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Convolution groups for quasihyperbolic systems of differential operators

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Abstract. In contrast to the usual treatment (see e.g. J. J. Duistermaat [3]) convolution groups are constructed for differential operators defined by non-homogeneous polynomials (Proposition 5) and for quasi-hyperbolic systems, i.e. systems "correct in the sense of Petrovsky" (Proposition 9). An explicit formula for the convolution group of the Lamé system in elastodynamics is presented in Proposition 11.

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Dedicated to the memory of Prof. Klaus Floret

1 Introduction: convolution groups for homogeneous, elliptic, hyperbolic and ultrahyperbolic operators

The explicit formulae for the electrostatic potential U caused by a charge density ϱ and for the displacement u of an elastic plate loaded by a pressure distribution p look quite differently:

$$\left.\begin{aligned} U &= -\frac{1}{4\pi|x|} * \varrho = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\varrho(x-\xi)}{|\xi|}\,\mathrm{d}\xi, \\ u &= \frac{|x|^2}{8\pi}\log|x| * p = \frac{1}{8\pi}\int\limits_{\mathbb{R}^2} p(x-\xi)|\xi|^2\log|\xi|\,\mathrm{d}\xi. \end{aligned}\right\} \qquad (1)$$

U as well as u are **convolutions** with fundamental solutions of the three-dimensional Laplacean Δ_3 and of the two-dimensional biharmonic operator Δ_2^2

respectively, i.e.,

$$\left.\begin{aligned} \underbrace{-\Delta_3\delta}_{R_{-2}} * \underbrace{\frac{1}{4\pi|x|}}_{R_2} &= \underbrace{\delta}_{R_0}\ , \\ \underbrace{\Delta_2^2\delta}_{R_{-4}} * \underbrace{\frac{|x|^2}{8\pi}\log|x|}_{R_4} &= \underbrace{\delta}_{R_0}\ . \end{aligned}\right\} \tag{2}$$

M. Riesz succeeded in representing the five occurring distributions as special values of a single distribution-valued function

$$\mathbb{C} \longrightarrow \mathcal{S}',\ \lambda \longmapsto R_\lambda$$

(cf. [25, p. 586]; [12, p. 146, 154]; [13, p. 47, 49]). For $0 < \operatorname{Re}\lambda < n$, the **elliptic kernel of M. Riesz** R_λ is defined by a locally integrable function, i.e.,

$$R_\lambda = \frac{\Gamma\left(\frac{n-\lambda}{2}\right)}{2^\lambda \pi^{n/2}\Gamma\left(\frac{\lambda}{2}\right)}|x|^{\lambda-n}.$$

For other complex values $\lambda,\ R_\lambda$ is defined by analytic continuation and, finally, at the poles $\lambda = n+2k,\ k \in \mathbb{N}_0$, as the finite part of the Laurent series of $\lambda \longmapsto R_\lambda$. It results

$$R_{-2k} = (-\Delta_n)^k\delta \text{ if } k \in \mathbb{N}_0.$$

Formulae (2) are special cases of the convolution relation $R_\lambda * R_\nu = R_{\lambda+\nu}$, valid if and only if $\operatorname{Re}(\lambda+\nu) < n$ or $-\lambda/2 \in \mathbb{N}_0$, or $-\nu/2 \in \mathbb{N}_0$ (cf. [19, p. 40, Satz 9]; [20, p. 12-05]).

Note that **convolvability** neither is implied by **support properties** since $\operatorname{supp} R_\lambda = \mathbb{R}^n$ if $\lambda \notin -2\mathbb{N}_0$ nor by **decay properties** since – in general – $R_\lambda \notin \mathcal{D}'_{L^p}$. Hence the **general concept of convolution** has to be used – as defined by L. Schwartz in his “Théorie des distributions à valeurs vectorielles” (cf. [28, p. 131,132] and [14, p. 185]).

For the iterated **wave operators** $(\partial_t^2 - \Delta_{n-1})^k$, M. Riesz defined the **hyperbolic Riesz kernels** Z_λ by

$$Z_\lambda = \frac{1}{\pi^{\frac{n}{2}-1}2^{\lambda-1}\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda-n}{2}+1\right)}s^{\lambda-n},\ s(t,x) = (t^2 - x_1^2 - \cdots - x_{n-1}^2)^{1/2}$$

if (t,x) belongs to the forward light cone

$$K = \left\{(t,x) \in \mathbb{R}^n;\ t^2 - x_1^2 - \cdots - x_{n-1}^2 \geq 0,\ t \geq 0\right\}$$

and $s = 0$ if $(t, x) \notin K$. Due to $\operatorname{supp} Z_\lambda \subset K$ the convolution relation

$$Z_\lambda * Z_\nu = Z_{\lambda+\nu} \text{ if } \lambda, \nu \in \mathbb{C}\,,$$

holds in the classical spaces $\mathcal{D}'_{+K}$ of L. Schwartz [27, p. 177, (VI,5;19)], [21].

The generalization to the **ultrahyperbolic operators** $(\partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots - \partial_n^2)^k$ follows the same pattern although the technical difficulties increase if $n - 1 > p > 1, n > 3$. For a modern treatment see [17] (cf. also [32, § 28.1, p. 555–562] or [18]).

The elliptic, hyperbolic and ultrahyperbolic kernels of M. Riesz have in common the property that both the iterated differential operators applied to δ and their fundamental solutions appear as special values of the generalized distance function

$$\lambda \longmapsto c(\lambda, n)(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 \pm \mathrm{i}\,0)^\lambda.$$

Essentially, this follows from the fact that Fourier transforms of powers of Euclidean or Lorentzian distances are powers of such distances. Since the Fourier transforms of powers of higher order homogeneous polynomials are no more powers of polynomials, the **construction of convolution groups for homogeneous higher order differential operators** is connected with the Fourier transform of their symbols – a "technique which M. Riesz disliked as being too indirect" [3, p. 100].

A generalization to **real-valued, homogeneous, elliptic polynomials** was given in [36].

1 Proposition. *Let P be a homogeneous polynomial of degree m in n variables with $P(\xi) > 0$ for $\xi \neq 0$. P ought not to be expressible as a power of another polynomial. Denoting by $T_\lambda := \mathcal{F}^{-1}(P^\lambda)$ the "convolution group" of P the following assertions are equivalent:*

(i) T_λ and T_ν are convolvable.

(ii) $\lambda \in \mathbb{N}_0$ or $\nu \in \mathbb{N}_0$ or $\operatorname{Re}(\lambda + \nu) > -\dfrac{n}{m}$.

In this case, we have $T_\lambda * T_\nu = T_{\lambda+\nu}$.

A generalization of the construction of hyperbolic Riesz kernels to homogeneous hyperbolic operators was given in [7], [1].

