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A new class of Gleason parts homeomorphic to the unit disk

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Abstract. We show that the Gleason part of every cluster point of an interpolating sequence of type 1 in the set of nontrivial points in the spectrum of H^∞ is homeomorphic to the unit disk.

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To the memory of Professor Klaus Floret

1 Introduction

Let A be a uniform algebra. Its spectrum, denoted by $M(A)$, is the set of all (nonzero) multiplicative linear functionals endowed with the weak-$*$-topology. $M(A)$ is a compact Hausdorff space. The algebras we are dealing with will be the algebra L^∞ of (equivalence classes) of essentially bounded, Lebesgue measurable functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$ and the Banach algebra H^∞ of all bounded analytic functions in the open unit disk $\mathbb{D}$. By the famous Corona-Theorem of Carleson, $\mathbb{D}$ can be considered as a dense subset of $M(H^\infty)$.

We refer the reader to the books of Browder [1] or Gamelin [4] for a detailed exposition of the theory of uniform algebras and to the books of Garnett [5] and Hoffman [9] for information about the algebras H^∞ and L^∞. In the sequel, we shall always identify $f \in H^\infty$ with its Gelfand transform $\hat{f}$ defined on $M(H^\infty)$ by $\hat{f}(m) = m(f)$.

In this paper we are concerned with the structure of the Gleason parts in $M(H^\infty)$. Recall that the Gleason part, $P(m)$, associated with a point $m \in M(H^\infty)$, is defined as follows:

$$P(m) = \{\, x \in M(H^\infty) : \rho(x, m) < 1 \,\},$$

where

$$\rho(x, m) = \sup\{\, |f(x)| : f \in H^\infty, ||f||_\infty \leq 1, f(m) = 0 \,\}$$

is the pseudohyperbolic distance on $M(H^\infty)$. We note that Schwarz's Lemma implies that

$$\rho(x,m) = \sup\{\, \rho_{\mathbb{D}}(f(x), f(m)) : f \in H^\infty, ||f||_\infty < 1\,\},$$

where $\rho_{\mathbb{D}}(z,w) = \left|\frac{z-w}{1-\overline{z}w}\right|$ is the usual pseudohyperbolic distance on the unit disk $\mathbb{D}$. Moreover, "$m \sim x$ if and only if $\rho(m,x) < 1$" defines an equivalence relation on $M(H^\infty)$. Hence the Gleason parts are exactly the equivalence classes of this relation.

K. Hoffman [10] showed that, within $M(H^\infty)$, the Gleason parts are either singletons or maximal analytic disks. Moreover, there exists a continuous map L_m of $\mathbb{D}$ onto the part $P(m)$ such that $f \circ L_m$ is analytic for every $f \in H^\infty$ and $L_m(0) = m$. This Hoffman map L_m is given by $L_m(z) = \lim \frac{z+z_\alpha}{1+\overline{z}_\alpha z}$, where (z_α) is any net in $\mathbb{D}$ converging to m, and where the limit is taken in the topology of $M(H^\infty)^{\mathbb{D}}$. The set of all points $m \in M(H^\infty)$ for which $P(m)$ is nontrivial, that is for which $P(m)$ is not a singleton, is denoted by G. Of course, $\mathbb{D}$ is a nontrivial Gleason part. Hoffman showed that $m \in G$ if and only if m lies in the closure of an interpolating sequence in $\mathbb{D}$, that is a sequence (a_n) satisfying

$$\inf_k \prod_{j:j\neq k} \left|\frac{a_j - a_k}{1-\overline{a}_j a_k}\right| \geq \delta > 0.$$

