Note di Matematica **25,** n. 2, 2005/2006, 77–96.

Mappings between Banach spaces that preserve convergence of series

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Abstract. We know that a numerical series is absolutely convergent, if, and only if, it is unconditionally convergent. Dvoretzky and Rogers proved in 1950 that in any infinite dimensional Banach space there are unconditionally convergent series not absolutely convergent. This result was the origin of the development of the study of the linear mappings between Banach spaces sending unconditionally summable sequences into absolutely summable sequences (the Theory of absolutely Summing Mappings). The Nonlinear Theory started with Pietsch in 1983, when he presented a few results for scalar multilinear mappings and homogeneous polynomials defined on Banach spaces. In 1989 this author started the study of absolutely summing holomorphic mappings between Banach spaces. In a minicourse, thaught in 1997 at a Seminário Brasileiro de Análise, this author presented several results dated of 1996 and 1997 on nonlinear absolutely summing mappings (not necessarily holomorphic mappings) between Banach spaces. This course included an interesting characterization of regularly summing mappings f between Banach spaces, that is those mappings such that $(f(a+x_j)-f(a))_{j=1}^{\infty}$ is absolutely summable whenever $(x_j)_{j=1}^{\infty}$ is absolutely summable. This result, applied in a convenient setting, implied a nice characterization result for nonlinear absolutely summing mappings, bearing striking similarities to the correspond result for linear absolutely summing mappings. These and other results, as well as historical references, can be found in M. C. Matos, Math. Nachr. 258, 71-89 (2003). In this work we introduce and develop the concept of uniformly regularly summing mappings between Banach spaces, allowing an interesting characterization of the uniformly absolutely summing mappings between Banach spaces, thus extending previous results on nolinear absolutely summing mappings. We also state a relation between uniform regularity and the Lipschitz property for mappings between Banach spaces.

Keywords: Banach spaces, uniformly absolutely summing mappings

MSC 2000 classification: primary 47H60, secondary 46G25, 46E50

1 First definitions and examples

In this work, E e F denote Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and A indicates a non empty open subset of E.

In [5], the concept of regular mapping between Banach spaces allowed the proof of several results on absolutely summing mappings.

1 Definition. For $\rho > 0$, a mapping f from A into F is *ρ-regular at the point* $a \in A$ if there are numbers $M(a) > 0$ and $r(a) > 0$, such that the closed ball $\overline{B}_{r(a)}(a)$ of center a and radius ρ is contained in A and

$$\|f(a+x)-f(a)\|^{\rho} \leq M(a)\|x\|, \qquad \forall x \in E,\ \|x\| \leq r(a).$$

If f is ρ-regular at every point a of B, $B \subset A$, it is said that f *is ρ-regular on* B. The 1-regular mappings at a point a are said to be *regular at* a.

The following extension of this definition will give us interesting results.

2 Definition. For $\rho > 0$, a mapping f from A into F is *uniformly ρ-regular on* $B \subset A$ if there are numbers $M(B) > 0$ and $r(B) > 0$, such that $B + \overline{B}_{r(B)}(0) \subset A$ and

$$\sup_{a \in B} \|f(a+x)-f(a)\|^{\rho} \leq M(B)\|x\|, \qquad \forall x \in E, \|x\| \leq r(B).$$

The uniformly 1-regular mappings on B are said to be *uniformly regular on* B.

3 Example. The function g, defined on $\mathbb{R}$ by

$$g(x) = x \sin \frac{1}{x},\ if\ x \neq 0, \text{ and } g(0) = 0,$$

is regular on $\mathbb{R}$, by the results of [5]. If it were uniformly regular on $[-\epsilon, +\epsilon]$, with $\epsilon > 0$, we would find $r(\epsilon) > 0$ and $M(\epsilon) > 0$, such that

$$\sup_{a \in [-\epsilon,+\epsilon]} |g(a+x)-g(a)| \leq M(\epsilon)|x|, \qquad \forall\ |x| \leq r(\epsilon).$$

This would imply

$$|g'(a)| \leq M(\epsilon), \qquad \forall\ |a| \leq \frac{r(\epsilon)}{2}, a \neq 0.$$

However $g'(a)$ is not bounded when $a \neq 0$ varies in a neighborhood of 0.

4 Proposition. *If $f : A \longrightarrow F$ is Fréchet differentiable on A and the differential $df : A \longrightarrow \mathcal{L}(E;F)$ is locally bounded on A, then f is uniformly regular on each compact subset of A. Here $\mathcal{L}(E;F)$ indicates, as usual, the Banach space of the continuous linear mappings from E into F, with its natural norm of the supremum over B_E = the closed ball in E of center 0 and radius 1. In particular, every mapping from A into F, analytic on A, is uniformly regular on the compact subsets of A.*

Proof. If $K \subset A$ is compact, for each $a \in K$, there is $\delta(a) > 0$, such that $\overline{B}_{\delta(a)}(a) \subset A$ and

$$\sup_{\|x-a\| \leq \delta(a)} \|df(x)\| = M(a) < +\infty.$$

We consider $2r(a) = \delta(a)$ when $a \in K$. There are $a_1, \dots, a_n \in K$, such that

$$K \subset \bigcup_{j=1}^{n} B_{r(a_j)}(a_j).$$

We take the numbers $r(K) = \min\{r(a_j); j = 1, \dots, n\} > 0$ and $M(K) = \max\{M(a_j); j = 1, \dots, n\} > 0$. For each $a \in K$, there is $j \in \{1, \dots, n\}$, such that $\|a - a_j\| < r(a_j)$. Hence, if $\|x\| \leq r(K)$, we have $a + tx \in B_{\delta(a_j)}(a_j)$, for every $t \in [0, 1]$. Thus

$$\sup_{t\in[0,1]} \|df(a + tx)\| \leq M(a_j) \leq M(K).$$

By the Mean Value Theorem, we have

$$\|f(a + x) - f(a)\| \leq \sup_{t\in[0,1]} \|df(a + tx)\|\|x\| \leq M(K)\|x\|.$$

This implies

$$\sup_{a\in K} \|f(a + x) - f(a)\| \leq M(K)\|x\|,$$

for every $\|x\| \leq r(K)$. $\boxed{QED}$

2 Characterization results

We recall the concept of locally Lipschitzian mapping.

5 Definition. For $\rho > 0$, a mapping $f : A \longrightarrow F$ is *locally ρ-Lipschitzian at the point* $a \in A$ if there are $N(a) > 0$ and $\delta(a) > 0$, such that $B_{\delta(a)}(a) \subset A$ and

$$\|f(x) - f(y)\|^\rho \leq N(a)\|x - y\|, \qquad \forall x, y \in B_{\delta(a)}(a).$$

It is said that the mapping f is *locally ρ-Lipschitzian on A* if f is locally ρ-Lipschitzian at each point of A.

