\bullet$. On $E(k_0)_\bullet \subset M_\bullet \times PBZ_\bullet$, there is the reflection isomorphism $(\rho, \epsilon(n)\lambda)$ which maps $E(k_0)_\bullet$ to itself because of $\epsilon(n)\lambda k_0 = k_0^\tau \rho$. By construction, $(\rho, \epsilon(n)\lambda)$ is involutive and homotopic to the identity. Pull back to $\tilde{M}_\bullet$ by $(p, \tilde{k}_0)$ defines the reflection isomorphism $\tilde{\rho}$. $\boxed{QED}$

It is a tempting question if ρ on $L_\bullet$ can be chosen such that $\rho(x * y) = \rho(x) * \rho(y)$. If this is possible and $*$ is commutative, any generalized cocycle $c : BG_\bullet \to L_\bullet$ can be deformed to an equivariant cocycle $c_0 := \frac{1}{2}(c * \rho^\tau c^\tau \gamma) : BG_\bullet \to L_\bullet$, i.e. $c_0 \simeq c$ and $\rho c_0 = c_0^\tau \gamma$.

Unfortunately, an attempt to prove inductively existence of such ρ as above runs against the following obstruction: If ρ on $M_\bullet$ satisfies $\rho(x*y) = \rho(x)*\rho(y)$, then (ρ, λ') on $E(k_0)_\bullet$ (with $\lambda' := \epsilon(n)\lambda$) satisfies

$$(\rho, \lambda')((x, v) * (y, w)) = (\rho(x * y), \lambda'(v + w + l(x, y)))$$

but

$$(\rho, \lambda')(x, v) * (\rho, \lambda')(y, w) = (\rho(x) * \rho(y), \lambda' v + \lambda' w + l(\rho(x), \rho(y))).$$

Thus there is the obstruction $\lambda' l(x, y) - l(\rho(x), \rho(y))$ for $\tilde{\rho}$ to be homomorphic with respect to $*$. Here, l depends on the choice of the bundle map $\tilde{k}_0$ over k_0. A different choice is given by applying an element of the gauge group of bundle automorphisms $Aut(\tilde{M}_\bullet \to M_\bullet)$. As $Z_\bullet$ is abelian, this group is given by $map_\Delta(M_\bullet, Z_\bullet)$. If one changes $\tilde{k}_0$ to $\tilde{k}_0 + fp$ with $f : M_\bullet \to Z_\bullet$, a straightforward calculation shows that the obstruction changes by

$$\lambda'(f(x * y) - f(x) - f(y)) - (f(\rho x * \rho y) - f(\rho x) - f(\rho y)).$$

In general, it seems not to be possible to find some f such that the obstruction vanishes. It would be interesting to know which simplicial loops or generalized cohomology theories admit a reflection involution ρ that is homotopic with respect to multiplication $*$.

Furthermore, it would be interesting to know if there is some generalization of our theorem to the case of simplicial loops $L_\bullet$ which are local away from p with p some odd prime. The corresponding generalization of the existence of a strict commutative loop structure was done in [8].

References

[1] D. J. Benson: Representations and Cohomology II: Cohomology of Groups and Modules, Cambridge Studies in Advanced Math. Vol. 31, Cambridge Univ. Press, Cambridge - New York 1991.

[2] G. E. Bredon: Introduction to Compact Transformation Groups, Pure and Applied Mathematics, Vol. 46, Academic Press, New York - London 1972.

[3] R. H. Bruck: A Survey of Binary Systems, Ergeb. Math. Grenzgeb. 20, Springer, Heidelberg 1958.

[4] O. Chein, H. O. Pflugfelder, J. D. H. Smith (Editors): Quasigroups and Loops. Theory and Applications, Sigma Series in Pure Math. Vol. 8, Heldermann Verlag, Berlin 1990.

[5] E. B. Curtis: *Simplicial homotopy theory*, Advances in Math., **6**, (1971), 107–209.

[6] K. Irye, A. Kono: *Mod p retracts of G-product spaces*, Math. Zeitschrift, **190**, (1985), 357–363.

[7] S. Klaus: *On simplicial loops and H-spaces*, Topology and its Applications, 112, (2001), 337–348.

[8] S. Klaus: *H-spaces with cyclic symmetry*, Topology and its Applications, **124**, (2002), 1–8.

[9] J. P. May: Simplicial Objects in Algebraic Topology, van Nostrand, Princeton 1967.