If $m \in G$, then the Hoffman map L_m is a bijection of $\mathbb{D}$ onto $P(m)$. However, L_m is, in general, not a homeomorphism. In [8] and [12] several characterizations of those parts $P(m)$ were given for which $P(m)$ is homeomorphic to $\mathbb{D}$. Note that $P(m)$ is homeomorphic to $\mathbb{D}$ if and only if L_m is a homeomorphism. This holds because continuous bijective mappings of $\mathbb{D}$ onto itself are automatically homeomorphisms. But there were only very few examples. The classical one of Hoffman states that if m lies in the closure of a thin interpolating sequence, that is a sequence (a_n) satisfying

$$\lim_{k\to\infty} \prod_{j:j\neq k} \left|\frac{a_j - a_k}{1-\overline{a}_j a_k}\right| = 1,$$

then $P(m)$ is homeomorphic to $\mathbb{D}$. In [7], it is shown that every cluster point x of a sequence (x_n) of nontrivial points lying in different fibers of the spectrum of H^∞ has the property that it belongs to a homeomorphic part. In this paper we shall now present a much bigger class of sequences in $M(H^\infty)$ whose cluster points enjoy that property.

1 Definition.

(i) A sequence $(x_n) \in M(H^\infty)^{\mathbb{N}}$ is said to be interpolating for H^∞ if for every bounded sequence $(w_n) \in \mathbb{C}^{\mathbb{N}}$ there exists a function $f \in H^\infty$ such that $f(x_n) = w_n$ for all n.

(ii) The interpolating sequence (x_n) is said to be of type 1 if the norm of f can be chosen to be 1 whenever (w_n) is in the unit ball of ℓ^∞.

Interpolating sequences of type 1 were characterized in [6] (see section 1). Due to the maximum principle, they are necessarily contained in the Corona $M(H^\infty)\setminus\mathbb{D}$ of H^∞. Sequences whose elements lie in different fibers, are examples (see [6]). Our result of the present paper will be that the Gleason part of every cluster point of an interpolating sequence of type 1 in G is an analytic disk which is homeomorphic to $\mathbb{D}$.

2 Prerequisites

For the reader's convenience, we recall here some facts and fix our notation. As usual, we shall identify the Shilov boundary of H^∞ with $M(L^\infty)$. Let $m \in M(H^\infty)$. A probability measure μ_m defined on the Borel sets of the Shilov boundary of H^∞ is called a representing measure for m if

$$m(f) = \int_{M(L^\infty)} f d\mu_m \text{ for every } f \in H^\infty.$$

It is well known (see [9, p. 182]) that every $m \in M(H^\infty)$ admits a unique representing measure μ_m. The smallest compact subset of $M(L^\infty)$ with μ_m-measure 1 is called the support set of μ_m, or simply m, and will be denoted by $\operatorname{supp} m$. We note that $\operatorname{supp}\Phi_{z_0} = M(L^\infty)$ for all $z_0 \in \mathbb{D}$, where $\Phi_{z_0} : f \mapsto f(z_0)$ is the evaluation functional at $z_0 \in \mathbb{D}$. All other support sets S are very thin, in the sense that there exists $\lambda \in \mathbb{T}$ such that $S \subseteq M(L^\infty) \cap M_\lambda$, where M_λ is the fiber

$$M_\lambda = \{\, m \in M(H^\infty) : \mathrm{id}(m) = \lambda \,\}.$$

(Here id denotes the coordinate function $\mathrm{id}(z) = z$.)

The following is a well known result from the theory of uniform algebras. We refer the reader to the books of Gamelin [4] and Leibowitz [11]. For part (b) and (c), see also [3].

2 Lemma. *Let $x \in M(H^\infty)$. Then*

(a) $L_x(\mathbb{D}) = P(x) \subseteq M(H^\infty|_{\operatorname{supp} x}) = \{\, m \in M(H^\infty) : \operatorname{supp} m \subseteq \operatorname{supp} x \,\}$

(b) *If* $||f||_\infty = 1$ *and* $|f(x)| = 1$*, then* f *is constant on* $M(H^\infty|_{\operatorname{supp} x})$*. In particular* $f \equiv f(x) = e^{i\sigma}$ *on* $P(x)$*.*

(c) *If* u *is an inner function invertible in* $H^\infty|_{\operatorname{supp} x}$*, then* $u \equiv e^{i\sigma}$ *on* $M(H^\infty|_{\operatorname{supp} x})$*.*

Recall that the zero set $Z(f)$ of $f \in H^\infty$ is the set of all $x \in M(H^\infty)$ for which $f(x) = 0$.