2 Proposition. *([7, Thm. 3.1, p. 33 and Thm. 3.4, p. 34]; [1, p. 146]) Let P be a homogeneous polynomial in n variables $(\tau, \xi) \in \mathbb{R}^n$, hyperbolic in the τ-direction, i.e., $P(1, 0) \neq 0$ and the polynomials in one variable, $\tau \longmapsto P(\tau, \xi)$ have only real zeros for $\xi \in \mathbb{R}^{n-1}$. Let Γ be the* ***hyperbolicity cone****, i.e., the connected component of $\{\, (\tau, \xi) \in \mathbb{R}^n;\ P(\tau, \xi) \neq 0 \,\}$ containing $(1, 0)$. The dual K of Γ, i.e. $K := \Gamma^*$ is the* ***propagation cone****. Then P does not vanish* *on the tube domain* $\mathbb{R}^n + \mathrm{i}\Gamma$*, which is simply connected, and hence* $\log P$ *can be defined continuously thereon, uniquely up to a constant* $2k\pi\mathrm{i}$, $k \in \mathbb{Z}$*. Set* $\log P(\tau, \xi) := \lim\limits_{\epsilon \searrow 0} \log P(\tau + \mathrm{i}\,\epsilon, \xi)$ *and* $P(\tau,\xi)^\lambda := \mathrm{e}^{\lambda \log P(\tau,\xi)}$, $T_\lambda := \mathcal{F}^{-1}(P^\lambda)$ *for* $\operatorname{Re}\lambda > 0$*. Then*

(i) P^λ *and* T_λ *can be continued to entire distribution-valued functions* $\mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ *with* $T_\lambda \in \mathcal{D}'_{+K}$*.*

(ii) $T_\lambda * T_\nu = T_{\lambda+\nu}$ *for* $\lambda, \nu \in \mathbb{C}$*.*

3 Remark. Note that the convolution groups in Proposition 1 and 2 are defined by $T_\lambda = \mathcal{F}^{-1}(P^\lambda)$ whereas $R_\lambda = \mathcal{F}^{-1}\big(|\xi|^{-\lambda}\big)$ and $Z_\lambda = \mathcal{F}^{-1}\big(\big(-(\tau - \mathrm{i}\,0)^2 + |\xi|^2\big)^{-\lambda/2}\big)$. **Riesz integrals for symmetric cones** associated with a simple Euclidean Jordan algebra are defined in [6, Thm. VII.2.2, p. 132, and the Notes, p. 143]. **Riemann-Liouville operators** for **homogeneous cones** are defined in [8, Thm.1, p. 99, Prop. 3 and Corollary, p. 105, Prop. 2, p. 118 and Thm. 2, p. 120].

The following considerations try to transfer the idea of convolution groups to non-homogeneous differential operators (Section 3), and more generally, to quasihyperbolic systems of differential operators (Section 4). As an example, we shall construct the convolution group for the system of elastic waves in isotropic media $(\partial_t^2 - M\Delta_3)I_3 - (\Lambda + M)\nabla\nabla^T$ (Section 5).

The **notations** are those of [27] with the exception that the Fourier transform is defined by

$$\mathcal{F}\varphi(\xi) = \mathcal{F}_{x\to\xi}\,\varphi = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle}\varphi(x)\ \mathrm{d}x \quad \text{for} \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Moreover,

$$\partial_t = \frac{\partial}{\partial t},\ \nabla = \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix},\ \partial = (\partial_t, \nabla^T),\ \Delta_n = \nabla^T\nabla,\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix};$$

Y is the Heaviside function.

A **preliminary version** of this contribution was presented at the “Third Workshop on Functional Analysis” at Trier (September 2001).

2 The construction of the convolution group for the heat operator by Laurent Schwartz

In his "Séminaire: Equations aux dérivées partielles" ([29, exposé 10]; [30, p. 44]), L. Schwartz defines

$$E^\lambda = \frac{t^{\lambda-1}}{\Gamma(\lambda)} E^1 \,,$$

if $\operatorname{Re}\lambda > 1$ and if E^1 denotes the tempered fundamental solution of $\partial_t - \Delta_{n-1}$ with support in $[0,\infty)_t \times \mathbb{R}^{n-1}_x$, i.e.,

$$E^1 = \frac{Y(t)}{(4\pi t)^{(n-1)/2}}\, \mathrm{e}^{-|x|^2/(4t)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

(cf. also: [32, p. 563, (28.36), (28.38)]).

The simple relation $(\partial_t - \Delta_{n-1})E^\lambda = E^{\lambda-1}$ for $\operatorname{Re}\lambda > 1$ [29, 10-03, (10)] shows that $\lambda \longmapsto E^\lambda$ can be extended to an entire distribution-valued function. In fact, $E^\lambda \in \mathcal{D}'_+(\mathbb{R}_t)\hat{\otimes}\mathcal{D}'_{L^1}(\mathbb{R}^{n-1}_x) = \mathcal{D}'_+(\mathcal{D}'_{L^1})$ [28, p. 52, Definition]. Using vector-valued convolution with respect to t and partial Fourier transform with respect to x he proves

$$E^\lambda * E^\nu = E^{\lambda+\nu} \text{ for } \lambda, \nu \in \mathbb{C}.$$

Our construction of the **convolution group for the operator** $\partial_t + R(-\mathrm{i}\,\partial_x)$ only slightly modifies Schwartz's procedure: We replace

- E^1 by the fundamental solution $\mathcal{F}^{-1}_{\xi\to x}\big(Y(t)\mathrm{e}^{-tR(\xi)}\big)$ of $\partial_t + R(-\mathrm{i}\,\partial_x)$,
- $\mathcal{D}'_+$ by the smaller space $\mathcal{D}'_{[0,\infty[}$,
- $\mathcal{D}'_{L^1}$ by the smaller space $\mathcal{O}'_C$,
- λ by $-\lambda$.

4 Remark. In [30, p. 44, Remark A], the resolvent $R^{(k)}_x$ of the k-times iterated equation $\left(\dfrac{\mathrm{d}}{\mathrm{d}t} + A_x(t)\right)^k U(t) = 0$ is represented as product of $\dfrac{t^{k-1}}{(k-1)!}$ and the resolvent of the equation itself, i.e.,

$$R^{(k)}_x(t,\tau) = \frac{(t-\tau)^{k-1}}{(k-1)!}\, R_x(t,\tau).$$

An analogous formula for a fundamental solution of $\big(\partial_t + P(\partial_x)\big)^k$ in terms of a fundamental solution of $\partial_t + P(\partial_x)$ is given in [35, Prop., p. 66].

3 Generalized heat kernels

The case of the heat operator $\partial_t - \Delta_{n-1} = \partial_t + R(-\mathrm{i}\,\partial_x)$, i.e. $R(\xi) = |\xi|^2 = \xi_1^2 + \cdots + \xi_{n-1}^2$, gives the idea to take for R a real-valued polynomial in $n-1$ variables $\xi = (\xi_1, \ldots, \xi_{n-1})$, homogeneous and positive definite, i.e. $R(\xi) > 0$ for $\xi \neq 0$.

