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The modulus semigroup for linear delay equations II

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Abstract. The main purpose of this paper is describing the generator of the modulus semigroup of the C_0-semigroup associated with the delay equation

$$\begin{cases} u'(t) = Au(t) + Lu_t & (t \geq 0)\,, \\ u(0) = x \in X, & u_0 = f \in L_p(-h, 0; X)\,, \end{cases}$$

in the Banach lattice $X \times L_p(-h, 0; X)$, where X is a Banach lattice with order continuous norm. As a preparation it is shown that $W_p^1(a, b; X)$ is a sublattice of $L_p(a, b; X)$, for $1 \leq p < \infty$. A further preparation is the computation of the modulus of the operator L appearing above. Also, we establish a result concerning the existence of the modulus semigroup for C_0-semigroups acting in KB-spaces.

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Dedicated to the memory of my friend Klaus Floret

1 Introduction

The starting point of the present paper was the question whether the modulus semigroup for a semigroup arising in linear differential equations with delay obtained in [6] could be obtained in more general situations.

If $(T(t))_{t\geq 0}$ is a C_0-semigroup on a Banach lattice X then the *modulus semigroup* $(T_\#(t))_{t\geq 0}$—if it exists—is the smallest positive C_0-semigroup dominating $(T(t))_{t\geq 0}$. Even if for a specific case the existence of the modulus semigroup can be concluded by theoretical reasons it is of interest to get more knowledge of the modulus semigroup, e.g. by describing its generator (see [5], [6], [7], [15]).

Assume that a C_0-semigroup $(T(t))$ is dominated by a C_0-semigroup $(S(t))$. If the growth (at ∞) of the dominating C_0-semigroup can be determined (e.g., by computing the spectrum of the generator) then one obtains information on the behavior of $(T(t))$. For an application of this method we refer to [10]. In such a situation it certainly is desirable to work with the smallest possible semigroup dominating $(T(t))$, i.e., the modulus semigroup of $(T(t))$.

The C_0-semigroup we will consider is associated with the Cauchy problem for a linear delay equation,

$$\begin{cases} u'(t) = Au(t) + Lu_t & (t \geq 0), \\ u(0) = x, \quad u_0 = f, \end{cases} \tag{DE}$$

in the L_p-context, for $1 \leq p < \infty$, with initial values $x \in X$, $f \in L_p(-h, 0; X)$, where X is a Banach lattice with order continuous norm. Here, $h = 1$ or $h = \infty$, $A \in L(X)$ is a regular operator, and $L\colon C([-h, 0]; X) \to X$ is the bounded linear operator given by

$$Lf := \int_{[-h,0]} d\eta(\vartheta) f(\vartheta) \quad (f \in C([-h, 0]; X)),$$

where $\eta\colon [-h, 0] \to L(X)$ is a function of bounded variation with no mass at zero. (We recall the notation

$$C([-\infty, 0]; X) := \{ f \in C((-\infty, 0]; X);\ f(-\infty) := \lim_{\tau \to -\infty} f(\tau) \text{ exists} \}).$$

Also, for a function $u\colon (-h, \infty) \to X$, we recall the notation

$$u_t(\vartheta) := u(t + \vartheta) \quad (-h < \vartheta < 0,\ t \geq 0).$$

The delay equation (DE) is equivalent to an abstract Cauchy problem

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) & (t \geq 0), \\ \mathcal{U}(0) = \binom{x}{f} \end{cases}$$

in the space $X \times L_p(-h, 0; X)$, where $\mathcal{A}$ is given by

$$\mathcal{A} := \begin{pmatrix} A & L \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \{ (x, f) \in X \times W_p^1(-h, 0; X);\ f(0) = x \}.$$

From [4], [8], [9], [11] it is known that the operator $\mathcal{A}$ generates a C_0-semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ on the Banach lattice $X \times L_p(-h, 0; X)$.

Since it is assumed that A is a regular operator it is known that the semigroup generated by A possesses a modulus semigroup whose generator $A_\#$ is equal to A 'on the diagonal' whereas its 'off-diagonal part' is the modulus of the 'off-diagonal part' of A; we refer to Section 4 for details. Also, we will pose assumptions implying that the operator L possesses a modulus. Then, in view of the result proved in [6], it seems reasonable to expect that the generator $\mathcal{A}_\#$ of the modulus semigroup of $(e^{t\mathcal{A}})_{t\geq 0}$ is given by

$$\mathcal{A}_\# = \begin{pmatrix} A_\# & |L| \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}.$$

It is the main object of this paper to confirm this conjecture; see Theorem 11.

The ideas of our proof in Section 4 are the natural extension of the ideas present in [6], for finite dimensional X. However, in order to make these ideas work it was necessary to develop the properties of Banach lattice valued Sobolev spaces as well as to compute the modulus of the operator L above. What might still seem unsatisfactory is that the operator A is supposed to be bounded (and then regular, because the generated semigroup is required to have a modulus semigroup). Instead, it is desirable to just assume that A is the (possibly unbounded) generator of a C_0-semigroup having a modulus semigroup. In fact, with the generator $A_\#$ of the latter one has the same result as sketched above. This result will be the content of the forthcoming paper [16]. We mention that the results of Sections 2 and 3 of the present paper are needed in [16], and that additional new ideas are needed in the proof.

In Section 2 it is shown that, for a Banach lattice X with order continuous norm, the Sobolev space $W_p^1(-h,0;X)$ is a vector lattice and moreover, for functions $f,g \in W_p^1(-h,0;X)$, $g \geq 0$, the function $(\operatorname{sgn} f)(|f| \wedge g)$ belongs to $W_p^1(-h,0;X)$. It is this latter property which is used for computing the modulus of the operator L on the subspace $W_p^1(-h,0;X)$ of $C([-h,0];X)$; see Remark 2 and the proof of Lemma 13. Besides serving as a tool for Section 4, the results of this section are of independent interest.

In Section 3 we show that the modulus of the operator L mentioned above is an operator of the same kind as L, associated with the 'variation' $\widetilde{\eta}$ of η in the regular operators, under suitable hypotheses.

Section 4 is devoted to showing the main result of the paper. In principle, the method used in [6] is extended to the present more general case. In order to achieve this we need the results of Sections 2 and 3, in the last part of the proof of Lemma 13. Also, we present a new proof of Lemma 13(b) avoiding the use of the finite dimensionality of $\mathbb{R}^n$.

In the Appendix we prove a supplementary result on the modulus of semigroups. This result, Proposition 16, generalizes the result [5, Prop. 2.5] concerning the existence of the modulus semigroup. The crucial point in our proof is showing the measurability of the object obtained as the natural candidate for the modulus semigroup. This is achieved by using a suitable concept of lower semicontinuity of Banach lattice valued functions.

2 On the Banach lattice valued Sobolev space W_p^1

For standard results and terminology concerning Banach lattices we refer the reader to [1], [12], [14].

If E is a (real or complex) Banach lattice (with the scalar field denoted by $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), and $x, y \in E$, $y \geq 0$, then there exists the *truncation of* x *by* y, denoted by $\tau_y x$. This element is uniquely determined by the properties

(i) $|\tau_y x| = |x| \wedge y$,

(ii) $(\operatorname{Re} \gamma\tau_y x)_+ \leq (\operatorname{Re} \gamma x)_+$ for all $\gamma \in \mathbb{K}$, $|\gamma| = 1$.