6 Theorem. *If f is a mapping from A into F, the following conditions are equivalent:*
(1) f is uniformly ρ-regular on each compact subset of A.
(2) Every $a \in A$ has a neighborhood where f is uniformly ρ-regular.
(3) f is locally ρ-Lipschtzian on A.

PROOF. (1) $\Longrightarrow$ (2).
We suppose the existence of $a \in A$ such that f is not uniformly ρ-regular on any of its neighborhoods. hence, for each $n \in \mathbb{N}$, there is $x_n \in A$, such that $\|x_n - a\| \leq \frac{1}{n}$ and $\|f(x_n) - f(a)\|^\rho > n\|x_n - a\|$. Since $K = \{a\} \cup \{x_n; n \in \mathbb{N}\}$

is compact, $\lim_{n\to\infty} x_n = a$ and $\|f(a+(x_n-a)-f(a)\|^\rho = \|f(x_n)-f(a)\|^\rho > n\|x_n-a\|$, for all $n\in\mathbb{N}$, it is clear that f cannot be uniformly ρ-regular on K.
(2) $\Longrightarrow$ (3).
For $a \in A$, since (2) is true, we have $r(a) > 0$, such that $B_{r(a)}(a)$ is a neighborhood of a where f is uniformly ρ-regular. Hence, there are $\delta(a) > 0$ and $M(a) > 0$, such that $B_{r(a)}(a) + \overline{B}_{\delta(a)}(0) \subset A$ and

$$\sup_{b\in B_{r(a)}(a)} \|f(b+x)-f(b)\|^\rho \leq M(a)\|x\| \qquad \forall x \in \overline{B}_{\delta(a)}(0).$$

If necessary, we can decrease the value of $r(a) > 0$, in such a way that $2r(a) \leq \delta(a)$. In this case we have

$$\|f(w)-f(y)\|^\rho = \|f(y+(w-y))-f(y)\|^\rho \leq M(\rho)\|w-y\| \qquad \forall w,y \in B_{r(a)}(a),$$

because $\|w-y\| \leq \|w-a\|+\|a-y\| \leq 2r(a)$, when $w,y \in B_{r(a)}(a)$. This shows that f is locally ρ-Lipschtzian at a.
(3) $\Longrightarrow$ (1).
Once a compact subset K of A is given, since (3) is true, for each $a \in K$, we can choose $2r(a) > 0$, such that f is ρ-Lipschtzian on $B_{2r(a)}(a)$. Thus there is $M(a) > 0$, in such a way that

$$\|f(w)-f(y)\|^\rho \leq M(a)\|w-y\| \qquad \forall w,y \in B_{2r(a)}(a).$$

We can cover K with a finite number of balls $B_{r(a_j)}(a_j)$, $a_j \in K$, $j = 1,\ldots,n$. Now we consider

$$r(K) = \min\{r(a_j); j=1,\ldots,n\} > 0 \text{ and } M(K) = \max\{M(a_j); j=1,\ldots,n\}.$$

If $b \in K$, there is $j \in \{1,\ldots,n\}$, such that $\|b-a_j\| < r(a_j)$. If $\|x\| \leq r(K)$, since $b, b+x \in B_{2r(a_j)}(a_j)$, we can write

$$\|f(b+x)-f(b)\|^\rho \leq M(a_j)\|x\| \leq M(K)\|x\|.$$

This show that f is uniformely ρ-regular on K. $\boxed{QED}$

In [5] we proved the following theorem.

7 Theorem. *For $p,q \in]0,+\infty[$, a mapping f from A into F is regularly $(p;q)$-summing at the point a of A if, and only if, f is p/q-regular at a.*

We recall the concept of regularly summing mapping as well as the notations used to state that concept.

8 Definition. For $p,q \in]0,+\infty[$, a mapping $f : A \longrightarrow F$ is *regularly $(p;q)$-summing at the point* $a \in A$ if $(f(a+x_j)-f(a))_{j=1}^\infty \in l_p(F)$, when $(x_j)_{j=1}^\infty \in l_q(E)$, with x_j belonging to a fixed neighborhood U of 0, for each $j \in \mathbb{N}$.

9 Definition. For $p \in]0, +\infty[$, a sequence $(x_n)_{n=1}^{\infty}$ of elements of E is *absolutely p-summable*, and this is indicated by $(x_n)_{n=1}^{\infty} \in l_p(E)$, when

$$\|(x_n)_{n=1}^{\infty}\|_p := \left(\sum_{n=1}^{+\infty} \|x_n\|^p\right)^{\frac{1}{p}} < +\infty.$$

We can prove an analogous result to Theorem 7 for uniformly regular mappings on subsets B of A having strictly positive distance to the boundary of A. We need the following concept.

10 Definition. If $p, q \in]0, +\infty[$, a mapping $f : A \longrightarrow F$ is *uniformly regularly $(p; q)$-summing on the subset B of A* if the distance from B to the boundary of A is strictly positive and

$$\sum_{j=1}^{\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^p < +\infty,$$

whenever $(x_j)_{j=1}^{\infty} \in l_q(E)$, with x_j in a fixed neighborhood U of 0, for each $j \in \mathbb{N}$. In the case $p = q$, it is said that f *is uniformly regularly p-summing on B*. When $p = q = 1$, it is said that f *is uniformly regularly summing on B*.

Now we can state our result.

11 Theorem. *If $p, q \in]0, +\infty[$, a mapping f from A into F is uniformly regularly $(p; q)$-summing on a subset B of A, with strictly positive distance to the boundary of A, if , and only if, f is uniformly p/q-regular on B.*

Proof. Since one implication is trivial, we only have to prove the other. We take $r = p/q$. We suppose that f is uniformly regularly (qr, q)-summing on B but it is not uniformly r-regular on B. We may consider $\rho > 0$ such that $B + B_\rho(0) \subset A$ and the condition of Definition 10 is true with $U = B_\rho(0)$. For each $j \in \mathbb{N}$ we can find an $x_j \in E$ such that

$$\|x_j\|^q < \frac{\rho}{j^3} \quad \textit{and} \quad \sup_{a \in B} \|f(a + x_j) - f(a)\|^{rq} > j\|x_j\|^q.$$

Since $(x_j)_{j=1}^{\infty} \in l_q(E)$, we have

$$\sum_{j=1}^{+\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty.$$

Thus,

$$\sum_{j=1}^{+\infty} j\|x_j\|^q \leq \sum_{j=1}^{+\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty.$$