These assumptions exclude interesting operators like

$$\partial_t + \partial_1 + \cdots + \partial_{n-1} + c = \partial_t + \mathrm{i}(-\mathrm{i}\,\partial_1 - \cdots - \mathrm{i}\,\partial_{n-1}) + c,\ \ c \in \mathbb{C},$$

or **Sobolev's operator**

$$\partial_t - \partial_x\partial_y\partial_z = \partial_t + \mathrm{i}(-\mathrm{i}\,\partial_x)(-\mathrm{i}\,\partial_y)(-\mathrm{i}\,\partial_z)$$

or **Schrödinger's operator**

$$\partial_t \pm \mathrm{i}\,\Delta_{n-1},$$

since the corresponding polynomials are not real-valued and/or not homogeneous.

Therefore, we shall only assume that R is a **complex-valued** polynomial satisfying the condition

$$\inf_{\xi\in\mathbb{R}^n} \operatorname{Re} R(\xi) > -\infty.$$

This condition is equivalent to the **quasihyperbolicity** of $\partial_t + R(-\mathrm{i}\,\partial_x)$ in the t-direction (cf. [22, p. 442] and the definition of quasihyperbolic systems in Section 4) and is also called **correctness in the sense of Petrovsky** ([16, p. 143, (12.8.2)]; [33, p. 262]; [10, (64), p. 167; p. 168, Definition]; [11, p. 7]; [5, p. 204]; [2, (3.15), p. 223]).

For $P(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x) = \partial_t + R(-\mathrm{i}\partial_x)$ let us now state the special case arising from Proposition 9 in Section 4, which refers to an arbitrary quasihyperbolic **system** $A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)$.

5 Proposition. *Let $R(\xi)$ be a complex-valued polynomial in $n-1$ variables $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ such that $\inf_{\xi\in\mathbb{R}^{n-1}} \operatorname{Re} R(\xi) > -\infty$. Then*

(i) the distribution

$$H_\lambda = \frac{t_+^{-\lambda-1}}{\Gamma(-\lambda)} \mathcal{F}^{-1}_{\xi\to x}(Y(t)\mathrm{e}^{-tR(\xi)}),$$

defined by

$$\langle \varphi, H_\lambda \rangle = \frac{1}{\Gamma(-\lambda)} \int_0^\infty t^{-\lambda-1}\,\mathrm{d}t \int_{\mathbb{R}^{n-1}} \mathcal{F}^{-1}_{x\to\xi}\,(\varphi(t,x))\,\mathrm{e}^{-tR(\xi)}\,\mathrm{d}\xi$$

for $\operatorname{Re}\lambda < 0$ *and* $\varphi \in \mathcal{D}(\mathbb{R}^n)$ *and by analytic continuation for* $\lambda \in \mathbb{C}$*, is well-defined and belongs to the space* $\mathcal{D}'_{[0,\infty[}\hat{\otimes}\mathcal{O}'_C(\mathbb{R}^{n-1})$*;*

(ii) For $\operatorname{Re}\lambda < -1$*,* H_λ *is the product of* $\dfrac{t_+^{-\lambda-1}}{\Gamma(-\lambda)} \otimes 1_x$ *and* $\mathcal{F}^{-1}_{\xi\to x}(Y(t)\mathrm{e}^{-tR(\xi)})$*;*

(iii) $H_k = \big(\partial_t + R(-\mathrm{i}\partial_x)\big)^k\delta$ *if* $k \in \mathbb{N}_0$*;*

(iv) the function $H : \mathbb{C} \longrightarrow \mathcal{D}'_{[0,\infty[}\hat{\otimes}\mathcal{O}'_C$ *is entire;*

(v) H_λ *and* H_ν *are convolvable for all* $\lambda, \nu \in \mathbb{C}$*;*

(vi) $H_\lambda * H_\nu = H_{\lambda+\nu}$ *for all* $\lambda, \nu \in \mathbb{C}$*;*

(vii) $u := H_{-k} * T \in \mathcal{D}'_{[c,\infty[}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1})$ *is the unique solution of the inhomogeneous equation*

$$\big(\partial_t + R(-\mathrm{i}\partial_x)\big)^k u = T \text{ if } T \in \mathcal{D}'_{[c,\infty[}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1})$$

for some $c \in \mathbb{R}$*,* $k \in \mathbb{N}_0$*. The mapping*

$$\begin{aligned} \mathcal{D}'_{[c,\infty)}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1}) &\longrightarrow \mathcal{D}'_{[c,\infty)}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1}), \\ u &\longmapsto \big(\partial_t + R(-\mathrm{i}\partial_x)\big)^k u \end{aligned}$$

is an isomorphism.

(viii) H_{-k} *is the uniquely determined* ***fundamental solution*** *of* $\big(\partial_t+R(-\mathrm{i}\partial_x)\big)^k$ *in* $\mathcal{D}'_{[0,\infty[}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1})$*.*

6 Example. The convolution group of the differential operator $\partial_t + a$ has the representation $H_\lambda = \dfrac{t_+^{-\lambda-1}}{\Gamma(-\lambda)}\mathrm{e}^{-at} \otimes \delta_x$ (cf. [27, (VI,5;15), p. 176]: "fractional differentiation and integration").

4 The convolution group for quasihyperbolic systems of differential operators

Let us define quasihyperbolic systems of linear differential operators (with constant coefficients):

7 Definition. (Cf. [23, p. 530]) The $m \times m$ matrix $A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)$ of differential operators is called **quasihyperbolic** in the t-direction iff

$$\exists\,\sigma_0 \in \mathbb{R} : \forall \sigma < \sigma_0,\ \forall \xi \in \mathbb{R}^{n-1} : \det A(\tau + \mathrm{i}\sigma, \xi) \neq 0.$$

8 Remark.

(1) Note that such systems are called "correct in the sense of Petrovsky" in [9, Ch. III, 2., p. 107].