(The existence and uniqueness of $\tau_y x$ can be shown by using the lattice isomorphism between E_z, the principal ideal generated by any element $z \in E$ dominating $|x| \vee y$, and $C(K)$, for suitable compact K; we refer to [13, C-I.8] for this procedure. In $C(K)$, the element $\tau_g f$ is given by $(\operatorname{sgn} f)(|f| \wedge g)$, for $f, g \in C(K)$, $g \geq 0$, where $\operatorname{sgn} f$ denotes the (possibly discontinuous) function $t \mapsto \operatorname{sgn}(f(t))$.)

For later use we note that, for $x_1, x_2 \in E$, $y_1, y_2 \in E_+$, one has

$$\left|\tau_{y_1} x_1 - \tau_{y_2} x_2\right| \leq |x_1 - x_2| + |y_1 - y_2|. \tag{1}$$

(This inequality is true for complex numbers:

$$\begin{aligned}
&\left|(\operatorname{sgn} x_1)(|x_1| \wedge y_1) - (\operatorname{sgn} x_2)(|x_2| \wedge y_2)\right| \\
&\leq \left|(\operatorname{sgn} x_1)(|x_1| \wedge y_1) - (\operatorname{sgn} x_2)(|x_2| \wedge y_1)\right| + \left||x_2| \wedge y_1 - |x_2| \wedge y_2\right| \\
&\leq |x_1 - x_2| + |y_1 - y_2|\,.
\end{aligned}$$

Since the principal ideal generated by $|x_1| + |x_2| + y_1 + y_2$ is lattice isomorphic to some $C(K)$, the inequality carries over to the general case.)

If X is a Banach lattice, K a compact space, and $E = C(K; X)$, then

$$\tau_g f(t) = \tau_{g(t)} f(t) \qquad (t \in K)$$

for all $f, g \in C(K; X)$, $g \geq 0$. (In particular, the continuity of the function $t \mapsto \tau_{g(t)} f(t)$ follows from (1).)

If E is countably order complete, and $x \in E$, we recall that the signum operator $\operatorname{sgn} x \in L(E)$ exists and is characterized by the properties

(i) $(\operatorname{sgn} x)\bar{x} = |x|$,

(ii) $|(\operatorname{sgn} x)y| \leq |y| \quad (y \in E)$,

(iii) $(\operatorname{sgn} x)y = 0$ for all $y \in E$, $y \perp x$

(cf. [13, C-I.8]). For all $y \in E_+$, one then obtains

$$\tau_y x = (\operatorname{sgn} x)(|x| \wedge y)\,.$$

1 Theorem. *Let X be a Banach lattice with order continuous norm, $1 \leq p < \infty$, and let $-\infty \leq a < b \leq \infty$. Let $f, g \in W_p^1(a,b;X)$, $g \geq 0$. Then $|f| \in W_p^1(a,b;X)$ and $(\operatorname{sgn} f)(|f| \wedge g) \in W_p^1(a,b;X)$, where $\operatorname{sgn} f$ denotes the ($L(X)$-valued) function $t \mapsto \operatorname{sgn} f(t)$.*

2 Remark. The last property mentioned in Theorem 1 is needed in the following context; see the proof of Lemma 13 below.

Let Y, Z be Banach lattices. Let $A \in L(Y,Z)$ be a regular operator possessing a modulus $|A|$ with

$$|A|x = \sup\{\, |Ay|\,;\ y \in Y,\ |y| \leq x\,\}$$

for $x \in Y_+$. Moreover, let X be a dense subspace of Y enjoying the property that $x, z \in X$, $x \geq 0$ implies $\tau_x z \in X$. Then

$$|A|x = \sup\{\, |Az|\,;\ z \in X,\ |z| \leq x\,\}$$

for all $0 \leq x \in X$. In order to show this let $0 \leq x \in X$. It clearly suffices to show that the set $\{\, z \in X\,;\ |z| \leq x\,\}$ is dense in $\{\, y \in Y\,;\ |y| \leq x\,\}$. Now, if $y \in Y$, $|y| \leq x$, then there exists a sequence $(x_n) \subseteq X$, $x_n \to y$. By the property mentioned above we have that $z_n := \tau_x x_n$ belongs to X ($n \in \mathbb{N}$), and (1) implies $z_n \to \tau_x y = y$ ($n \to \infty$).

In the proof of Theorem 1 we need the following lemma.

3 Lemma. *Let X be a Banach space, $1 \leq p < \infty$, $f \in L_p(\mathbb{R};X)$. Assume that*

$$\Big\{\, \frac{1}{h}(f(\cdot + h) - f)\,;\ 0 < h \leq 1\,\Big\}$$

is relatively weakly compact in $L_p(\mathbb{R};X)$. Then f is differentiable a.e., $f' \in L_p(\mathbb{R};X)$, f is continuous (after a suitable modification a.e.), and $f(t) = f(0) + \int_0^t f'(s)\,ds$.

Proof. From the hypothesis and the Eberlein-Šmulyan theorem we obtain that there exists a null sequence $(h_j)_j \subseteq (0,1]$ such that $g := \operatorname{w-lim} \frac{1}{h_j}(f(\cdot + h_j) - f)$ exists. This implies that for all $\varphi \in C_c^1(\mathbb{R})$ one obtains

$$\int f\varphi'\,dt = -\int g\varphi\,dt\,.$$

Using standard methods of distribution theory we conclude that

$$f(t) - \int_0^t g(s)\,ds$$

is constant (a.e.), i.e., f is continuous and

$$f(t) = f(0) + \int_0^t g(s)\, ds\,.$$

This implies that f is differentiable a.e., and $f' = g$ (cf. [2, Prop. 1.2.2]). $\boxed{QED}$

Proof of Theorem 1. Since differentiability is a local property we may assume that $(a,b) = \mathbb{R}$.

The hypothesis $f \in W_p^1(\mathbb{R};X)$ implies that $\frac{1}{h}(f(\cdot+h) - f) \to f'$ $(h \to 0)$, and therefore the set $\left\{ \frac{1}{h}(f(\cdot+h) - f)\,;\, 0 < h \le 1 \right\}$ is relatively compact in $L_p(\mathbb{R};X)$. Now the inequality

$$\left|\frac{1}{h}(|f(\cdot+h)| - |f|)\right| \le \left|\frac{1}{h}(f(\cdot+h) - f)\right|$$

and the order continuity of the norm of $L_p(\mathbb{R};X)$ imply that the set

$$\left\{ \frac{1}{h}(\left|f(\cdot+h)\right| - \left|f\right|)\,;\, 0 < h \le 1 \right\}$$

is relatively weakly compact in $L_p(\mathbb{R};X)$; cf. [1, chap. 4, Thm 13.8]. Therefore Lemma 3 implies $|f| \in W_p^1(\mathbb{R};X)$.