12 Remark. if $(k_j)_{j=1}^{\infty}$ is a sequence of natural numbers such that

$$\sum_{j=1}^{+\infty} k_j \|y_j\|^q < +\infty,$$

then

$$\sum_{j=1}^{+\infty} k_j \sup_{a \in B} \|f(a+y_j) - f(a)\|^{rq} < +\infty.$$

In our case, for the sequence $(x_j)_{j=1}^{\infty}$ we have chosen above, we have

$$\sum_{j=1}^{+\infty} k_j \|x_j\|^q < +\infty \Longrightarrow \sum_{j=1}^{+\infty} jk_j \|x_j\|^q \leq \sum_{j=1}^{+\infty} k_j \sup_{a \in B} \|f(a+x_j) - f(a)\|^{qr} < +\infty.$$

Now, if we apply the remark, with jk_j replacing k_j, we obtain

$$\sum_{j=1}^{+\infty} jk_j \|x_j\|^q < +\infty \Longrightarrow \sum_{j=1}^{+\infty} jk_j \sup_{a \in B} \|f(a+x_j) - f(a)\|^{qr} < +\infty.$$

Finally, we can write

$$\sum_{j=1}^{+\infty} j^2 k_j \|x_j\|^q \leq \sum_{j=1}^{+\infty} jk_j \sup_{a \in B} \|f(a+x_j) - f(a)\|^{qr} < +\infty,$$

whenever $\sum_{j=1}^{+\infty} k_j \|x_j\|^q < +\infty$. We choose

$$k_j = \left[\frac{1}{j^2 \|x_j\|^q}\right] := \sup\left\{m \in \mathbb{N}; m \leq \frac{1}{j^2 \|x_j\|^q}\right\},$$

for each $j \in \mathbb{N}$. Since we have

$$\sum_{j=1}^{+\infty} \left[\frac{1}{j^2 \|x_j\|^q}\right] \|x_j\|^q \leq \sum_{j=1}^{+\infty} \frac{1}{j^2} < +\infty,$$

we get

$$\sum_{j=1}^{+\infty} j^2 \left[\frac{1}{j^2 \|x_j\|^q}\right] \|x_j\|^q < +\infty \tag{*}$$

We have

$$\frac{1}{j^2 \|x_j\|^q} - 1 \leq \left[\frac{1}{j^2 \|x_j\|^q}\right] \leq \frac{1}{j^2 \|x_j\|^q}$$

and, after multiplication by $j^2\|x_j\|^q$,

$$1 - j^2\|x_j\|^q \leq \left[\frac{1}{j^2\|x_j\|^q}\right] j^2\|x_j\|^q \leq 1 \tag{**}$$

We note that x_j was chosen in such a way that $j^2\|x_j\|^q \leq \frac{\varrho}{j}$. Now, if we consider the limit in (**), for j going to ∞, we obtain

$$\lim_{j\to\infty}\left[\frac{1}{j^2\|x_j\|^q}\right] j^2\|x_j\|^q = 1.$$

This is a contradiction to (*). $\boxed{QED}$

In next section we shall have the oportunity to show that this result has nice consequences in the theory of absolutely summing mappings.

13 Corollary. *A mapping f from A into F is uniformly regularly $(p;q)$-summing on the compact subsets of A if, and only if, f is locally p/q-Lipschtzian on A.*

3 Uniformly absolutely summing mappings

We need to recall some concepts and notations of [5].

14 Definition. A sequence $(x_n)_{n=1}^{\infty}$ of elements of E is *weakly absolutely p-summable* if $(< x', x_n >)_{n=1}^{\infty} \in l_p(\mathbb{K}) = l_p$, for every x' in the toplogical dual E' of E. This is indicated by $(x_n)_{n=1}^{\infty} \in l_p^w(E)$.

If $(x_n)_{n=1}^{\infty} \in l_p^w(E)$, an application of the Banach-Steinhaus Theorem implies

$$\|(x_n)_{n=1}^{\infty}\|_{w,p} := \sup_{\|x'\|\leq 1}\left(\sum_{n=1}^{+\infty} | < x', x > |^p\right)^{\frac{1}{p}} < +\infty.$$

15 Definition. A weakly absolutely p-summable sequence $(x_n)_{n=1}^{\infty}$ of elements of E is *unconditionally p-summable* if $\lim_{n\to\infty}\|(x_j)_{j=n}^{\infty}\|_{w,p} = 0$. This is indicated by $(x_n)_{n=1}^{\infty} \in l_p^u(E)$.

In order to justify the term “unconditionally” used in Definition 15, we note that, for $p = 1$, the above definition is equivalent to require the convergence of $\sum_{n=1}^{\infty} x_{\pi(n)}$ in E, for every bijection π from $\mathbb{N}$ onto itself.

16 Definition. For $p, q \in]0, \infty[$, a mapping f from A into F is *absolutely $(p;q)$-summing at the point $a \in A$* if $(f(a + x_j) - f(a))_{j=1}^{\infty} \in l_p(F)$, whenever $(x_j)_{j=1}^{\infty} \in l_q^u(E)$ and $a + x_j \in A$, with x_j in a fixed neighborhood U of 0, for all $j \in \mathbb{N}$. If f is absolutely $(p;q)$-summing at each $a \in A$, it is said that f is *absolutely* $(p;q)$*-summing on* A. If $p=q$ it is usual to say that f is *absolutely* p *-summing on* A. When $p=q=1$ it is said simply that f is *absolutely summing on* A

As we already know from the linear theory, the concepts of regularly $(p;q)$-summing mapping and of absolutely $(p;q)$-summing mapping at a point $a \in A \subset E$, with values in E, coincide if, and only if, E is finite dimensional.

We note that $\|.\|_{w,q}$ defines a norm, if $q \geq 1$, (or a q-norm, if $0<q<1$) on $l_q^u(E)$. Also, $\|.\|_p$ defines a norm, if $p \geq 1$, (or a p-norm, if $0<p<1$) on $l_p(F)$. In any of these cases we obtain a complete metrizable topological vector sapce. We also note that, for each $a \in A$,

$$V_{q,A}(a) := \{(x_j)_{j=1}^{\infty} \in l_q^u(E); a+x_j \in A, \forall j \in \mathbb{N}\}$$

is an open subset of $l_q^u(E)$.

If f is mapping from A into F, absolutely $(p;q)$-summing at a point a of A, there is no loss of generality if we consider $A=a+U$ in Definition 16. In this case we have the natural associate well defined mapping:

$$\psi_{a,p,q}(f) : V_{q,A}(a) \longrightarrow l_p(F).$$

given by

$$\psi_{a,p,q}(f)((x_j)_{j=1}^{\infty}) = (f(a+x_j)-f(a))_{j=1}^{\infty}$$

In [5] we proved the following characterization theorem for absolutely $(p, : q)$-summing mappings.