(2) If $A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)$ is quasihyperbolic, then the matrix-valued function

$$A : U \longrightarrow Gl_m(\mathbb{C}),\ (z,\xi) \longmapsto A(z,\xi),$$

is well-defined and continuous on the simply connected domain $U := \{\,(z,\xi) \in \mathbb{C} \times \mathbb{R}^{n-1};\ \operatorname{Im} z < \sigma_0\,\}$. Let us suppose that the algebraic multiplicities of the eigenvalues of $A(z,\xi)$ do not change when (z,ξ) varies in U, i.e. $A(z,\xi)$ has p different eigenvalues $\mu_1(z,\xi), \dots, \mu_p(z,\xi)$ of respective multiplicity $r_1, \dots, r_p$. Then $\sum_{j=1}^p r_j = m$ and $\mu_j(z,\xi)$ depend analytically on (z,ξ). In this case $\log A(z,\xi)$ and thus $A(z,\xi)^\lambda,\ \lambda \in \mathbb{C}$, can be continued analytically throughout U from some chosen starting value at $(z_0,\xi_0) \in U$. In fact, for each $(z,\xi) \in U$ we define $\log A(z,\xi)$ by using a Jordan decomposition of the matrix $A(z,\xi)$:
If $e_1, \dots, e_k \in \mathbb{C}^m$ span an irreducible generalized eigenspace for the eigenvalue $\mu \in \mathbb{C}$, i.e.

$$\begin{aligned} A(z,\xi)e_1 &= \mu e_1, \\ A(z,\xi)e_j &= \mu e_j + e_{j-1},\ j = 2, \dots, k, \end{aligned}$$

then (cf. also [15, Thm. 2.6 h, p. 131])

$$\bigl(\log A(z,\xi)\bigr)e_j := \sum_{r=0}^{j-1} \frac{1}{r!} g^{(r)}(\mu) e_{j-r},$$

where $g(\mu) = \log \mu$ is assigned first for the p different eigenvalues $\mu_1(z_0,\xi_0), \dots, \mu_p(z_0,\xi_0)$ and then continued analytically into $(z,\xi) \in U$. Explicitly we then obtain

$$A(z,\xi)^\lambda e_j = \sum_{r=0}^{j-1} \binom{\lambda}{r} \mu^{\lambda-r} e_{j-r}$$

with $\mu^{\lambda-r} = \mathrm{e}^{(\lambda-r)\log\mu}$.

Scholium.
Note that the differential of the exponential map

$$\exp : \mathbb{C}^{m\times m} \longrightarrow \mathbb{C}^{m\times m}$$

satisfies

$$\det(\mathrm{d}_A \exp) = \prod_{i=1}^{p} \mathrm{e}^{\mu_i r_i^2} \prod_{1\le i<j\le p} \left(\frac{\mathrm{e}^{\mu_i} - \mathrm{e}^{\mu_j}}{\mu_i - \mu_j}\right)^{2r_i r_j},$$

where $\mu_1, \dots, \mu_p$ are the different eigenvalues of A with multiplicities r_j. Therefore, log cannot be defined analytically at $B = \mathrm{e}^A$ such that $A = \log B$ if $\det(\mathrm{d}_A \exp)$ vanishes, i.e. if two different eigenvalues $\mu_i \neq \mu_j$ of A fulfill $\mathrm{e}^{\mu_i} = \mathrm{e}^{\mu_j}$, i.e. if they differ by a multiple of $2\pi\mathrm{i}$. E.g. there is no analytic inverse $\widetilde{\log}$ of exp at

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which fulfills

$$\widetilde{\log} I = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi\mathrm{i} \end{pmatrix}.$$

Therefore log cannot be defined continuously on the curve

$$\begin{pmatrix} 1 & t \\ 0 & \mathrm{e}^{\mathrm{i}t} \end{pmatrix} \in \mathbb{C}^{2\times 2},\ t \in [0, 2\pi]$$

starting with $\log I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since the double eigenvalue 1 at $t = 0$ splits for $t > 0$ (e.g., for the hyperbolic system $A(-\mathrm{i}\partial_t) = \begin{pmatrix} -1 & \partial_t \\ 0 & \partial_t^4 \end{pmatrix}$ it is impossible to define $\log A(z) = \log \begin{pmatrix} -1 & \mathrm{i}z \\ 0 & z^4 \end{pmatrix}$ in the half-plane $\sigma = \operatorname{Re} z < 0$ in a continuous let alone an analytic way). That is why we assumed the multiplicities of the eigenvalues of $A(z, \xi)$ to remain constant in U, in order to ensure the analyticity of

$$U \longrightarrow \mathbb{C}^{m\times m},\ (z, \xi) \longmapsto \log A(z, \xi)$$

(cf. [26, in particular p. 404 and p. 412]).

The next proposition generalizes the propositions 2 and 5.

9 Proposition. *Let the $m \times m$ matrix of linear differential operators (with constant coefficients) be quasihyperbolic in the t-direction and let σ_0 be as in the definition of quasihyperbolicity. We assume that the algebraic multiplicities of the eigenvalues of $A(z, \xi)$ remain constant for $\operatorname{Im} z < \sigma_0$, $\xi \in \mathbb{R}^{n-1}$. Let us define $A(\tau + \mathrm{i}\sigma, \xi)^\lambda$, $\sigma < \sigma_0$, $(\tau, \xi) \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$, as in Remark 3, and finally,*

$$T_\lambda := \mathrm{e}^{-\sigma t} \mathcal{F}^{-1}_{(\tau,\xi)\mapsto(t,x)} \big(A(\tau + \mathrm{i}\sigma, \xi)^\lambda\big).$$

Then:

(i) *For all* $\lambda \in \mathbb{C}$, T_λ *is well-defined;*

(ii) $\forall \lambda \in \mathbb{C}$*:*
$T_\lambda \in (\mathcal{D}'_{[0,\infty)} \hat{\otimes} \mathcal{O}'_C)^{m\times m}$*;*
T_λ *does not depend on* $\sigma < \sigma_0$*;*
the function $\mathbb{C} \longrightarrow (\mathcal{D}'_{[0,\infty)} \hat{\otimes} \mathcal{O}'_C)^{m\times m}$, $\lambda \longmapsto T_\lambda$, *is an entire distribution-valued function;*

(iii) $T_k = A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)^k \delta,\ k \in \mathbb{N}_0$*;*

(iv) T_λ *and* T_ν *are convolvable for all* $\lambda, \nu \in \mathbb{C}$*;*

(v) $T_\lambda * T_\nu = T_{\lambda+\nu}$ *for all* $\lambda, \nu \in \mathbb{C}$*;*

(vi) $u := T_{-k} * f \in \big(\mathcal{D}'_{[c,\infty[} \otimes \mathcal{S}'(\mathbb{R}^{n-1})\big)^m$ *is the unique solution of the inhomogeneous system*

$$A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)^k u = f$$

if $f \in \big(\mathcal{D}'_{[c,\infty[} \hat{\otimes} \mathcal{S}'(\mathbb{R}^{n-1})\big)^m$ *for some* $c \in \mathbb{R},\ k \in \mathbb{N}_0$.
The mapping $f \longmapsto u$ *is a continuous right-inverse of* $A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)^k$.

(vii) T_{-k} *is the only fundamental matrix of* $A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)^k,\ k \in \mathbb{N}_0$ *satisfying* $\mathrm{e}^{\sigma t} T_{-k} \in \big(\mathcal{S}'(\mathbb{R}^n)\big)^{m\times m}$ *for some* $\sigma < \sigma_0$.