A Blaschke product with zero sequence (a_n) in the open unit disk $\mathbb{D}$ is a function of the form

$$B(z) = e^{i\theta} z^N \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z},$$

where (a_n) satisfy $\sum_n (1 - |a_n|) < \infty$. This infinite product converges unconditionally and locally uniformly in $\mathbb{D}$.

The function B is called *normalized*, if $B(0) > 0$.

A Blaschke product B for which the zero sequence is an interpolating sequence is called an interpolating Blaschke product with *uniform separation constant* $\delta(B)$ defined by

$$\delta(B) := \inf_{k \in \mathbb{N}} \prod_{j: j \neq k} \left| \frac{a_j - a_k}{1 - \overline{a}_j a_k} \right|.$$

We note that

$$\delta(B) = \inf_n (1 - |a_n|^2) |B'(a_n)|.$$

3 Lemma (Hoffman's Lemma. See [10], p. 86, 106 and [5] p. 404, 310)**.** *Let* $\varepsilon, \eta, \delta$ *be numbers satisfying* $0 < \varepsilon < \eta < \delta < 1$*,* $\frac{2\eta}{1+\eta^2} < \delta$ *and* $0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}$*. Suppose that* B *is an interpolating Blaschke product with zeros* $\{ z_n : n \in \mathbb{N} \}$ *such that*

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2) |B'(z_n)| \geq \delta.$$

Then

(1) $Z(B)$ *is the closure of the zero set of* B *in* $\mathbb{D}$*,*

(2) $\rho(x, y) \geq \delta$ *for any* $x, y \in Z(B), x \neq y$*, and*

(3) $\{ m \in M(H^\infty) : |B(m)| < \varepsilon \} \subseteq \{ m \in M(H^\infty) : \rho(m, Z(B)) < \eta \} \subseteq \{ m \in M(H^\infty) : |B(m)| < \eta \}$*.*

4 Lemma. *Let* E *be a closed subset in* $M(H^\infty)$ *and suppose that* $x \in G \setminus E$*. Then for every* $\sigma \in]0, 1[$*, there exists an interpolating Blaschke product* B *such that* $B(x) = 0$ *and* $|B| \geq \sigma \rho^2(x, E)$ *on* E*.*

Proof. Let $\eta = \sqrt{\sigma}\rho(x, E)$. Since the pseudohyperbolic distance $\rho(\cdot, E)$ is lower semi-continuous ([10, p. 103]), there exists a neighborhood U of x in $M(H^\infty)$ such that $\rho(U, E) > \eta$. Choose $\delta \in]0, 1[$ such that $\frac{2\eta}{1+\eta^2} < \delta$. By Hoffman ([10, p. 90]) there exists an interpolating Blaschke product B with $\delta(B) > \delta$, $Z(B) \subseteq U$ and $B(x) = 0$. In particular $\rho(Z(B), E) > \eta$. Hence, by Lemma 3, $|B| > \eta(\delta - \eta)/(1 - \eta\delta) > \eta^2$ on E. QED

5 Lemma. *Let λ and β be complex numbers of modulus 1. Then, for every r with $0 < r < 1$ there exists a normalized Blaschke factor $L_a(z) = \frac{|a|}{a}\frac{a - z}{1 - \overline{a}z}$ such that $|a| = r$ and $L_a(\lambda) = \beta$.*

Proof. Let r be chosen with $0 < r < 1$, and let $a = re^{i\theta}$. Then we have to solve $\frac{r}{re^{i\theta}}\frac{re^{i\theta} - \lambda}{1 - re^{-i\theta}\lambda} = \beta$ for $e^{i\theta}$. Our equation is equivalent to $L_r(\lambda e^{-i\theta}) = \beta$. Since L_r is its own inverse, we let $e^{i\theta} = \lambda L_r(\overline{\beta})$ to obtain our solution. QED

3 Homeomorphic Gleason parts

The following two facts will be the major tools for the proof of our main result. The first deals with a description of the interpolating sequences of type 1.

6 Theorem ([6]). *A sequence (x_n) in $M(H^\infty)$ is an interpolating sequence of type 1 for H^∞ if and only if*

$$M(H^\infty|_{\operatorname{supp} x_n}) \cap \overline{\bigcup_{j \neq n} M(H^\infty|_{\operatorname{supp} x_j})} = \emptyset.$$

The second fact, proved in [8], characterizes the class of homeomorphic Gleason parts.