17 Theorem. *If* f *is a mapping from an open subset of* E *into* F *and* a *is in the domain of* f*, then the following conditions are equivalent:*
(1) f *is absolutely* $(p;q)$*-summing at* a*.*
(2) $\psi_{a,p,q}(f)$ *is a well defined mapping from* $V_{q,A}(a)$ *into* $l_p(F)$*, for some open neighborhood* A *of* a*.*
(3) There are $M>0$ *and* $\delta>0$*, such that*

$$\sum_{j=1}^{n} \|f(a+x_j)-f(a)\|^p \leq M^q \sup_{x' \in B_{E'}} \sum_{j=1}^{n} |<x', x_j>|^q,$$

for all $n \in \mathbb{N}$ *and any* $x_j \in E$*, such that* $\|(x_j)_{j=1}^{n}\|_{w,q} < \delta$*.*
(4) There are $M>0$ *and* $\delta>0$*, such that*

$$\sum_{j=1}^{+\infty} \|f(a+x_j)-f(a)\|^p \leq M^q \sup_{x' \in B_{E'}} \sum_{j=1}^{+\infty} |<x', x_j>|^q,$$

for every $(x_j)_{j=1}^{\infty} \in l_q^u(E)$*, such that* $\|(x_j)_{j=1}^{n}\|_{w,q} < \delta$*.*
(5) $\psi_{a,p,q}(f)$ *is a well defined mapping from* $V_{q,A}(a)$ *(*A *being an open neighborhood of* a*) into* $l_p(F)$*, regularly* $(p;q)$*-summing at 0.*

Now we want to obtain an analogous characterization result for an uniformly absolutely $(p;q)$-summing mapping defined as follows.

18 Definition. For $p, q \in]0, +\infty[$, a mapping $f : A \longrightarrow F$ is *uniformly absolutely* $(p;q)$*-summing on a subset* B *of* A (having strictly positive distance to the boundary of A) if

$$\sum_{j=1}^{\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^p < +\infty,$$

whenever $(x_j)_{j=1}^{\infty} \in l_p^u(E)$, with $B + \{x_j\} \subset A$, x_j in some fixed neighborhood U of 0, for all $j \in \mathbb{N}$. If $p = q$ it is said that f *is uniformly absolutely* p*-summing on* B. When $p = q = 1$, it also said that f *is uniformly absolutely summing on* B.

We want to fix some notations. If I is a non empty set, we indicate by $l_\infty(I;E)$ the vector space of all bounded mappings g from I into E. This space is a Banach space if we consider on it the norm given by $\|g\|_I = \sup_{t \in I} \|g(t)\|$, when $g \in l_\infty(I;E)$. If E is complete r-normed space, then $l_\infty(I;E)$ is a complete r-normed space. We also use the notation $(g(t))_{t \in I}$ for g in $l_\infty(I;E)$.

If f defined in an open subset of E, with values in F is uniformly absolutely $(p;q)$-summing on a subset B of the domain of F (having strictly positive distance to the boundary the domain of f), there is no loss of generality if we consider $A = B + U$ with the notations of Definition 18. In this case we have the following well defined mapping:

$$\psi_{B,p,q}(f) : (x_j)_{j=1}^{\infty} \in V_{q,A}(B) \longrightarrow \psi_{B,p,q}((x_j)_{j=1}^{\infty}) \in l_p(l_\infty(B;F),$$

where $\psi_{B,q}((x_j)_{j=1}^{\infty}) = ((f(b + x_j) - f(b))_{b \in B})_{j=1}^{\infty}$ and

$$V_{q,A}(B) := \{(x_j)_{j=1}^{\infty} \in l_q^u(E); B + x_j \in A, \forall j \in \mathbb{N}\}.$$

We note that $V_{q,A}(B)$ is a neighborhood of the origin of $l_q^u(E)$.

19 Theorem. *Let* f *be an application from* A *into* F *that is uniformly absolutely* $(p;q)$*-summing on* $B \subset A$ *having strictly positive distance to the boundary of* A*. With no loss of generality it is possible to consider* $A = B + U$*, under the notation of Definition 18. Then* $\psi_{B,p,q}(f)$ *is regularly* $(p;q)$*-summing at* $0 \in V_{q,A}(B)$*.*

Proof. We consider $(X_j)_{j=1}^{\infty} \in l_q(l_q^u(E))$, with $X_j = (x_{j,k})_{k=1}^{\infty} \in V_{q,A}(B)$, for every $j \in \mathbb{N}$. We have

$$\sup_{x' \in B_{E'}} \sum_{j,k=1}^{\infty} | < x', x_{j.k} > |^q \leq \sum_{j=1}^{\infty} (\|X_j\|_{w,q})^q < +\infty.$$

This shows that the sequence $(x_{j,k})_{j,k=1}^{\infty}$ is in $l_q^w(E)$. We also have $B+x_{j,k} \in A$, for all $j,k \in \mathbb{N}$. For $\epsilon > 0$, there is $j_0 \in \mathbb{N}$, such that

$$\sum_{j>j_0} (\|X_j\|_{w,q})^q \leq \frac{\epsilon}{2}.$$

Since $X_1, \ldots, X_{j_0} \in l_q^u(E)$, there is $k_0 \in \mathbb{N}$ such that

$$(\|(x_{j,k})_{k>k_0}\|_{w,q})^q \leq \frac{\epsilon}{2j_0} \qquad (\forall j = 1, \ldots, j_0):$$

If $J = \{(j,k) \in \mathbb{N} \times \mathbb{N}; j \leq j_0, k \leq k_0\}$, we have $\left(\|(x_{j,k})_{(j.k)\notin J}\|_{w,q}\right)^q \leq \epsilon$. Thus $(x_{j,k})_{j,k=1}^{\infty}$ is in $l_q^u(E)$. Since f is uniformly absolutely $(p;q)$-summing on B, we obtain

$$\sum_{j=1}^{+\infty} (\|\psi_{B,p,q}(f)(X_j)\|_B)^p = \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} \sup_{a\in B} \|f(a+x_{j,k}) - f(a)\|^p < +\infty.$$

This finishes the proof of our result. $\boxed{QED}$

20 Remark. If we examine the proof of Theorem 11, we see that it can be stated for a complete r-normed space E and a complete s-normed space F. Hence, the conclusion of Theorem 19 is equivalent to state that $\psi_{B,p,q}(f)$ is p/q-regular at 0.

Now we can state the following charaterization result.