Proof. (i) To show first that T_λ is well-defined, let us apply Seidenberg-Tarski's theorem [16, Thm. A.2.2, p. 364] to the zeros of $\det\big(\mu I_m - A(\tau + \mathrm{i}\sigma, \xi)\big)$ to conclude that the eigenvalues μ of $A(\tau + \mathrm{i}\sigma, \xi)$ can converge to 0 or ∞ only algebraically, i.e.

$$\begin{aligned}&\exists\, c_1, c_2, k > 0 : \forall (\tau, \xi) \in \mathbb{R}^n, \forall \sigma < \sigma_0, \forall \text{ eigenvalues } \mu \text{ of } A(\tau + \mathrm{i}\sigma, \xi) : \\ &\qquad c_1(1 + |\xi|^2 + \sigma^2 + \tau^2)^{-k}(\sigma_0 - \sigma)^k \le |\mu| \le c_2(1 + |\xi|^2 + \sigma^2 + \tau^2)^k.\end{aligned}$$

The same argument applies to the eigenvalues of $A(\tau + \mathrm{i}\sigma, \xi)^* A(\tau + \mathrm{i}\sigma, \xi)$ and thereby shows the at most algebraic growth of $\big(\|A(\tau + \mathrm{i}\sigma, \xi)\|_2\big)^{\pm 1}$ with respect to (τ, σ, ξ)—since $\|B\|_2^2$ is the maximal eigenvalue of $B^* B$ for $B \in \mathbb{C}^{m\times m}$.

Next, we note that Lagrange's interpolation formula [26, formula (2.1), p. 397] implies for $f(B),\ B \in \mathbb{C}^{m\times m},\ f\ \mathcal{C}^{m-1}$ at the eigenvalues of B, the following estimate: $\exists\, C_m > 0 : \forall B \in \mathbb{C}^{m\times m}$:

$$\big\|f(B)\big\| \le C_m \max(\|B\|, 1)^{m-1} \cdot \max_{\substack{j=1,\dots,p \\ 1\le k\le r_j - 1}} \big|f^{(k)}(\mu_j)\big| \cdot \min_{i\ne j}(1, |\mu_i - \mu_j|)^{1-m}$$

where B has the different eigenvalues $\mu_j,\ j=1,\dots,p$, with the respective algebraic multiplicities $r_j,\ j=1,\dots,p,\ \sum\limits_{j=1}^{p} r_j = m$. Applying Seidenberg-Tarski's theorem to $|\mu_i-\mu_j|$ for different eigenvalues μ_i,μ_j of $A(\tau+\mathrm{i}\sigma,\xi)$ finally yields that

$$\forall\sigma<\sigma_0:\forall\lambda\in\mathbb{C}:A(\tau+\mathrm{i}\sigma,\xi)^\lambda\in\big(\mathcal{O}_M(\mathbb{R}^n_{\tau,\xi})\big)^{m\times m}$$

since

$$\big|\mu^{\lambda-k}\big|=|\mu|^{\operatorname{Re}\lambda-k}\mathrm{e}^{-\operatorname{Im}\lambda\arg\mu}$$

and $\arg\mu$ remains bounded for $\sigma<\sigma_0,(\tau,\xi)\in\mathbb{R}^n$. Because of $\mathcal{O}_M\subset\mathcal{S}'$ we infer that

$$T_\lambda=\mathrm{e}^{-\sigma t}\mathcal{F}^{-1}_{(\tau,\xi)\to(t,x)}\big(A(\tau+\mathrm{i}\sigma,\xi)^\lambda\big)$$

is well-defined, and, furthermore,

$$\mathrm{e}^{\sigma t}T_\lambda\in\mathcal{O}'_C(\mathbb{R}^n)^{m\times m}\simeq\big(\mathcal{O}'_C(\mathbb{R}^1_t)\hat{\otimes}\mathcal{O}'_C(\mathbb{R}^{n-1}_x)\big)^{m\times m}$$

by the nuclearity of the spaces $\mathcal{O}'_C$ [28, Prop. 28, p. 98].

(ii) Let us next show that T_λ does not depend on $\sigma<\sigma_0$. We fix $\lambda\in\mathbb{C}$ and use the analyticity of the function $z\longmapsto A(\tau+\mathrm{i}z,\xi)^\lambda\in\mathbb{C}^{m\times m}$ in the half-plane $\operatorname{Re}z<\sigma_0$ (for fixed (τ,ξ)). Hence

$$\int\limits_{\mathbb{R}^n}\varphi(\tau,\xi)A(\tau+\mathrm{i}z,\xi)^\lambda\,\mathrm{d}\tau\mathrm{d}\xi$$

depends analytically on z if $\varphi\in\mathcal{S}(\mathbb{R}^n)$ (by making use of the estimate in (i) and Lebesgue's theorem on dominated convergence), and we conclude that $\{z\in\mathbb{C};\ \operatorname{Re}z<\sigma_0\}\longrightarrow\mathcal{S}'(\mathbb{R}^n_{\tau,\xi}),\ z\longmapsto A(\tau+\mathrm{i}z,\xi)^\lambda$ is weakly holomorphic and thus holomorphic [13, Theorem 1.1.4, p. 57]. The same is true then for $z\longmapsto\mathrm{e}^{-zt}\mathcal{F}^{-1}\big(A(\tau+\mathrm{i}z,\xi)^\lambda\big)=:S(z)$, and since, obviously, S depends only on $\operatorname{Re}z=\sigma<\sigma_0$, this implies that S is constant and thus T_λ is independent of the choice of $\sigma<\sigma_0$. In order to prove that $T_\lambda=0$ on $H:=(-\infty,0)\times\mathbb{R}^{n-1}$, it is sufficient to show that

$$\lim_{\sigma\to-\infty}\mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\big(A(\tau+\mathrm{i}\sigma,\xi)^\lambda\big)=0\text{ in }\mathcal{D}'(H).$$

But this follows from the facts that $\lim\limits_{\sigma\to-\infty}\mathrm{e}^{-\sigma t}\sigma^k=0$ in $\mathcal{E}(H)$ for $k\in\mathbb{N}$ and that the set $\big\{\sigma^{-k}A(\tau+\mathrm{i}\sigma,\xi)^\lambda;\ \sigma<\sigma_0-1\big\}$ is bounded in $\mathcal{S}'(\mathbb{R}^n_{\tau,\xi})$ for suitable $k \in \mathbb{N}$ (again using the estimates in (i)). (We also use the hypocontinuity of the multiplication mapping $\mathcal{E}(H)\times\mathcal{D}'(H) \longrightarrow \mathcal{D}'(H)$, $(\psi, T) \longmapsto \psi \cdot T$: [27, Theorem III, p. 119]). Hence,

$$T_\lambda \in \left(\mathcal{D}'_{[0,\infty[}\hat{\otimes}\mathcal{O}'_C(\mathbb{R}^{n-1}_x)\right)^{m\times m}.$$

Finally, the holomorphy of $\lambda \longmapsto T_\lambda$ follows from that of

$$\mathbb{C} \longrightarrow \mathcal{O}_M(\mathbb{R}^n_{\tau,\xi})^{m\times m}, \lambda \longmapsto A(\tau + \mathrm{i}\sigma, \xi)^\lambda,$$

which in turn is implied by the Seidenberg-Tarski estimates in (i).