In order to prove the second assertion we note that (1) implies

$$\begin{aligned} &\frac{1}{h}\left|(\operatorname{sgn} f)(|f| \wedge g)(\cdot+h) - (\operatorname{sgn} f)(|f| \wedge g)\right| \\ &\le \frac{1}{h}\left|f(\cdot+h) - f\right| + \frac{1}{h}\left|g(\cdot+h) - g\right|. \end{aligned}$$

Arguing as above for $|f|$ we obtain $(\operatorname{sgn} f)(|f| \wedge g) \in W_p^1(\mathbb{R};X)$. $\boxed{QED}$

4 Remark. Let X be a Banach lattice with order continuous norm, $1 \le p < \infty$. Then it is not difficult to show that the Banach lattice $L_p(a,b;X)$ has order continuous norm.

Let $f, g \in L_p(a,b;X)$. Then $(\operatorname{sgn} f)g \in L_p(a,b;X)$. (This is obvious if f, g are simple functions. The general case is then treated by approximation.)

If $f \in L_p(a,b;X)$ then the signum operator $\operatorname{sgn}_{L_p(a,b;X)} f$ is given by pointwise application of the ($L(X)$-valued) function $t \mapsto \operatorname{sgn} f(t)$. (This operator has the properties of the signum operator in $L_p(a,b;X)$, and the uniqueness of the signum operator shows the assertion.)

As a supplementary information we want to indicate an explicit expression for the derivative of $|f|$ in Theorem 1. In order to formulate the result we need one further piece of notation. As at the beginning of this section, let E be a countably order complete Banach lattice. For $z \in E_+$ we denote by P_z the band projection onto the band $\{z\}^{dd}$ generated by z. We then define $\widehat{\operatorname{sgn}}\, x, \widetilde{\operatorname{sgn}}\, x \colon E \to E$ by

$$\begin{aligned}(\widehat{\operatorname{sgn}}\, x)y &:= (\operatorname{sgn} x)y + (I - P_{|x|})|y|\,,\\ (\widetilde{\operatorname{sgn}}\, x)y &:= (\operatorname{sgn} x)y - (I - P_{|x|})|y| \quad (= -(\widehat{\operatorname{sgn}}\, x)(-y))\end{aligned}$$

(cf. [13, p. 257]). In general, $\widehat{\operatorname{sgn}}\, x$ and $\widetilde{\operatorname{sgn}}\, x$ are non-linear.

5 Remark. Under the hypotheses of Theorem 1 one has

$$\begin{aligned}|f|'(t) &= \operatorname{Re}((\widehat{\operatorname{sgn}}\, \overline{f(t)})f'(t)) = \operatorname{Re}((\widetilde{\operatorname{sgn}}\, \overline{f(t)})f'(t))\\ &= \operatorname{Re}((\operatorname{sgn} \overline{f(t)})f'(t)) \qquad \text{a.e.}\end{aligned}$$

In order to see this we recall from the proof of Theorem 1 that Lemma 3 implies that $|f|$ is differentiable a.e. On the other hand, f is differentiable a.e., and the modulus in X is Lipschitz continuous and right Gâteaux differentiable (cf. [13, C-II, Prop. 5.6]), with $D_r|x|(y) = \operatorname{Re}((\widehat{\operatorname{sgn}}\, \overline{x})(y))$, and therefore the chain rule shown in [13, B-II, Prop. 2.3] implies

$$\begin{aligned}\left(\frac{d}{dt}\right)_r |f(t)| &= \operatorname{Re}\left((\widehat{\operatorname{sgn}}\, \overline{f(t)})(f'(t))\right) \qquad \text{a.e.,}\\ \left(\frac{d}{dt}\right)_l |f(t)| &= \operatorname{Re}\left((\widetilde{\operatorname{sgn}}\, \overline{f(t)})(f'(t))\right) \qquad \text{a.e.}\end{aligned}$$

Since the left hand sides are equal a.e., and obviously

$$(\widetilde{\operatorname{sgn}}\, x)(y) \le (\operatorname{sgn} x)(y) \le (\widehat{\operatorname{sgn}}\, x)(y) \qquad (x, y \in X)\,,$$

we obtain the assertion.

3 The modulus of L

Let X be a Banach space. Let $h = 1$ or $h = \infty$, and let $\eta \colon [-h, 0] \to L(X)$ be of bounded variation. The latter assumption implies that η has one-sided limits at all points of $[-h, 0]$. For notational convenience, we introduce $\eta(I)$ for (open, half-open, and closed) subintervals $I \subseteq [-h, 0]$,

$$\eta((a, b)) := \eta(b-) - \eta(a+)\,,\ \eta([a, b)) := \eta(b-) - \eta(a-)\,,\ \text{etc.}$$

(Here, $(-h)-$ and $0+$ should be interpreted as $-h$ and 0, respectively).

We associate with η an operator $L_\eta \colon C([-h, 0]; X) \to X$ in the following way. By $T = T([-h, 0]; X)$ we denote the set of step functions belonging to subintervals of $[-h, 0]$. We introduce

$$\mathcal{I} := \left\{ (I_1, \ldots, I_n);\ (I_1, \ldots, I_n) \text{ partition of } [-h, 0] \text{ into subintervals, } n \in \mathbb{N} \right\}.$$

Each $\varphi \in T$ can be written as $\varphi = \sum_{j=1}^n \mathbf{1}_{I_j} c_j$, where $(I_1, \dots, I_n) \in \mathcal{I}$, $c_1, \dots, c_n \in X$. Then $\widetilde{L}_\eta \varphi$ is defined by

$$\widetilde{L}_\eta \varphi := \sum_{j=1}^n \eta(I_j) c_j \,.$$

In this way, we have defined a linear operator $\widetilde{L}_\eta \colon T \to X$. Providing T with the supremum norm we obviously have $\|\widetilde{L}_\eta\| \le |\eta|([-h,0])$, where, for a subinterval $I \subseteq [-h,0]$, we denote the variation of η on I by

$$|\eta|(I) := \sup\Big\{ \sum_{j=1}^n \|\eta(I \cap I_j)\| \,;\, (I_1, \dots, I_n) \in \mathcal{I} \Big\}.$$

It is not difficult to show that the function $\vartheta \mapsto |\eta|([-h,\vartheta))$ is left continuous, and thus

$$\lim_{\vartheta \to \vartheta_0 -} |\eta|([\vartheta, \vartheta_0)) = \lim_{\vartheta \to \vartheta_0 -} \big(|\eta|([-h,\vartheta_0)) - |\eta|([-h,\vartheta)) \big) = 0 \,, \tag{2}$$

for all $\vartheta_0 \in (-h, 0]$. Similarly,

$$\lim_{\vartheta \to \vartheta_0 +} |\eta|((\vartheta_0, \vartheta]) = 0 \tag{3}$$

for all $\vartheta_0 \in [-h, 0)$.

Further we denote by $R = R([-h,0];X) := \overline{T([-h,0];X)}^{\ell_\infty([-h,0];X)}$ the *regulated functions* and by $\widehat{L}_\eta \colon R \to X$ the unique continuous extension of $\widetilde{L}_\eta$ to R.

Then, finally, L_η is defined as the restriction of $\widehat{L}_\eta$ to $C := C([-h,0];X)$.