21 Theorem. *Let f be an F-valued mapping defined on an open subset of E. Let B be a subset of the domain of f, with strictly positive distance to the boundary of this domain. Then the following conditions are equivalent:*
(1) f is uniformly absolutely $(p;q)$-summing on B.
(2) $\psi_{B,p,q}(f)$ is a well defined mapping from $V_{q,A}(B)$ into $l_p(l_\infty(B;F))$, for some $A = B + U$, U an open neighborhood of 0 in E.
(3) There are $M > 0$ and $\delta > 0$ such that

$$\sum_{j=1}^{n} \sup_{a\in B} \|f(a+x_j) - f(a)\|^p \leq M^q \sup_{x'\in B_{E'}} \sum_{j=1}^{n} |<x', x_j>|^q,$$

for each $n \in \mathbb{N}$ and every $x_j \in E$, $j \in \mathbb{N}$, satisfying $\|(x_j)_{j=1}^n\|_{w,q} < \delta$.
(4) There are $M > 0$ and $\delta > 0$, such that

$$\sum_{j=1}^{+\infty} \sup_{a\in B} \|f(a+x_j) - f(a)\|^p \leq M^q \sup_{x'\in B_{E'}} \sum_{j=1}^{+\infty} |<x', x_j>|^q,$$

for each $(x_j)_{j=1}^{\infty} \in l_q^u(E)$*, sutisfying* $\|(x_j)_{j=1}^{n}\|_{w,q} < \delta$.
(5) $\psi_{B,p,q}(f)$ *is a well defined mapping from* $V_{p,A}(B)$ *into* $l_p(l_\infty(B;F))$*, with* $A = B + U$*,* U *an open neighborhood of 0 in* E*, that is regularly* $(p;q)$*-summing at 0.*

Proof. We note that (2) is a reformulation of (1). It is clear that (5) implies (2). Theorem 19 shows that (1) implies (5). Remark 20 tell us that (4) and (5) are equivalent. The equivalence between (3) and (4) is a simple exercise. $\boxed{QED}$

22 Remark. If in Definition 18 we replace tha condition "$(x_j)_{j=1}^{\infty} \in l_q^u(E)$" by "$(x_j)_{j=1}^{\infty} \in l_q^w(E)$", it is not difficult to see that, with the obvious modifications, Theorem 19, Remark 20 and Theorem 21 are still true. Since the condition (3) of Theorem 21 is the same in both cases, we can say that the two definitions are equivalent.

4 Examples and counter-examples of uniformly absolutely summing mappings

The results of this section show that the existence of uniformly absolutely summing mappings is not a rare phenomenon.

We begin with a polynomial from l_1 into l_2 that is absolutely summing on l_1 but is not uniformly absolutely summing on some compact subset of l_1.

23 Example. As we proved in [5], if f is a mapping from an open subset A of l_1 into l_2 such that $d^2 f$, its Fréchet differential of order 2, is locally bounded on A, then f is absolutely summong on A. In particular, each continuous 2-homogeneous polynomial contínuo from l_1 into l_2 is absolutely summing on l_1. Let S be the bilinear mapping from $l_1 \times l_1$ into l_2 given by $S(e_k, e_j) = e^{i\theta(j,k)} e_j$, for $j, k \in \mathbb{N}$. Here we are considering $\theta(j,k) \in \mathbb{R}$. Hence, for $x, y \in l_1$, we have

$$S(x,y) = \sum_{j=1}^{\infty} y_j \sum_{k=1}^{\infty} x_k e^{i\theta(j,k)} e_j.$$

We can write

$$\|S(x,y)\|_2 = \left(\sum_{j=1}^{\infty} \left| y_j \sum_{k=1}^{\infty} x_k e^{i\theta(j,k)} \right|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{\infty} |y_j|^2 \left(\sum_{k=1}^{\infty} |x_k| \right)^2 \right)^{\frac{1}{2}}$$
$$\leq \|x\|_1 \|y\|_2 \leq \|x\|_1 \|y\|_1,$$

for all $x, y \in l_1$. This shows that S is continuous and has norm equal to 1. For

$(x^{(j)})_{j=1}^{\infty} \in l_1^u(l_1) \setminus l_1(l_1)$ we choose $\theta(j,k)$ such that $e^{i\theta(j,k)} x_k^{(j)} = |x_k^{(j)}|$ for all $j, k \in \mathbb{N}$. Since

$$\sum_{j=1}^{\infty} \|x^{(j)}\|_1 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |x_k^{(j)}| = +\infty,$$

we can find $(c_j)_{j=1}^{\infty} \in c_0$, $c_j \geq 0$, $j \in \mathbb{N}$, such that

$$\sum_{j=1}^{\infty} c_j \left(\sum_{k=1}^{\infty} |x_k^{(j)}| \right) = +\infty.$$

In order to see this it is enough to show that exists $(c_j)_{j=1}^{\infty} \in c_0$ in such a way that the series

$$\sum_{j=1}^{\infty} c_j \left(\sum_{k=1}^{\infty} |x_k^{(j)}| \right)$$

is divergent. If this did not occur, we would have a linear functional on c_0 defined by

$$T(c) = \sum_{j=1}^{\infty} c_j \left(\sum_{k=1}^{\infty} |x_k^{(j)}| \right)$$

for each $c = (c_j)_{j=1}^{\infty} \in c_0$. But, for each $n \in \mathbb{N}$, we know that the linear functional T_n, defined on c_0 by

$$T_n(c) = \sum_{j=1}^{n} c_j \left(\sum_{k=1}^{n} |x_k^{(j)}| \right)$$

for $c = (c_j)_{j=1}^{\infty} \in c_0$, is continuous. Since $(T_n)_{n=1}^{\infty}$ is pointwise convergent to T on c_0, a consequence of the Banach-Steinhaus Theorem implies that T is continuous. By the characterization of the topological dual of c_0, we must have

$$\sum_{j=1}^{\infty} \|x^{(j)}\|_1 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |x_k^{(j)}| < +\infty.$$

But we know that this is not true.

We consider $P(x) = S(x,x)$, for $x \in l_1$, where S is defined in terms of the $\theta(j,k)$ chosen as above. We are going to show that the 2-homogeneous polynomial P is not uniformly absolutely summing on the compact set $K = \{c_j e_j; j \in \mathbb{N}\} \cup \{0\}$. We note that

$$\begin{aligned}\|P(c_j e_j + x^{(j)}) - P(c_j e_j)\|_2 &= \|S(c_j e_j, x^{(j)}) + S(x^{(j)}, c_j e_j) + P(x^{(j)})\|_2 \\ &\geq \|S(x^{(j)}, c_j e_j)\|_2 - \|S(c_j e_j, x^{(j)}) + P(x^{(j)})\|_2\end{aligned}$$

$$\geq \left|\sum_{k=1}^{\infty} x_k^{(j)} c_j e^{i\theta(j,k)}\right| - \left(\sum_{n=1}^{\infty} |c_j x_n^{(j)}|^2\right)^{\frac{1}{2}} - \|P(x^{(j)})\|_2.$$

Hence we have

$$\sum_{j=1}^{\infty} \sup_{a\in K} \|P(a + x^{(j)}) - P(a)\|_2 \geq$$

$$\geq \sum_{j=1}^{\infty} \left(\left|\sum_{k=1}^{\infty} x_k^{(j)} c_j e^{i\theta(j,k)}\right| - \left(\sum_{n=1}^{\infty} |c_j x_n^{(j)}|^2\right)^{\frac{1}{2}}\right) - \sum_{j=1}^{\infty} \|P(x^{(j)}\|_2.$$