(iii) follows from

$$\begin{aligned} T_k = \mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\big(A(\tau + \mathrm{i}\sigma, \xi)^k\big) &= \mathrm{e}^{-\sigma t} A\big(-\mathrm{i}(\partial_t - \sigma), -\mathrm{i}\partial_x\big)^k \delta \\ &= A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)^k(\mathrm{e}^{-\sigma t}\delta) = A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)^k\delta. \end{aligned}$$

(iv),(v) The spaces $\mathcal{D}'_{[c,\infty)}, \mathcal{S}', \mathcal{O}'_C$ are nuclear (cf. [34, Corollary, p. 530, and Prop. 50.1, 50.3, p. 514]; [28, Prop. 28, p. 98]) and thus, e.g.,

$$\mathcal{D}'_{[c,\infty[}\hat{\otimes}_\pi\mathcal{S}' = \mathcal{D}'_{[c,\infty[}\hat{\otimes}_\epsilon\mathcal{S}' =: \mathcal{D}'_{[c,\infty)}\hat{\otimes}\mathcal{S}'$$

by Theorem 50.1 in [34, p. 511]. Two distributions in $\mathcal{D}'_{[0,\infty[}\hat{\otimes}\mathcal{O}'_C(\mathbb{R}^{n-1})$ and $\mathcal{D}'_{[c,\infty[}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1})$ are convolvable and their convolution product belongs to $\mathcal{D}'_{[c,\infty[}\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n-1})$. This implies (iv) and (vi).

(vi) By the Fourier exchange theorem for $\mathcal{O}_M$ and $\mathcal{O}'_C$, we have

$$\begin{aligned} T_\lambda * T_\nu &= \mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\big(A(\tau + \mathrm{i}\sigma, \xi)^\lambda \cdot A(\tau + \mathrm{i}\sigma, \xi)^\nu\big) \\ &= \mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\big(A(\tau + \mathrm{i}\sigma, \xi)^{\lambda+\nu}\big) = T_{\lambda+\nu}, \end{aligned}$$

since

$$A(\tau + \mathrm{i}\sigma, \xi)^z = \mathrm{e}^{z \log A(\tau+\mathrm{i}\sigma,\xi)} \quad \text{for} \quad z \in \mathbb{C}$$

by definition, and

$$\mathrm{e}^{zB} \cdot \mathrm{e}^{wB} = \mathrm{e}^{(z+w)B} \quad \text{for} \quad z, w \in \mathbb{C}.$$

(vii) If $S \in \mathcal{D}'(\mathbb{R}^n)^{m\times m}$ fulfills

$$\mathrm{e}^{\sigma t}S \in \mathcal{S}'(\mathbb{R}^n)^{m\times m} \text{ for } \sigma < \sigma_0 \text{ and } A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x)S = 0,$$

then also $A(-\mathrm{i}(\partial_t - \sigma), -\mathrm{i}\partial_x)(\mathrm{e}^{\sigma t}S) = 0$ and hence $A(\tau + \mathrm{i}\sigma, \xi)\mathcal{F}(\mathrm{e}^{\sigma t}S) = 0$ which implies that $\mathcal{F}(\mathrm{e}^{\sigma t}S)$ and thus S vanishes.

$\boxed{QED}$

10 Remark. Let us observe that $A(z,\xi)^\lambda$ can also be defined in certain cases where the algebraic multiplicities of the eigenvalues of $A(z,\xi)$ vary for $\operatorname{Im} z < \sigma_0, \xi \in \mathbb{R}^{n-1}$. E.g., if "Agmon's ray condition" is fulfilled, i.e. there exists a fixed ray $\mathbb{R}_+\omega, \omega \in \mathbb{C} \setminus 0$, such that no eigenvalue of $A(z,\xi)$ lies on this ray for $\operatorname{Im} z < \sigma_0, \xi \in \mathbb{R}^{n-1}$ (cf. [31, Def. 2, p. 890]). In fact, we can fix a branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_+ \cdot \omega$ and define $\log A(z,\xi)$ analytically for $\operatorname{Im} z < \sigma_0, \xi \in \mathbb{R}^{n-1}$ by applying this branch of the logarithm to the eigenvalues of $A(z,\xi)$.

5 The convolution group of the isotropic elastodynamic system

Let us consider the matrix of differential operators describing waves in linear, isotropic homogeneous elastic media, i.e.

$$A(-\mathrm{i}\partial_t, -\mathrm{i}\partial_x) = (\partial_t^2 - M\Delta_3)I_3 - (\Lambda + M)\nabla \cdot \nabla^T$$

where $\Lambda, M > 0$ are the Lamé constants.

5.1 Definition of the convolution group T_λ

For the elastodynamic system,

$$A(\tau,\xi) = (-\tau^2 + M|\xi|^2)I_3 + (\Lambda + M)\xi\xi^T$$

is a hyperbolic 3×3-matrix, since

$$\det A(\tau + \mathrm{i}\sigma, \xi) = \bigl(-(\tau + \mathrm{i}\sigma)^2 + M|\xi|^2\bigr)^2\bigl(-(\tau + \mathrm{i}\sigma)^2 + (\Lambda + 2M) \cdot |\xi|^2\bigr) \neq 0$$

for $\sigma < \sigma_0 := 0$ and $(\tau,\xi) \in \mathbb{R}^4$. On the other hand, the eigenvalues of $A(z,\xi)$ are

$$\mu_1(z,\xi) = -z^2 + M|\xi|^2 \text{ and } \mu_2(z,\xi) = -z^2 + (\Lambda + 2M)|\xi|^2$$

with multiplicities $r_1 = 2, r_2 = 1$, respectively, if $\xi \neq 0$, whereas $A(z,0) = -z^2 I_3$ and hence μ_1, μ_2 coincide for $\xi = 0$, Therefore, the assumption on constant multiplicities of the eigenvalues in Proposition 9 is not fulfilled. But

$$\mu_j(z,\xi) \in \mathbb{C} \setminus (-\infty, 0] = \mathbb{C} \setminus (-\mathbb{R}_+),\ j = 1,2,$$

and hence Agmon's ray condition is satisfied, cf. the Remark 10. Thus the logarithm and the powers of $A(z,\xi)$ are defined as follows. For $\xi \neq 0$ we diagonalize

$A(z,\xi)$ with respect to a basis η,ζ,ξ with $\eta,\zeta\perp\xi$ and we obtain the diagonal matrix

$$\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_1 & 0\\ 0 & 0 & \mu_2\end{pmatrix}.$$