Next, assume in addition that X is an order complete Banach lattice, that η takes its values in the regular operators, that the 'regular variation'

$$\widetilde{\eta}(\vartheta) := \begin{cases} \sup\Big\{ \sum\limits_{j=1}^n |\eta([-h,\vartheta) \cap I_j)| \,;\, (I_1, \dots, I_n) \in \mathcal{I} \Big\} & \text{for } -h \le \vartheta < 0 \,, \\ \sup\Big\{ \sum\limits_{j=1}^n |\eta(I_j)| \,;\, (I_1, \dots, I_n) \in \mathcal{I} \Big\} & \text{for } \vartheta = 0 \end{cases} \tag{4}$$

exists, and that $\widetilde{\eta}$ is of bounded variation. Then $\widetilde{L}_{\widetilde{\eta}}$, $\widehat{L}_{\widetilde{\eta}}$, and $L_{\widetilde{\eta}}$ are defined as above.

The aim of this section is to show that L_η possesses a modulus, and that $|L_\eta| = L_{\widetilde{\eta}}$.

6 Lemma. *Let $\varphi, \psi \in R$, $|\psi| \le \varphi$. Then* $\bigl|\widehat{L}_\eta\psi\bigr| \le \widehat{L}_{\widetilde{\eta}}\varphi$.

PROOF. First, let $\varphi, \psi \in T$. There exists $(I_1, \ldots, I_n) \in \mathcal{I}$ such that $\psi = \sum_{j=1}^n \mathbf{1}_{I_j} d_j$, $\varphi = \sum_{j=1}^n \mathbf{1}_{I_j} c_j$. Then

$$|\widetilde{L}_\eta\psi| = \Bigl|\sum_{j=1}^n \eta(I_j)d_j\Bigr| \le \sum_{j=1}^n |\eta(I_j)|c_j \le \sum_{j=1}^n \widetilde{\eta}(I_j)c_j = \widetilde{L}_{\widetilde{\eta}}\varphi\,.$$

For the general case, let (φ_n), (ψ_n) in T, $\varphi_n \to \varphi$, $\psi_n \to \psi$ uniformly; without loss of generality $|\psi_n| \le \varphi_n$ $(n \in \mathbb{N})$ (otherwise replace φ_n by $\varphi_n \vee |\psi_n|$). Then

$$|\widehat{L}_\eta\psi| \leftarrow |\widetilde{L}_\eta\psi_n| \le \widetilde{L}_{\widetilde{\eta}}\varphi_n \to \widehat{L}_{\widetilde{\eta}}\varphi \qquad (n \to \infty)\,. \qquad \boxed{QED}$$

7 Lemma. *(a) Let $\varphi \in T_+$. Then* $\widetilde{L}_{\widetilde{\eta}}\varphi = \sup\bigl\{\operatorname{Re}\widetilde{L}_\eta\psi;\ \psi \in T,\ |\psi| \le \varphi\bigr\}$.

(b) $|\widehat{L}_\eta| = \widehat{L}_{\widetilde{\eta}}$.

PROOF. (a) Let I be a subinterval of $[-h, 0]$. If $c \in X_+$, $\varphi = \mathbf{1}_I c$, then

$$\begin{aligned}
\widetilde{L}_{\widetilde{\eta}}\varphi &= \widetilde{\eta}(I)c = \sup\Bigl\{\sum_{j=1}^n |\eta(I \cap I_j)|c;\ (I_1, \ldots, I_n) \in \mathcal{I}\Bigr\} \\
&= \sup\Bigl\{\operatorname{Re}\sum_{j=1}^n \eta(I \cap I_j)d_j\,;\ (I_1, \ldots, I_n) \in \mathcal{I},\ d_j \in X,\ |d_j| \le c \ \ (j = 1, \ldots, n)\Bigr\} \\
&= \sup\bigl\{\operatorname{Re}\widetilde{L}_\eta\psi\,;\ \psi \in T,\ |\psi| \le \varphi\bigr\}\,.
\end{aligned}$$

This inequality carries over to general $\varphi \in T_+$.

(b) For $\varphi \in T_+$ we obtain

$$\begin{aligned}
\widehat{L}_{\widetilde{\eta}}\varphi = \widetilde{L}_{\widetilde{\eta}}\varphi &= \sup\bigl\{\operatorname{Re}\widetilde{L}_\eta\psi\,;\ \psi \in T,\ |\psi| \le \varphi\bigr\} \\
&\le \sup\bigl\{\operatorname{Re}\widehat{L}_\eta\psi\,;\ \psi \in R,\ |\psi| \le \varphi\bigr\} \le \widehat{L}_{\widetilde{\eta}}\varphi\,,
\end{aligned}$$

using Lemma 6 in the last step. This shows $|\widehat{L}_\eta|\varphi = \widehat{L}_{\widetilde{\eta}}\varphi$. Since $T = \operatorname{lin} T_+$ is dense in R, and $|\widehat{L}_\eta|$, $\widehat{L}_{\widetilde{\eta}}$ are continuous we obtain the assertion. $\boxed{QED}$

8 Lemma. *Let $\psi \in R$, $\varepsilon > 0$. Then there exists $\psi_\varepsilon \in C$ $(= C([-h, 0]; X))$ such that* $\bigl\|\widehat{L}_{\widetilde{\eta}}|\psi - \psi_\varepsilon|\bigr\| \le \varepsilon$.

PROOF. Since T is dense in R it suffices to show the assertion for $\psi \in T$.

(i) Let $-h \le a < b \le 0$, $x \in X$, $\varepsilon > 0$. Then there exists $\zeta \in C([-h, 0]; \mathbb{R})$, $0 \le \zeta \le \mathbf{1}_{(a,b)}$ such that

$$\bigl\|\widehat{L}_{\widetilde{\eta}}|\mathbf{1}_{(a,b)}x - \zeta x|\bigr\| \le \varepsilon.$$

Indeed, by (2), (3) there exist $a < a' < b' < b$ such that

$$|\widetilde{\eta}|((a,a')) \le \frac{\varepsilon}{2\|x\|}, \quad |\widetilde{\eta}|((b',b)) \le \frac{\varepsilon}{2\|x\|}.$$

Put $\zeta := 1$ on $[a',b']$, $\zeta := 0$ on $[-h,0] \setminus (a,b)$, and connect these values continuously and such that $0 \le \zeta \le 1$. Then

$$\big\|\widehat{L}_{\widetilde{\eta}}|\mathbf{1}_{(a,b)}x - \zeta x|\big\| \le |\widetilde{\eta}|((a,a'))\|x\| + |\widetilde{\eta}|((b',b))\|x\| \le \varepsilon\,.$$

(ii) Let $a \in [-h,0]$, $\varepsilon > 0$. Then there exists $\zeta \in C([-h,0];\mathbb{R})$, $\mathbf{1}_{\{a\}} \le \zeta \le 1$ such that

$$\|\widehat{L}_{\widetilde{\eta}}|\mathbf{1}_{\{a\}}x - \zeta x|\| \le \varepsilon.$$

Indeed, by (2), (3) there exists an open interval J containing a such that $|\widetilde{\eta}|(J) - |\widetilde{\eta}|(\{a\}) \le \frac{\varepsilon}{\|x\|}$. Put $\zeta := 1$ on $\{a\}$, $\zeta := 0$ on $[-h,0] \setminus J$, and connect these values continuously and such that $0 \le \zeta \le 1$. As above we obtain the asserted inequality.