But

$$\sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} x_k^{(j)} c_j e^{i\theta(j,k)}\right| = \sum_{j=1}^{\infty} c_j \left(\sum_{k=1}^{\infty} |x_k^{(j)}|\right) = +\infty$$

and

$$\sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} |c_j x_n^{(j)}|^2\right)^{\frac{1}{2}} = \sum_{j=1}^{\infty} c_j \|x^{(j)}\|_2 < +\infty,$$

since $(c_j)_{j=1}^{\infty} \in c_0$ and $(x^{(j)})_{j=1}^{\infty} \in l_1(l_2)$ (the inclusion mapping from l_1 into l_2 is absolutely summing). Since P is absolutely summing at the origin, we have

$$\sum_{j=1}^{\infty} \|P(x^{(j)}\|_2 < +\infty.$$

Thus

$$\sum_{j=1}^{\infty} \sup_{a\in K} \|P(a + x^{(j)}) - P(a)\|_2 = +\infty,$$

and P cannot be uniformly absolutely summing on K.

24 Proposition. *If g is an absolutely p-summing linear mapping from E into F and f is an uniformly regularly p-summing mapping on each compact subset of an open subset B of F with values in a Banach space G, then $f \circ g$ is uniformly absolutely p-summing on each compact subset of the open subset $A = g^{-1}(B)$ of E.*

The proof is simple since $(g(x_j))_{j=1}^{\infty} \in l_p(F)$, whenever $(x_j)_{j=1}^{\infty} \in l_p^u(E)$.

25 Example. **(1)** Let p be in $[1, 2]$ and T be a continuous linear mapping from c_0 into l_p. If f is a mappinga defined on an open subset B of l_p, with values in F such that *df* is locally bounded on B, then $f \circ T$ is uniformly absolutely 2-summing on each compact subset of $A = T^{-1}(B)$. In particular, if f is analytic on B, $f \circ T$ is uniformly absolutely 2-summing on each compact subset of A.

This follows from Proposition 24 since T is absolutely 2-summing (see [4]) and f is uniformly regularly 2-summing by Proposition 4 and Theorem 11.

(2) For $2 < p < r < +\infty$, let T be a continuous linear mapping from c_0 into l_p. If f is a mappinga defined on an open subset B of l_p, with values in F such that df is locally bounded on B, then $f \circ T$ is uniformly absolutely r-summing on each compact subset of $A = T^{-1}(B)$. In particular, if f is analytic on B, $f \circ T$ is uniformly absolutely r-summing on each compact subset of A.

This follows from Proposition 24 since T is absolutly r-summing (see [3] and [7]) and f is uniformly regularly r-summing on the compact subsets of A by Proposition 4 and Theorem 11.

(3) For $2 \leq p < +\infty$, if E has cotype p and f is a mapping defined on an open subset B of E with values in F, such that df is locally bounded on B, then f is uniformly absolutely $(p;1)$-summing on the compact subsets of B.

This follows from Proposition 24 since id_E is absolutely $(p;1)$-summing and f is uniformly regularly p-summing on the compact subsets of B by Proposition 4 and Theorem 11.

Example 23 shows that there are continuous polynomials from l_1 into l_2 that are not uniformly absolutely summing on the compact subsets of L_1. The above result implies that the continuous polynomials from l_1 into l_2 or, more generally, the mappings f defined on open subsets B of l_1, with values in l_2, having df locally bounded on B, are uniformly absolutely $(2;1)$-summing on the compact subsets of l_1.

5 Examples dealing with holomorphic mappings

In this section all Banach spaces are considered over the field $\mathbb{C}$. Also, we use the usual notations of the Infinite Dimensional Holomorphy Theory (see Nachbin [6]).

We describe the generalized de Rademacher functions introduced by Aron and Grobevnik [1]. For a fixed $n \in \mathbb{N}$, $n \geq 2$, we take the n-th roots of the unity $1 = \lambda_1, \lambda_2, \ldots, \lambda_n$, considered in the increasing order of the principal values of their arguments. The closed interval $[0,1]$ is divided in n intervals of equal lenght $I_1, \ldots, I_n$, written in the order they appear from left to right. We consider the complex function $s_1^{(n)}$, defined on $[0,1]$ as follows: $s_1^{(n)}(t) = \lambda_j$, for t in the interior of I_j and $s_1^{(n)}(t) = 1$, if t is one of the extremities of I_j, $j = 1, \ldots, n$. If $k \geq 1$, we considered defined the functions $s_1^{(n)}, \ldots, s_k^{(n)}$ and we construct the function $s_{k+1}^{(n)}$ in the following way. Each interval J used in the definition of $s_k^{(n)}$ is divided in n intervals of equal lenght $J_1, \ldots, J_n$, written in the order they appear from left to right. Now we define $s_{k+1}^{(n)}$ by $s_{k+1}^{(n)}(t) = \lambda_j$, if t is in the interior of J_j, and $s_{k+1}^{(n)}(t) = 1$, when t is one of the extremities of J_j, $j = 1, \ldots, n$. since $\sum_{j=1}^{n} \lambda_j^m = 0$ for each $m < n$, we have

$$\int_0^1 s_{j_1}^{(n)}(t) \ldots s_{j_n}^{(n)}(t)dt = \delta_{j_1,\ldots,j_n},$$

where $\delta_{j_1,\ldots,j_n} = 1$, if $j_1 = \cdots = j_n$, and $\delta_{j_1,\ldots,j_n} = 0$ in the other cases. We say that $(s_j^{(n)})_{j=1}^{\infty}$ is the *sequence of the Rademacher n-functions.*

26 Theorem. *If f is a holomorphic function on an open subset A of E, with values in $\mathbb{C}$, then f is uniformly absolutely summing on each compact subset of A.*