Hence $A^\lambda,\ \lambda\in\mathbb{C}$, fulfills $A^\lambda\eta=\mu_1^\lambda\eta,\ A^\lambda\zeta=\mu_1^\lambda\zeta,\ A^\lambda\xi=\mu_2^\lambda\xi$ and thus is given by

$$A(z,\xi)^\lambda=\mu_1^\lambda I_3+(\mu_2^\lambda-\mu_1^\lambda)\frac{\xi\xi^T}{|\xi|^2}.$$

If $\xi=0$, then $\mu_1=\mu_2$ and $A(z,\xi)=\mu_1^\lambda I_3$. According to Remark 10, all statements of Proposition 9 then hold for

$$T_\lambda:=\mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\big(A(\tau+\mathrm{i}\sigma,\xi)^\lambda\big),\ \sigma<0.$$

5.2 Calculation of T_λ

The above representation of $A(z,\xi)^\lambda$ yields $T_\lambda=T_\lambda^1+T_\lambda^2$, where $T_\lambda^1:=\mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\big(\mu_1(\tau+\mathrm{i}\sigma,\xi)^\lambda\big)I_3$ is built up from the convolution group of the wave operator $\partial_t^2-M\Delta_3$, i.e.

$$\begin{aligned}T_\lambda^1 &= \mathcal{F}^{-1}\Big(\big(-\tau^2+\mathrm{i}\tau\cdot 0+M|\xi|^2\big)^\lambda\Big)I_3=\\ &= M^{-3/2}Z_{-2\lambda}\Big(t,\frac{x}{\sqrt{M}}\Big)I_3,\end{aligned}$$

cf. [27, (VII, 7; 37), p. 264], and explicitly

$$T_\lambda^1=\frac{2^{2\lambda+1}\big(t^2-\frac{|x|^2}{M}\big)^{-\lambda-2}Y\big(t-\frac{|x|}{\sqrt{M}}\big)}{\pi M^{3/2}\Gamma(-\lambda)\Gamma(-\lambda-1)}I_3$$

for $\operatorname{Re}\lambda<-1$, cf. [27, (VII, 7; 36), p. 263]. On the other hand,

$$\begin{aligned}T_\lambda^2&=\mathrm{e}^{-\sigma t}\mathcal{F}^{-1}\Big(\big(\mu_2(\tau+\mathrm{i}\sigma,\xi)^\lambda-\mu_1(\tau+\mathrm{i}\sigma,\xi)^\lambda\big)\frac{\xi\xi^T}{|\xi|^2}\Big)\\ &=-\mathrm{e}^{-\sigma t}\nabla\nabla^T\mathcal{F}^{-1}\Big(\frac{\big(-(\tau+\mathrm{i}\sigma)^2+(\Lambda+2M)|\xi|^2\big)^\lambda-\big(-(\tau+\mathrm{i}\sigma)^2+M|\xi|^2\big)^\lambda}{|\xi|^2}\Big)\\ &=-\mathrm{e}^{-\sigma t}\nabla\nabla^T\mathcal{F}^{-1}\Big(\lambda\int_M^{\Lambda+2M}\big(-(\tau+\mathrm{i}\sigma)^2+\varrho|\xi^2|\big)^{\lambda-1}\,\mathrm{d}\varrho\Big)\end{aligned}$$

$$= -\lambda \nabla\nabla^T \int_M^{\Lambda+2M} \varrho^{-3/2} Z_{-2\lambda+2}\Big(t, \frac{|x|}{\sqrt{\varrho}}\Big)\, \mathrm{d}\varrho$$

$$= \frac{-\lambda 2^{2\lambda-1}}{\pi\Gamma(1-\lambda)\Gamma(-\lambda)} \nabla\nabla^T \int_M^{\Lambda+2M} \varrho^{-3/2}\Big(t^2 - \frac{|x|^2}{\varrho}\Big)^{-\lambda-1} Y\Big(t - \frac{|x|}{\sqrt{\varrho}}\Big)\, \mathrm{d}\varrho.$$

Still supposing $\operatorname{Re}\lambda < -1$ the substitution $\sigma = \dfrac{|x|}{t\sqrt{\varrho}}$ yields

$$T_\lambda^2 = \frac{2^{2\lambda}}{\pi\Gamma(-\lambda)^2} \nabla\nabla^T \Bigg(\frac{t^{-2\lambda-1}}{|x|} \int_{\frac{|x|}{t\sqrt{\Lambda+2M}}}^{\frac{|x|}{t\sqrt{M}}} (1-\sigma^2)^{-\lambda-1} Y(1-\sigma)\, \mathrm{d}\sigma \Bigg)$$

$$= \frac{2^{2\lambda} t^{-2\lambda-1}}{\pi\Gamma(-\lambda)^2} \nabla\cdot\nabla^T \Bigg\{ \frac{1}{|x|} \sum_{j=0}^{\infty} \binom{-\lambda-1}{j} \frac{(-1)^j}{2j+1}$$

$$\cdot \Bigg[\Big[1 - \big(\frac{|x|}{t\sqrt{\Lambda+2M}}\big)^{2j+1}\Big] Y\big(t - \frac{|x|}{\sqrt{\Lambda+2M}}\big) - \Big[1 - \big(\tfrac{|x|}{t\sqrt{M}}\big)^{2j+1}\Big] Y\big(t - \tfrac{|x|}{\sqrt{M}}\big)\Bigg]\Bigg\}.$$

Finally, let us perform the differentiations, still supposing $\operatorname{Re}\lambda < -1$. Using $\nabla^T\big(f(|x|)\big) = f'(|x|)\cdot \dfrac{x^T}{|x|}$ we obtain

$$T_\lambda^2 = \frac{2^{2\lambda} t^{-2\lambda-1}}{\pi\Gamma(-\lambda)^2} \nabla \Bigg\{ \frac{x^T}{|x|^3} \sum_{j=0}^{\infty} \binom{-\lambda-1}{j} \frac{(-1)^{j+1}}{2j+1}$$

$$\cdot \Big[\Big(1 + 2j\big(\frac{|x|}{t\sqrt{\Lambda+2M}}\big)^{2j+1}\Big) Y\big(t - \frac{|x|}{\sqrt{\Lambda+2M}}\big)$$

$$- \Big(1 + 2j\big(\frac{|x|}{t\sqrt{M}}\big)^{2j+1}\Big) Y\big(t - \frac{|x|}{\sqrt{M}}\big)\Big]\Bigg\}. \quad (*)$$