(iii) Since ψ is a linear combination of functions of the kind treated in (i) and (ii) we obtain the assertion. $\boxed{QED}$

9 Proposition. $\big|L_\eta\big| = L_{\widetilde{\eta}}$.

Proof. Let $\varphi \in C$, $\varphi \ge 0$. We have to show

$$\widehat{L}_{\widetilde{\eta}}\varphi\ \big(= \sup\{\, |\widehat{L}_\eta\psi|\,;\ \psi \in R,\ |\psi| \le \varphi \,\}\big) = \sup\{\, |\widehat{L}_\eta\psi|\,;\ \psi \in C,\ |\psi| \le \varphi \,\}.$$

(For the following, compare the argument given in Remark 2.) It clearly suffices to show that the set $\{\, \widehat{L}_\eta\chi;\ \chi \in C,\ |\chi| \le \varphi \,\}$ is dense in $\{\, \widehat{L}_\eta\psi;\ \psi \in R,\ |\psi| \le \varphi \,\}$.

Let $\psi \in R$, $|\psi| \le \varphi$, $\varepsilon > 0$. By Lemma 8 there exists $\widetilde{\chi} \in C$ such that $\big\|\widehat{L}_{\widetilde{\eta}}|\psi - \widetilde{\chi}|\big\| \le \varepsilon$.

We define $\chi := \tau_\varphi\widetilde{\chi}$ ($\in C$); cf. Section 2. Then $|\chi| \le \varphi$, and (1) implies

$$|\psi(\vartheta) - \chi(\vartheta)| \le |\psi(\vartheta) - \widetilde{\chi}(\vartheta)|\,.$$

Therefore

$$\big\|\widehat{L}_\eta\psi - \widehat{L}_\eta\chi\big\| \le \big\|\widehat{L}_{\widetilde{\eta}}|\psi - \chi|\big\| \le \big\|\widehat{L}_{\widetilde{\eta}}|\psi - \widetilde{\chi}|\big\| \le \varepsilon\,.$$ $\boxed{QED}$

4 The modulus semigroup for a delay semigroup

We assume that X is a Banach lattice with order continuous norm, that $h = 1$ or $h = \infty$, and we assume that A, L, η are as in [6], where only the case $X = \mathbb{R}^n$ is considered.

More precisely, let $A \in L(X)$ be a regular operator. Recall that $A \in L(X)$ is regular if and only if the C_0-semigroup $(e^{tA})_{t\geq 0}$ possesses a modulus semigroup (cf. [13, C-II, Thm 4.17], [3, Cor.0.2]). The generator of the modulus semigroup is then given by $A_{\#} = \operatorname{Re} M + |B|$ where $A = M + B$ is the decomposition of A with $M \in \mathcal{Z}(X)$, $B \in \mathcal{Z}(X)^d$, with the center $\mathcal{Z}(X)$ of operators on X.

As in Section 3 we assume that the function $\eta\colon [-h, 0] \to L(X)$ is of bounded variation, takes its values in the regular operators, that $\widetilde{\eta}(\vartheta)$, given by equation (4), exists for all $-h \leq \vartheta \leq 0$, and that $\widetilde{\eta}$ is of bounded variation. (Note that this hypothesis is satisfied automatically in the context of [6].) Defining $L := L_\eta$ (in terms of the notation of Section 3), we know from Proposition 9 that $|L| = L_{\widetilde{\eta}}$.

Further, we assume that $\eta(\vartheta) \to \eta(0)$ $(\vartheta \to 0)$ in operator norm. This implies that the function η does not give rise to mass at zero and, by (2), that $|\eta|([\vartheta, 0]) \to 0$ as $\vartheta \to 0$. The following lemma shows that the corresponding property also holds for $\widetilde{\eta}$.

10 Lemma. *Under the previous assumptions on η one has $\widetilde{\eta}(0) = \lim_{\vartheta\to 0} \widetilde{\eta}(\vartheta)$.*

Proof. Without loss of generality we may assume that η is left continuous on $(-h, 0]$.

The assumption that $\widetilde{\eta}$ is of bounded variation implies that η has one-sided limits with respect to the regular norm at all points of $[-h, 0]$. Evidently, these limits coincide with the operator norm limits. In particular, one obtains

$$\lim_{\vartheta\to 0} \big\| |\eta(0) - \eta(\vartheta)| \big\| = 0 \,. \tag{5}$$

Assume $\widetilde{\eta}(0) \neq \widetilde{\eta}(0-)$. Since $\widetilde{\eta}$ is increasing we get $0 < \alpha := \|\widetilde{\eta}(0) - \widetilde{\eta}(0-)\| \leq \|\widetilde{\eta}(0) - \widetilde{\eta}(\vartheta)\|$, for all $\vartheta \in [-h, 0)$. In view of (5) there exists $\delta > 0$ such that $\big\| |\eta(0) - \eta(\vartheta)| \big\| \leq \alpha/4$ for $-\delta \leq \vartheta < 0$. Choose $\widetilde{\vartheta}_1 := -\delta$. Since $\big\|\widetilde{\eta}(0) - \widetilde{\eta}(\widetilde{\vartheta}_1)\big\| \geq \alpha$ there exist $\widetilde{\vartheta}_1 = \vartheta_0 < \cdots < \vartheta_n = 0$ such that $\Big\| \sum_{j=1}^{n} \big|\eta(\vartheta_j) - \eta(\vartheta_{j-1})\big| \Big\| \geq 3\alpha/4$. (Note that the set $\big\{ \sum_{j=1}^{n} \big|\eta(\vartheta_j) - \eta(\vartheta_{j-1})\big| \,;\, \widetilde{\vartheta}_1 = \vartheta_0 < \cdots < \vartheta_n = 0,\ n \in \mathbb{N} \big\}$ is directed, and use the order continuity of the norm.) Therefore

$$\Big\|\widetilde{\eta}(\tfrac{\vartheta_{n-1}}{2}-) - \widetilde{\eta}(\widetilde{\vartheta}_1-)\Big\| \geq \Big\|\widetilde{\eta}(\vartheta_{n-1}) - \widetilde{\eta}(\widetilde{\vartheta}_1)\Big\| \geq \Big\| \sum_{j=1}^{n-1} \big|\eta(\vartheta_j) - \eta(\vartheta_{j-1})\big| \Big\| \geq \alpha/2 \,.$$

Choose $\widetilde{\vartheta}_2 := \vartheta_{n-1}/2$. Using the same procedure one obtains $\widetilde{\vartheta}_3 \in (\widetilde{\vartheta}_2, 0)$ such that $\big\|\widetilde{\eta}(\widetilde{\vartheta}_3-) - \widetilde{\eta}(\widetilde{\vartheta}_2-)\big\| \geq \alpha/2$. Continuing in this way we obtain a contradiction to $\widetilde{\eta}$ being of bounded variation. $\boxed{QED}$