Proof. For a given compacto subset K of A, we know that there is $\delta > 0$ such that

$$f(a + x) - f(a) = \sum_{k=1}^{+\infty} \frac{\hat{d}^k f(a)}{k!}(x),$$

uniformly for $a \in K$ and $x \in \overline{B}_r(0)$, whenever $0 < r < \delta$. Moreover, we may suppose that

$$M(r) = \sup\{|f(z)| : z \in K + \overline{B}_r(0)\} < +\infty.$$

In order to simplify our notations we write $P_k(a) = \hat{d}^k f(a)/k!$. The above convergence and the Cauchy Integral Formulas imply

$$\sup_{a\in K} \|P_k(a)\| = \sup_{a\in K} \sup_{\|x\|\leq 1} \|P_k(a)(x)\| \leq M(r)\frac{1}{r^k},$$

for each $k \in \mathbb{N}$. We consider $x_1, \ldots, x_m \in E$, such that $\|(x_j)_{j=1}^m\|_{w,1} \leq r/2 \leq \delta/4$. We have

$$\sum_{j=1}^{m} |f(a + x_j) - f(a)| = \sum_{j=1}^{m} \left|\sum_{k=1}^{\infty} P_k(a)(x_j)\right|$$

$$\leq \sum_{k=1}^{\infty}\sum_{j=1}^{m} |P_k(a)(x_j)| \leq \sum_{k=1}^{\infty} \left|\sum_{j=1}^{m} P_k(a)(\alpha_{k,j}x_j)\right|$$

where $\alpha_{k,j}$ is a complex number of absolute value 1 such that $\alpha_{k,j}P_k(a)(x_j) = |P_k(a)(x_j)|$. Now we note that

$$\left|\sum_{j=1}^{m} P_k(a)(\alpha_{k,j}x_j)\right| = \left|\int_0^1 P_k(a)\left(\sum_{j=1}^{m} \alpha_{k,j}s_j^{(k)}(t)x_j\right) dt\right|$$

$$\leq \|P_k(a)\| \sup_{|t|=1} \left\| \sum_{j=1}^{m} \alpha_{k,j} s_j^{(k)}(t) x_j \right\|^k .$$

In the first equality of the above expression we used the multilinearity of the k-linear mapping associate to $P_k(a)$ and the properties of the Rademacher k-functions. Since

$$\sup_{|t|=1} \left\| \sum_{j=1}^{m} \alpha_{k,j} s_j^{(k)}(t) x_j \right\|^k \leq \left(\|(x_j)_{j=1}^m\|_{w,1}\right)^k ,$$

we can write

$$\sum_{j=1}^{m} \sup_{a\in K} |f(a+x_j) - f(a)| \leq \sum_{k=1}^{+\infty} \sup_{a\in K} \|P_k(a)\| \left(\|(x_j)_{j=1}^m\|_{w,1}\right)^k$$

$$\leq \sum_{k=1}^{+\infty} M(r) r^{-k} \left(\|(x_j)_{j=1}^m\|_{w,1}\right)^k$$

$$\leq r^{-1} M(r) \|(x_j)_{j=1}^m\|_{w,1} \sum_{k=0}^{+\infty} r^{-k} \left(\|(x_j)_{j=1}^m\|_{w,1}\right)^k .$$

Since $\|(x_j)_{j=1}^m\|_{w,1} \leq r/2 \leq \delta/4$, we have

$$\sum_{j=1}^{m} \sup_{a\in K} |f(a+x_j) - f(a)| \leq 2r^{-1} M(r) \|(x_j)_{j=1}^m\|_{w,1}.$$

By Theorem 21, our result is proved. $\boxed{QED}$

The proof of the preceding result also shows the following theorem.

27 Theorem. *If f is a holomorphic function on an open subset A of E with complex values, then, for each compact subset K of A there is $r > 0$ such that*

$$\sup_{a\in K} \sum_{j=1}^{+\infty} |f(a+x_j) - f(a)| \leq 2r^{-1} \sup_{z\in K+\overline{B}_r(0)} |f(z)| \|(x_j)_{j=1}^\infty\|_{w,1},$$

whenever $(x_j)_{j=1}^\infty \in l_1^u(E)$ and $\|(x_j)_{j=1}^\infty\|_{w,1} \leq \dfrac{r}{2}$.

28 Proposition. *If f is a holomorphic maping on an open subset A of E, with values in F, then, for each $a \in E$, we have $(f(a+x_j) - f(a))_{j=1}^\infty \in l_1^u(F)$, whenever $(x_j)_{j=1}^\infty \in l_1^u(E)$ and $a + x_j \in A$, x_j in some fixed neighborhood U of 0, $j \in \mathbb{N}$.*

PROOF. for each $\phi \in F'$, we have $\phi \circ f$ absolutly summing on A by Theorem 26. By Theorem 27, we have

$$\sum_{j=1}^{+\infty} |\phi \circ f(a + x_j) - \phi \circ f(a)| \leq 2r^{-1} \sup_{|z-a|=r} |\phi \circ f(z)| \|(x_j)_{j=1}^{\infty}\|_{w,1},$$

whenever $(x_j)_{j=1}^{\infty} \in l_1^u(E)$ and $\|(x_j)_{j=1}^{\infty}\|_{w,1} \leq \dfrac{r}{2}$. Thus, for each $\|\phi\| \leq 1$, we can write

$$\sum_{j=1}^{+\infty} |\phi \circ f(a + x_j) - \phi \circ f(a)| \leq 2r^{-1} \sup_{|z-a|=r} \|f(z)\| \|(x_j)_{j=1}^{\infty}\|_{w,1},$$

whenever $(x_j)_{j=1}^{\infty} \in l_1^u(E)$ and $\|(x_j)_{j=1}^{\infty}\|_{w,1} \leq \dfrac{r}{2}$. Now our thesis follows easily from this inequality. $\boxed{QED}$

29 Theorem. *Let $f : A \longrightarrow F$ be holomorphic on A and let g be a maping from an open subset B of F, containing $f(A)$, with values in a Banach space$G$, that is uniformly absolutely $(p;1)$-summing on each compact subset of B. Then $g \circ f$ is uniformly absolutely $(p;1)$-summing on each compact subset of A.*

PROOF. Let K be a acompact subset of A and $(x_j)_{j=1}^{\infty} \in l_1^u(E)$ such that $K + \{x_j\} \subset A$, $j \in \mathbb{N}$. By Proposition 28, we have $(f(a+x_j) - f(a))_{j=1}^{\infty} \in l_1^u(F)$, for each $a \in K$ and x_j in some fixed open neighborhood U of 0, for all $j \in \mathbb{N}$. Thus, since $f(K)$ is compact and g is uniformly absolutely $(p;1)$-summing over $f(K)$, we have

$$\sum_{j=1}^{\infty} \sup_{a \in K} \|g \circ f(a + x_j) - g \circ f(a))\|^p < +\infty,$$

because $g \circ f(a + x_j) - g \circ f(a)) = g(f(a) + (f(a + x_j) - f(a)) - g(f(a))$. $\boxed{QED}$

30 Definition. A holomorphic mapping f on E, with values in F,is said to be *exponentially p-dominated* if there are $C \geq 0$, $c \geq 0$, $\mu \in W(B_{E'})$, such that

$$\|f(x)\| \leq C \exp\left[c\left(\int_{B_{E'}} |\phi(x)|^p d\mu(\phi)\right)^{\frac{1}{p}}\right] \qquad \forall x \in E.$$

Here $W(B_{E'})$ indicates the set of all regular probabilities measures on the Borel subsets of $B_{E'}$ endowed with the weak star topology.