7 Theorem ([8]). *Let $m \in M(H^\infty) \setminus \mathbb{D}$. Then the following assertions are equivalent:*

(a) $P(m)$ is homeomorphic to $\mathbb{D}$;

(b) There exists an interpolating Blaschke product B with $Z(B) \cap P(m) = \{ m \}$.

(c) There exists a function $f \in H^\infty$ with $Z(f) \cap P(m) = \{ m \}$.

Actually only the equivalence of (a) and (b) are explicitly in [8]. But using Hoffman's [10] theory it easily follows that if $Z(f) \cap P(m) = \{ m \}$, then the order of the zero of $f \circ L_m$ is finite, and so f must have a Blaschke factor b which is interpolating and satisfies $b(m) = 0$ (see [10, p. 100]). So (c) implies (b). That (b) implies (c), is trivial.

We are now ready to prove our main Theorem.

8 Theorem. *Let $\{ x_n : n \in \mathbb{N} \}$, $x_n \in G$, be an interpolating sequence of type 1. Then the Gleason part of every cluster point of that sequence is homeomorphic to $\mathbb{D}$*

PROOF. Let x be a cluster point of $\{ x_n : n \in \mathbb{N} \}$. In order to show that $P(x)$ is homeomorphic to $\mathbb{D}$, we apply Theorem 7 and show that there exists a function $f \in H^\infty$ such that $Z(f) \cap P(x) = \{ x \}$.

Let $0 < \varepsilon_n < 1$ be so that $\prod_n \varepsilon_n$ converges. Using Theorem 6 we may choose neighborhoods U_n of $M(H^\infty|_{\text{supp } x_n})$ in $M(H^\infty)$ so that

$$\overline{U}_n \bigcap \overline{\bigcup_{j \neq n} U_j} = \emptyset.$$

Fix n. Since $\rho\Big(x_n, \overline{\bigcup_{j \neq n} U_j}\Big) = 1$ and $x_n \in G$, there is, by Lemma 4, a normalized interpolating Blaschke product b_n with $b_n(x_n) = 0$ and $|b_n| > \varepsilon_n$ on $\overline{\bigcup_{j \neq n} U_j}$. We may assume that

$$\sum_{j=1}^{\infty} (1 - |a_{j,n}|) \leq 2^{-n}, \tag{1}$$

where $(a_{j,n})_j$ is the zero sequence of b_n in $\mathbb{D}$ (otherwise delete a finite number of zeros of each b_n).

By ([10, p. 91]), there is a sequence of unimodular constants and normalized factors $b_j^{(n)}$ of b_n such that $b_j^{(n)}(x_n) = 0$ and such that $e^{i\theta_{n,j}} b_j^{(n)} \circ L_{x_n}$ converges, with j to infinity, locally uniformly on $\mathbb{D}$ to the identity function id. Hence there exists a normalized interpolating Blaschke product B_n dividing b_n such that $B_n(x_n) = 0$ and

$$\sup \left\{ \Big| (e^{i\theta_n} B_n \circ L_{x_n})(z) - z \Big| : |z| < 1 - \frac{1}{n} \right\} \leq \frac{1}{n}. \tag{2}$$

By Lemma 5 we can get rid of the constants $e^{i\theta_n}$ by replacing them with factors of the form $L_n(z) = \dfrac{|a_n|}{a_n} \dfrac{a_n - z}{1 - \overline{a}_n z}$, where the a_n are chosen so that $\sum_n (1 - |a_n|) < \infty$ and $L_n(\lambda_n) = e^{i\theta_n}$. Here $\lambda_n \in \mathbb{T}$ is that number for which $x_n \in M_{\lambda_n}$. In particular $L_n \circ L_{x_n} \equiv e^{i\theta_n}$.