Let $1 \le p < \infty$. From [6] we further recall the notations $\mathcal{A}_0$, $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ where

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}$$

is an operator in $X \times L_p(-h, 0; X)$ with domain

$$D(\mathcal{A}_0) = D(\mathcal{A}) = \{ (x, f) \in X \times W_p^1(-h, 0; X);\ f(0) = x \},$$

and

$$\mathcal{B} = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}.$$

Also

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A_\# & |L| \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}$$

with $D(\widetilde{\mathcal{A}}) = D(\mathcal{A})$. The proof that $(e^{t\widetilde{\mathcal{A}}})_{t\ge0}$ is a positive C_0-semigroup dominating $(e^{t\mathcal{A}})_{t\ge0}$ works as for [6, Lemma 2.1]. We recall from [5, Theorem 2.1] that the order continuity of the norm of the space $L_p(-h, 0; X)$ implies that the C_0-semigroup $(e^{t\mathcal{A}})_{t\ge0}$ possesses a modulus semigroup; we denote its generator by $\mathcal{A}_\#$.

The main result of our paper is contained in the following theorem.

11 Theorem. *The C_0-semigroup $\left(e^{t\widetilde{\mathcal{A}}}\right)_{t\ge0}$ is the modulus semigroup of the C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\ge0}$, i.e., $\mathcal{A}_\# = \widetilde{\mathcal{A}}$.*

12 Lemma. *(a) There exists $C \ge 0$ such that*

$$\left\| \left(e^{t\mathcal{A}} - e^{t\mathcal{A}_0}\right) J_1 \right\| \le Ct|\eta|([-t, 0]) \qquad \text{for all} \quad 0 < t < 1$$

(where J_1 denotes the canonical injection of X into $X \times L_p(-h, 0; X)$).

(b) $\frac{1}{t}(P_1 e^{t\mathcal{A}} J_1 - I_X) \to A$, $\frac{1}{t}(P_1 e^{t\widetilde{\mathcal{A}}} J_1 - I_X) \to A_\#$ $(t \to 0)$ (where P_1 is the canonical projection from $X \times L_p(-h, 0; X)$ onto the first component X, and I_X is the identity on X).

(c) $\frac{1}{t} P_1 e^{t\mathcal{A}} \left(\begin{smallmatrix} 0 \\ \varphi \end{smallmatrix}\right) \to L\varphi$, $\frac{1}{t} P_1 e^{t\widetilde{\mathcal{A}}} \left(\begin{smallmatrix} 0 \\ \varphi \end{smallmatrix}\right) \to |L|\varphi$ $(t \to 0)$ for all $\varphi \in W_p^1(-h, 0; X)$.

The proof is the same as for [6, Lemma 2.2].

The following lemma strengthens [6, Lemma 2.3] in as far as the limits exist as $t \to 0$ and not only for suitable sequences tending to 0.

13 Lemma. *(a) $D(\mathcal{A}) \subseteq D(\mathcal{A}_\#)$.*

(b) $\frac{1}{t}\left(P_1 e^{t\mathcal{A}_\#} \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right) - z\right) \to A_\# z$ weakly as $t \to 0$, for all $z \in X$, and $\frac{1}{t}\left(P_1 e^{t\mathcal{A}_\#} \left(\begin{smallmatrix} 0 \\ \varphi \end{smallmatrix}\right)\right) \to |L|\varphi$ weakly as $t \to 0$, for all $\varphi \in W_p^1(-h, 0; X)$.

For the proof we need the following lemma as an auxiliary statement.

14 Lemma. *As before, let X be a Banach lattice with order continuous norm. Assume that (A_k), (B_k), (C_k) are sequences of regular operators, $|A_k| \le B_k \le C_k$ ($k \in \mathbb{N}$), $C_k \to C \in L(X)$ in the weak operator topology, $A_k \to A \in L(X)$ in the strong operator topology ($k \to \infty$), and $|A| = C$. Then $B_k \to C$ ($k \to \infty$) in the weak operator topology.*

Proof. Let $x \in X_+$. The weak convergence $C_k x \to Cx$ (together with $\{ C_k x;\, k \in \mathbb{N} \} \subseteq X_+$) implies that the solid hull of the set $\{ C_k x;\, k \in \mathbb{N} \}$ is relatively weakly compact (cf. [1, chap. 4, Thm 13.8]), and therefore the set $\{ B_k x;\, k \in \mathbb{N} \}$ is relatively weakly compact. There exists a weakly convergent subsequence $(B_{k_j} x)_{j \in \mathbb{N}}$, $z :=$ w-lim $B_{k_j} x \le Cx$. For $y \in X$, $|y| \le x$ we obtain

$$|Ay| = \lim_j |A_{k_j} y| \le \text{w-lim}_j\, B_{k_j} x = z\,,$$

and therefore $|A|x \le z$. Thus $|A|x \le z \le Cx = |A|x$, $z = Cx$. Since this reasoning can be applied to any subsequence of $(B_k)_k$ we obtain $Cx = \text{w-lim}_k\, B_k x$. $\boxed{QED}$

15 Remark. We were not able to decide whether the hypothesis that X has order continuous norm is really needed for the validity of Lemma 14. Also, assuming that $A_k \to A$, $C_k \to C$ in operator norm did not help us to show the convergence of (B_k) in a stronger sense than asserted in Lemma 14.

Proof of Lemma 13. From the definition of the modulus semigroup we have the inequalities

$$|e^{t\mathcal{A}}| \le e^{t\mathcal{A}_\#} \le e^{t\widetilde{\mathcal{A}}} \tag{6}$$

for all $t \ge 0$. The proof of part (a) is obtained in the same way as in [6, proof of Lemma 2.3 (a)].

For the proof of (b) we make use of the band projection $\mathcal{P}$ in $L_r(X)$ onto the center $\mathcal{Z}(X)$, where $L_r(X)$ are the regular operators in X. We recall from [17] (see also [12, Theorem 3.1.22]) that $\mathcal{P}$ is contractive with respect to the operator norm. Inequality (6) implies

$$\frac{1}{t} \operatorname{Re}\big(\mathcal{P}(P_1 e^{t\mathcal{A}} J_1 - I_X)\big) \le \frac{1}{t}\mathcal{P}(P_1 e^{t\mathcal{A}_\#} J_1 - I_X) \le \frac{1}{t}\mathcal{P}(P_1 e^{t\widetilde{\mathcal{A}}} J_1 - I_X) \tag{7}$$

for all $t \ge 0$. According to Lemma 12(b) and the continuity of $\mathcal{P}$, the left and right hand sides of (7) converge to $\operatorname{Re}\mathcal{P}A = \mathcal{P}A_\#$, and therefore

$$\text{w-lim}_{t\to 0} \frac{1}{t}\mathcal{P}(P_1 e^{t\mathcal{A}_\#} J_1 - I_X) = \mathcal{P}A_\#, \tag{8}$$