31 Theorem. *Every exponentially p-dominated mapping from E into F is uniformly absolutely p-summing on the bounded subsets of E.*

We need a preliminary result.

32 Theorem. *Let f be exponentially p-dominated from E into F with*

$$\|f(x)\| \leq C \exp\left[c\left(\int_{B_{E'}} |\phi(x)|^p d\mu(\phi)\right)^{\frac{1}{p}}\right] \qquad \forall x \in E.$$

For each bounded subset B of E and each $\epsilon > 0$ there is $M(B,\epsilon) \geq 0$ such that

$$\sup_{a\in B}\left\|\hat{d}^k f(a)(x)\right\| \leq M(B,\epsilon)(c+\epsilon)^k\left(\int_{B_{E'}} |\phi(x)|^p d\mu(\phi)\right)^{\frac{k}{p}} \qquad \forall x \in E, k \in \mathbb{N}.$$

Proof. With no loss of generality we may consider $B = \overline{B}_r(0)$, for some $r > 0$. for each $\rho > 0$ and $a \in E$ we can write

$$\left\|\frac{\hat{d}^k f(a)}{k!}(x)\right\| = \left\|\frac{1}{2\pi i}\int_{|t|=\rho}\frac{f(a+tx)}{t^{k+1}}dt\right\| \leq \frac{1}{\rho^k}\sup_{|t|=\rho}\|f(a+tx)\|$$

$$\leq \frac{1}{\rho^k}\sup_{|t|=\rho} C\exp\left[c\left(\int_{B_{E'}} |\phi(a+tx)|^p d\mu(\phi)\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{1}{\rho^k} C\exp(c\|a\|)\exp\left[c\rho\left(\int_{B_{E'}} |\phi(x)|^p d\mu(\phi)\right)^{\frac{1}{p}}\right].$$

In order to simplify our notations we consider

$$\nu_p(x) = \left(\int_{B_{E'}} |\phi(x)|^p d\mu(\phi)\right)^{\frac{1}{p}}.$$

By the previous inequalities, if $\nu_p(x) \neq 0$, we have

$$\left\|\frac{\hat{d}^k f(a)}{k!}\left(\frac{x}{\nu_p(x)}\right)\right\| \leq \frac{1}{\rho^k} C\exp(c\|a\|)\exp(c\rho),$$

for each $a \in E$ and $\rho > 0$. If we choose $c\rho = k$, we obtain

$$\sup_{\|a\|\leq r}\left\|\hat{d}^k f(a)\left(\frac{x}{\nu_p(x)}\right)\right\| \leq \frac{k!c^k}{k^k} C\exp(cr)\exp(k). \tag{*}$$

From Sterling's formula we have

$$\lim_{k\to\infty}\frac{k!}{k^k\exp(-k)\sqrt{2\pi i}} = 1.$$

Now, by extracting the k-th root of the two sides of (*) and passing to the limit superior for k tending to ∞, we have

$$\limsup_{k\to\infty}\left(\sup_{\|a\|\le r}\left\|\hat{d}^k f(a)\left(\frac{x}{\nu_p(x)}\right)\right\|^{\frac{1}{k}}\right)\le c,$$

for $\nu_p(x)\neq 0$. Thus, for each $\epsilon>0$, there is $M(r,\epsilon)\ge 0$ such that

$$\|\hat{d}^k f(a)(x)\|\le M(r,\epsilon)(c+\epsilon)^k(\nu_p(x))^k,$$

for each $a\in\overline{B}_r(0)$, $x\in E$ and $k\in\mathbb{N}$. Hence the theorem is proved. $\boxed{QED}$

Proof of Theorem 31. By Theorem 32, if $(x_j)_{j=1}^\infty\in l_p^u(E)$ and $B\subset E$ is bounded, we can write

$$\begin{aligned}
\left(\sum_{j=1}^\infty\sup_{a\in B}\|\hat{d}^k f(a)(x_j)\|^p\right)^{\frac{1}{p}} &\le M(B,\epsilon)(c+\epsilon)^k\left(\sum_{j=1}^\infty(\nu_p(x_j))^{kp}\right)^{\frac{1}{kp}k}\\
&\le M(B,\epsilon)(c+\epsilon)^k\left(\sum_{j=1}^\infty(\nu_p(x_j))^{p}\right)^{\frac{k}{p}}\\
&\le M(B,\epsilon)(c+\epsilon)^k\left(\int_{B_{E'}}\sum_{j=1}^\infty|\phi(x_j)|^p d\mu(\phi)\right)^{\frac{k}{p}}\\
&\le M(B,\epsilon)(c+\epsilon)^k\left(\|(x_j)_{j=1}^\infty\|_{w,p}\right)^k.
\end{aligned}$$

Now we have:

$$\begin{aligned}
\left(\sum_{j=1}^\infty\sup_{a\in B}\|f(a+x_j)-f(a)\|^p\right)^{\frac{1}{p}} &\le \left(\sum_{j=1}^\infty\left(\sum_{k=1}^\infty\sup_{a\in B}\left\| frac1k!\hat{d}^k f(a)(x_j)\right\|\right)^p\right)^{\frac{1}{p}}\\
&\le\sum_{k=1}^\infty\frac{1}{k!}M(B,\epsilon)(c+\epsilon)^k\left(\|(x_j)_{j=1}^\infty\|_{w,p}\right)^k\\
&\le M(B,\epsilon)\exp\left((c+\epsilon)\|(x_j)_{j=1}^\infty\|_{w,p}\right)<+\infty.
\end{aligned}$$

This proves our result. $\boxed{QED}$

We also have the following result.

33 Proposition. *A holomorphic mapping f from E into F is exponentially p-dominated, if, and only if, there are $D\ge 0$, $d\ge 0$ and $\mu\in W(B_{E'})$ such that*

$$\|\hat{d}^k f(0)(x)\|\le Dd^k\left(\int_{B_{E'}}|\phi(x)|^p d\mu(\phi)\right)^{\frac{k}{p}}\qquad\forall k\in\mathbb{N}, x\in E.$$

Proof. One implication follows from Theorem 32 with $B = \{0\}$. The reverse implication is imediate. QED

34 Corollary. *Every p-dominated k-homogeneous polynomial from E into F is uniformly absolutely p-summing on the bounded subsets of E.*

Proof. We recall that a k-homogeneous polynomial P from E into F is p-dominated if there are $D \geq 0$, and $\mu \in W(B_{E'})$ such that

$$\|P(x)\| \leq D \left(\int_{B_{E'}} |\phi(x)|^p d\mu(\phi) \right)^{\frac{k}{p}} \qquad \forall k \in \mathbb{N}, x \in E.$$

From Corollary 34 it follows that P is exponentially p-dominated. Hence, by Theorem 31, P is uniformly absolutely p-summing on the bounded subsets of E. QED

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