Thus we obtain normalized Blaschke products $B_n^* = L_n B_n$ satisfying

$$\sup \left\{ \Big| (B_n^* \circ L_{x_n})(z) - z \Big| : |z| < 1 - \frac{1}{n} \right\} \leq \frac{1}{n}. \tag{3}$$

(1) and the choice of (a_n) imply that the collection of all zeros $\{ a_{j,n} : j, n \in \mathbb{N} \} \cup \{ a_n : n \in \mathbb{N} \}$ is a Blaschke sequence. Since B_n and L_n are normalized,

the unconditional convergence of Blaschke products implies that the infinite product $B=\prod_n B_n^*$ converges locally uniformly in $\mathbb{D}$ to a Blaschke product B.

Next we note that $|L_n|=1$ on $M(H^\infty)\setminus\mathbb{D}$ and that $|B_n|\geq|b_n|$. Since for $j\neq n$ we have $|B_j^*|>\varepsilon_j$ on $\overline{\bigcup_{k\neq j}U_k}\supset U_n$, we therefore get for every $z\in\mathbb{D}\cap U_n$ that

$$\prod_{j\neq n}|B_j^*(z)|\geq\prod_{j\neq n}\varepsilon_j=:\varepsilon>0. \tag{4}$$

Note that by the Corona-theorem $\overline{U}_n=\overline{U_n\cap\mathbb{D}}$. Hence, by (4)

$$\left|\prod_{j\neq n}B_j^*\right|\geq\varepsilon \text{ on } U_n.$$

Since $M(H^\infty|_{\operatorname{supp}x_n})\subseteq U_n$, we get that $\prod_{j\neq n}B_j^*$ is invertible in the restriction algebra $H^\infty|_{\operatorname{supp}x_n}$. Therefore, by Lemma 2, $\prod_{j\neq n}B_j^*$ is constant $e^{i\sigma_n}$ on $M(H^\infty|_{\operatorname{supp}x_n})$ for some $\sigma_n\in\mathbb{R}$.

Thus $B=e^{i\sigma_n}B_n^*$ on $M(H^\infty|_{\operatorname{supp}x_n})$. Since we cannot control the factors $e^{i\sigma_n}$, we have to get rid of them. Since $\{x_n : n\in\mathbb{N}\}$ is an interpolating sequence of type 1, there exists a norm one function $h\in H^\infty$ such that $h(x_n)=e^{-i\sigma_n}$ for every n. Hence $f:=hB$ is a norm one function with

$$f\circ L_{x_n}=B_n^*\circ L_{x_n}\,.$$

By (3) we get that $f\circ L_{x_n}$ converges locally uniformly in $\mathbb{D}$ to the identity function.

Let x be cluster point x of $\{x_n : n\in\mathbb{N}\}$. Suppose $x_{n(\alpha)}\to x$. Then, by ([10, p. 92]) or [2], $L_{x_{n(\alpha)}}\to L_x$ in the topology of $M(H^\infty)^{\mathbb{D}}$. In particular we see that $(f\circ L_{x_{n(\alpha)}})(z)\to(f\circ L_x)(z)$ for every $z\in\mathbb{D}$. Thus $(f\circ L_x)(z)=z$ from which we conclude that $Z(f)\cap P(x)=\{x\}$. Thus, by Theorem 7, $P(x)$ is homeomorphic to $\mathbb{D}$. QED

If we merely assume that (x_n) is an interpolating sequence, then the assertion of the theorem is no longer true: indeed, any point $m\in G$ lies in the closure of an interpolating sequence in $\mathbb{D}$. On the other hand, the assumption of being an interpolating sequence of type one, is not necessary, either. Any thin interpolating sequence in $\mathbb{D}$ fulfills the conclusion of the Theorem, but is not of type 1. Since the thin sequences are exactly the asymptotic interpolating sequences of type one in $\mathbb{D}$, (see [6]), we ask the following question:

Let (x_n) be an asymptotic interpolating sequence of type one, for short (asi1), in $M(H^\infty)$. Is the Gleason part of every cluster point of (x_n) homeomorphic to $\mathbb{D}$ whenever $x_n\in G$ for all n? Recall that (x_n) is an (asi1), if for every

(w_n) in the unit ball of ℓ^∞ there exists a function $f \in H^\infty$ with norm less than or equal to one such that $|f(x_n) - w_n| \to 0$.

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