by Lemma 14. On the other hand, (6) implies

$$\begin{aligned}&\Big|\frac{1}{t}(\mathcal{I}-\mathcal{P})(P_1 e^{t\mathcal{A}}J_1)\Big| = \frac{1}{t}(\mathcal{I}-\mathcal{P})\big|P_1 e^{t\mathcal{A}}J_1\big| \le \frac{1}{t}(\mathcal{I}-\mathcal{P})(P_1 e^{t\mathcal{A}_\#}J_1)\\ &\le \frac{1}{t}(\mathcal{I}-\mathcal{P})(P_1 e^{t\widetilde{\mathcal{A}}}J_1)\end{aligned}$$

for all $t > 0$. (We denote by $\mathcal{I}$ the identity on $L_r(X)$; observe that $\mathcal{I}-\mathcal{P}$ is a band projection.) Letting $t \to 0$, observing $(\mathcal{I}-\mathcal{P})(I_X) = 0$, and applying Lemma 14 we obtain

$$\text{w-lim}_{t\to 0}\frac{1}{t}(\mathcal{I}-\mathcal{P})(P_1 e^{t\mathcal{A}_\#}J_1) = (\mathcal{I}-\mathcal{P})A_\# = |(\mathcal{I}-\mathcal{P})A|\,. \tag{9}$$

Taking (8) and (9) together we obtain

$$\text{w-lim}_{t\to 0}\frac{1}{t}(P_1 e^{t\mathcal{A}_\#}J_1 - I_X) = A_\#\,.$$

For the proof of the last statement of the lemma we note that, as in [6, last part of the proof of Lemma 2.3], we obtain the existence of

$$L_\#\varphi := \text{w-lim}_{t\to 0}\frac{1}{t}\big(P_1 e^{t\mathcal{A}_\#}\big(\begin{smallmatrix}0\\ \varphi\end{smallmatrix}\big)\big)$$

for all $\varphi \in W_p^1(-h,0;X)$, and (6) implies

$$|L\varphi| \le L_\#|\varphi| \le |L||\varphi|$$

(recall $|\varphi| \in W_p^1(-h,0;X)$, by Theorem 1). Let $0 \le \varphi \in W_p^1(-h,0;X)$. Using Remark 2 in conjunction with the second part of Theorem 1 we obtain

$$|L|\varphi = \sup\{\,|L\psi|\,;\ \psi \in W_p^1(-h,0;X), |\psi| \le \varphi\,\} \le L_\#\varphi\,,$$

thus $|L|\varphi = L_\#\varphi$. Since $W_p^1(-h,0;X)$ is the linear span of $W_p^1(-h,0;X)_+$ we obtain $|L|\varphi = L_\#\varphi$ for all $\varphi \in W_p^1(-h,0;X)$. $\boxed{QED}$

Proof of Theorem 11. Having established Lemma 13 which is analogous to [6, Lemma 2.3] we conclude $\mathcal{A}_\# \supseteq \widetilde{\mathcal{A}}$ in the same way as in [6, proof of Theorem 2.4], and then obtain $\mathcal{A}_\# = \widetilde{\mathcal{A}}$ since $\mathcal{A}_\#$ and $\widetilde{\mathcal{A}}$ are generators. $\boxed{QED}$

Appendix

In this appendix we are going to present a supplementary result concerning modulus semigroups. This result deals with the existence of the modulus if no dominating semigroup is required to exist a priori. This is a generalization of [5, Prop. 2.5] where the case $X = L_p$, for $1 \le p < \infty$, is treated.

16 Proposition. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a KB-space X (i.e., a Banach lattice having the property that every norm bounded monotone sequence is convergent). Assume that $(T(t))_{t\geq 0}$ is quasi-contractive with respect to the regular norm (i.e., there exists ω such that $\|T(t)\|_r = \big\||T(t)|\big\| \leq e^{\omega t}$ for all $t \geq 0$). Then $(T(t))_{t\geq 0}$ possesses a modulus semigroup.*

Surprisingly enough, the difficult part of the proof of this result is proving the measurability of the object obtained as the candidate for the modulus semigroup. The following considerations are preparations for the proof of this part.

In the remainder of the appendix—except for the proof of Proposition 16, where X is assumed to be a KB-space—we assume that X is a Banach lattice with order continuous norm. A function $f\colon J \to X_+$, where $J \subseteq \mathbb{R}$ is an interval, will be called *lower semicontinuous* if $\lim_{s\to t} f(s) \wedge f(t) = f(t)$ for all $t \in J$.

Let $\mathcal{G}$ be a set of lower semicontinuous functions $g\colon J \to X_+$, with $\big\{ g(t)\,;\, g \in \mathcal{G} \big\}$ order bounded for all $t \in J$. Then the pointwise supremum

$$t \mapsto f(t) := \sup\{ g(t)\,;\, g \in \mathcal{G} \}$$

is again lower semicontinuous. Indeed, if $g, h \in \mathcal{G}$ then $g \vee h$ is lower semicontinuous; therefore we may assume that $\mathcal{G}$ is directed upward. The order continuity of the norm then implies that $f(t) = \lim_{g\in\mathcal{G}} g(t)$. If $t \in J$ then $g(s)\wedge g(t) \leq f(s)\wedge f(t) \leq f(t)$ for all $g \in \mathcal{G}$, $s \in J$, and $\lim_{s\to t} g(s)\wedge g(t) = g(t)$. From $f(t) = \lim_{g\in\mathcal{G}} g(t)$ we then obtain $\lim_{s\to t} f(s) \wedge f(t) = f(t)$.

17 Proposition. *Let $f\colon [0,1] \to X_+$ be lower semicontinuous. Then f is Bochner measurable.*

Proof. We construct a sequence (f_n) of simple functions converging to f pointwise.

Let $n \in \mathbb{N}$. For $t \in [0,1]$ there exists $\delta_t \in \big(0, \frac{1}{n}\big]$ such that $\|f(t) - f(s)\wedge f(t)\| \leq \frac{1}{n}$ for $t - \delta_t < s < t + \delta_t$. By compactness, we obtain $t_1, \dots, t_m \in [0,1]$ such that $[0,1] \subseteq \bigcup_{k=1}^m (t_k - \delta_{t_k}, t_k + \delta_{t_k})$. We set

$$A_k := \big((t_k - \delta_{t_k}, t_k + \delta_{t_k})[0,1]\big) \setminus \bigcup_{k'=1}^{k-1} A_{k'}$$

for $k = 1, \dots, m$ and define $f_n(t) := f(t_k)$ for $t \in A_k$, $k = 1, \dots, m$.

We claim that $f_n(t) \to f(t)$ $(n \to \infty)$ pointwise. It is clear from the construction that $\|f_n(t) - f(t) \wedge f_n(t)\| \leq \frac{1}{n}$ for all $t \in [0,1]$.

Let $t \in [0,1]$, $\varepsilon > 0$. By the lower semicontinuity of f there exists $n_0 \in \mathbb{N}$ such that $\|f(t) - f(s)\wedge f(t)\| \leq \varepsilon$ for $t - \frac{1}{n_0} < s < t + \frac{1}{n_0}$. Let $n \geq n_0$. The construction yields $f_n(t) = f(s)$ for some $s \in \big(t - \frac{1}{n}, t + \frac{1}{n}\big)$, and thus $\|f(t) - f_n(t)\wedge f(t)\| \leq \varepsilon$.