Since $\sum_{j=0}^{\infty}(-1)^{j+1}\binom{-\lambda-1}{j} = -(1-1)^{-\lambda-1} = 0$ for $\operatorname{Re}\lambda < -1$, the remaining differentiation does not yield any delta-terms, and hence, using

$$\sum_{j=0}^{\infty} \binom{-\lambda-1}{j} \frac{(-1)^j}{2j+1} = \frac{\sqrt{\pi}\,\Gamma(-\lambda)}{2\Gamma\big(-\lambda+\frac{1}{2}\big)},$$

we infer, still for $\operatorname{Re}\lambda < -1$,

$$
\begin{aligned}
T_\lambda^2 &= \frac{t^{-2\lambda-1}}{4\pi\Gamma(-2\lambda)}\Big(-\frac{I_3}{|x|^3}+\frac{3xx^T}{|x|^5}\Big)\Big[Y\Big(t-\frac{|x|}{\sqrt{\Lambda+2M}}\Big)-Y\Big(t-\frac{|x|}{\sqrt{M}}\Big)\Big]\\
&+\frac{2^{2\lambda+1}t^{-2\lambda-1}}{\pi\Gamma(-\lambda)\Gamma(-\lambda-1)}\sum_{j=1}^{\infty}\binom{-\lambda-2}{j-1}\frac{(-1)^{j+1}|x|^{2j-2}t^{-2j-1}}{2j+1}\Big(I_3+2(j-1)\frac{xx^T}{|x|^2}\Big)\\
&\quad\cdot\Big[(\Lambda+2M)^{-j-1/2}Y\Big(t-\frac{|x|}{\sqrt{\Lambda+2M}}\Big)-M^{-j-1/2}Y\Big(t-\frac{|x|}{\sqrt{M}}\Big)\Big].
\end{aligned}
$$

Note that $\sum_{k=0}^{\infty}\binom{\mu}{k}(-x)^k$ converges in $L^1\big([0,1]\big)$ to $(1-x)^\mu$ (for $\operatorname{Re}\mu > -1$).

5.3 Final result

Let us summarize the above calculation in the following proposition:

11 Proposition. *The convolution group T_λ of the isotropic elastodynamic system*

$$(\partial_t^2 - M\Delta_3)I_3 - (\Lambda+M)\nabla\cdot\nabla^T,\ \ \Lambda, M > 0,$$

satisfies:

(1) for $\operatorname{Re}\lambda < -1$, T_λ is locally integrable and has the representation

$$
\begin{aligned}
T_\lambda(t,x) &= \frac{2^{2\lambda+1}\big(t^2-\frac{|x|^2}{M}\big)^{-\lambda-2}Y\big(t-\frac{|x|}{\sqrt{M}}\big)}{\pi M^{3/2}\Gamma(-\lambda)\Gamma(-\lambda-1)}I_3\\
&+\frac{t^{-2\lambda-1}}{4\pi\Gamma(-2\lambda)}\Big(\frac{3xx^T}{|x|^5}-\frac{I_3}{|x|^3}\Big)\Big[Y\Big(t-\frac{|x|}{\sqrt{\Lambda+2M}}\Big)-Y\Big(t-\frac{|x|}{\sqrt{M}}\Big)\Big]\\
&\quad+\frac{2^{2\lambda+1}t^{-2\lambda-1}}{\pi\Gamma(-\lambda)\Gamma(-\lambda-1)}\sum_{j=1}^{\infty}\binom{-\lambda-2}{j-1}\frac{(-1)^{j+1}|x|^{2j-2}t^{-2j-1}}{2j+1}\\
&\qquad\cdot\Big(I_3+2(j-1)\frac{xx^T}{|x|^2}\Big)\\
&\quad\cdot\Big[(\Lambda+2M)^{-j-1/2}Y\Big(t-\frac{|x|}{\sqrt{\Lambda+2M}}\Big)-M^{-j-1/2}Y\Big(t-\frac{|x|}{\sqrt{M}}\Big)\Big];
\end{aligned}
$$

(2) for $k=2,3,\ldots$, the fundamental matrices T_{-k} of

$$\big[(\partial_t^2 - M\Delta_3)I_3 - (\Lambda+M)\nabla\nabla^T\big]^k$$

are given by the locally integrable functions

$$
\begin{aligned}
T_{-k}(t,x) &= \frac{2^{1-2k}\big(t^2-\frac{|x|^2}{M}\big)^{k-2}Y\big(t-\frac{|x|}{\sqrt{M}}\big)}{\pi M^{3/2}(k-1)!(k-2)!}I_3 \\
&+\frac{t^{2k-1}}{4\pi(2k-1)!}\left(\frac{3xx^T}{|x|^5}-\frac{I_3}{|x|^3}\right)\left[Y\left(t-\frac{|x|}{\sqrt{\Lambda+2M}}\right)-Y\left(t-\frac{|x|}{\sqrt{M}}\right)\right] \\
&+\frac{t^{2k-1}2^{1-2k}}{\pi(k-1)!}\sum_{j=1}^{k-1}\frac{(-1)^{j+1}|x|^{2j-2}t^{-2j-1}}{(j-1)!(2j+1)(k-j-1)!}\left(I_3+2(j-1)\frac{xx^T}{|x|^2}\right) \\
&\cdot\left[(\Lambda+2M)^{-j-1/2}Y\left(t-\frac{|x|}{\sqrt{\Lambda+2M}}\right)-M^{-j-1/2}Y\left(t-\frac{|x|}{\sqrt{M}}\right)\right].
\end{aligned}
$$

12 Remark. T_{-1} coincides with the well-known **Stokes' fundamental matrix** of

$$(\partial_t^2-M\Delta_3)I_3-(\Lambda+M)\nabla\cdot\nabla^T.$$

T_{-1} can be inferred by letting λ tend to -1 from below. More precisely, $T_{-1}=T_{-1}^1+T_{-1}^2$ with

$$T_{-1}^1(t,x)=M^{-3/2}Z_2\left(t,\frac{x}{\sqrt{M}}\right)I_3=\frac{1}{4\pi M|x|}\delta\left(t-\frac{|x|}{\sqrt{M}}\right)I_3,$$

and on the other hand, from formula $(*)$, we have

$$
\begin{aligned}
T_{-1}^2(t,x) &= \frac{t}{4\pi}\nabla\left(-\frac{x^T}{|x|^3}\left[Y\left(t-\frac{|x|}{\sqrt{\Lambda+2M}}\right)-Y\left(t-\frac{|x|}{\sqrt{M}}\right)\right]\right) \\
&= \frac{t}{4\pi}\left(\frac{3xx^T}{|x|^5}-\frac{I_3}{|x|^3}\right)\left[Y\left(t-\frac{|x|}{\sqrt{\Lambda+2M}}\right)-Y\left(t-\frac{|x|}{\sqrt{M}}\right)\right] \\
&\qquad+\frac{t^2xx^T}{4\pi|x|^5}\left[\delta\left(t-\frac{|x|}{\sqrt{\Lambda+2M}}\right)-\delta\left(t-\frac{|x|}{\sqrt{M}}\right)\right]
\end{aligned}
$$

(compare [4, (5.10.30)–(5.10.32), p. 400]; [37, (8.15), p. 282]; [24, Section 4.3]).

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