The two inequalities together yield

$$\|f_n(t) - f(t)\| \le \|f_n(t) - f(t) \wedge f_n(t)\| + \|f_n(t) \wedge f(t) - f(t)\| \le \frac{1}{n} + \varepsilon\,. \qquad \boxed{QED}$$

18 Proposition. *Let $f\colon [0,\infty) \to X_+$ be lower semicontinuous, and let $S\colon [0,\infty) \to L(X)_+$ be such that $t \mapsto S(t)x$ is lower semicontinuous for all $x \in X_+$. Then $t \mapsto S(t)f(t)$ is lower semicontinuous.*

Clearly $\lim_{s\to t} f(s) \wedge f(t) = f(t)$ if and only if $\lim_{n\to\infty} f(t_n) \wedge f(t) = f(t)$ for all sequences (t_n) tending to t, or, slightly more subtle, if each sequence (t_n) tending to t contains a subsequence (t_{n_j}) such that $\lim_{j\to\infty} f(t_{n_j}) \wedge f(t) = f(t)$. This remark makes it possible to reduce the statement of Proposition 18 to the treatment of sequences.

19 Lemma. *Let (x_n) be a sequence in X_+, $x \in X_+$. The following are equivalent:*

(i) $\lim_{n\to\infty} x_n \wedge x = x$;

(ii) each subsequence (x'_n) of (x_n) contains a subsequence (x''_n) satisfying

$$\lim_{n\to\infty} \inf\{\, x''_j \wedge x\,;\; j \ge n \,\} = x\,.$$

Proof. (i) $\Longrightarrow$ (ii). For a subsequence (x'_n) there exists a subsequence (x''_n) satisfying $\sum_{n=1}^{\infty} \|x - x''_n \wedge x\| < \infty$. For all $n \in \mathbb{N}$ we obtain

$$x \ge \inf_{j\ge n} x''_j \wedge x \ge x - \sum_{j=n}^{\infty} (x - x''_j \wedge x)\,.$$

Since the right hand side tends to x we obtain the assertion.

(ii) $\Longrightarrow$ (i). If a subsequence (x''_n) satisfies $\lim\limits_{n\to\infty} \inf\{\, x''_j \wedge x\,;\; j \ge n \,\} = x$ then evidently $\lim_{n\to\infty} x''_n \wedge x = x$. Thus, the standard subsequence argument shows the assertion. $\boxed{QED}$

20 Lemma. *Let (x_n) be a sequence in X_+, $x \in X_+$, $\lim_{n\to\infty} x_n \wedge x = x$. Let (S_n) be a sequence in $L(X)_+$, $S \in L(X)_+$, and $\lim_{n\to\infty} S_n y \wedge Sy = Sy$ for all $y \in X_+$. Then $\lim_{n\to\infty} S_n x_n \wedge Sx = Sx$.*

Proof. Let (n_j) be a subsequence of the natural numbers, and let $\varepsilon > 0$. By Lemma 19 there exists $0 \le x_\varepsilon \le x$ such that $\|x - x_\varepsilon\| \le \varepsilon$ and $x_{n'_j} \ge x_\varepsilon$ $(j \in \mathbb{N})$, for a suitable subsequence (n'_j) of (n_j). Using $\lim_{n\to\infty} S_n x_\varepsilon \wedge Sx_\varepsilon = Sx_\varepsilon$ and applying Lemma 19 once more we obtain $0 \le y_\varepsilon \le Sx_\varepsilon$ such that $\|Sx_\varepsilon - y_\varepsilon\| \le \varepsilon$ and $S_{n''_j} x_\varepsilon \ge y_\varepsilon$ $(j \in \mathbb{N})$, for a suitable subsequence (n''_j) of (n'_j). Then $S_{n''_j} x_{n''_j} \ge S_{n''_j} x_\varepsilon \ge y_\varepsilon$ for all $j \in \mathbb{N}$, and $\|Sx - y_\varepsilon\| \le \|Sx - Sx_\varepsilon\| + \|Sx_\varepsilon - y_\varepsilon\| \le \|S\|\varepsilon + \varepsilon$.

Applying this argument repeatedly we obtain a subsequence (n_j''') of (n_j) such that

$$\lim_{n\to\infty} \inf\{\, S_{n_j'''} x_{n_j'''} \wedge Sx \,;\, j \geq n \,\} = Sx\,.$$

Now an appeal to Lemma 19 yields the assertion. $\boxed{QED}$

In view of the remark after Proposition 18, the assertion of Proposition 18 is an immediate consequence of Lemma 20.

Proof of Proposition 16. As in [5, proof of Prop. 2.5] one may assume that $|T(t)|$ is a contraction for all $t \geq 0$.

The modulus semigroup $(T_{\#}(t))_{t\geq 0}$ is obtained as follows. We denote by Γ the set of all subdivisions of 1,

$$\Gamma = \{\, \gamma \in (0,1]^n \,;\, \gamma_1 + \cdots + \gamma_n = 1,\ n \in \mathbb{N} \,\}\,.$$

For $t \geq 0$, $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$ we define

$$T_\gamma(t) := |T(\gamma_n t)| \cdots |T(\gamma_1 t)|$$

and, for $x \in X_+$, obtain

$$T_{\#}(t)x := \sup_{\gamma\in\Gamma} T_\gamma(t)x = \lim_{\gamma\in\Gamma} T_\gamma(t)x\,.$$

(It is at this point where we use that X is a KB-space. Note that the net $(T_\gamma(t)x)_{\gamma\in\Gamma}$ is directed upward.) These statements are proved in [5, proof of Theorem 2.1].

For $x \in X_+$, the function

$$t \mapsto |T(t)|x = \sup\{\, |T(t)y| \,;\, y \in X, |y| \leq x \,\}$$

is lower semicontinuous as the supremum of continuous functions. Repeated application of Proposition 18 shows that $t \mapsto T_\gamma(t)x$ is lower semicontinuous for all $\gamma \in \Gamma$. We conclude that the function $t \mapsto T_{\#}(t)x$ is lower semicontinuous as the supremum of lower semicontinuous functions, and therefore is Bochner measurable by Proposition 17. This shows that $(T_{\#}(t))_{t\geq 0}$ is strongly measurable, and, as a consequence, is strongly continuous on $(0,\infty)$.

In order to show the strong continuity of $(T_{\#}(t))_{t\geq 0}$ on $[0,\infty)$ we use [18, Theorem 6]. According to this result it now is sufficient to show $T_{\#}(t)x \wedge x \to x$ as $t \to 0$, for all $x \in X_+$. This, however, is an immediate consequence of the inequalities $|T(t)x| \wedge x \leq T_{\#}(t)x \wedge x \leq x$ and the convergence $|T(t)x| \to x$ as $t \to 0$. $\boxed{QED}$

21 Remark. In Proposition 16, instead of assuming $(T(t))_{t\geq 0}$ to be quasi-contractive with respect to the regular norm, it is clearly sufficient to assume

$$\sup\{\, \|T_\gamma(t)\| \,;\, 0 \leq t \leq 1,\ \gamma \in \Gamma \,\} < \infty\,.$$

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