

Quantum gravity and “singularities”

Anastasios Mallios

*Department of Mathematics, University of Athens,
Greece*
amallios@math.uoa.gr

Abstract. It is our aim in the present article to point out that the phenomenal disagreement of the classical differential geometry (CDG, viz. differential geometry of smooth manifolds), when applied in the quantum domain (quantum field theory, for instance) is just due to the entanglement in that procedure of the underlying \mathcal{C}^∞ (: smooth)-manifold, hence, the appearance of the so-called “*singularities*”, while the inherent, quite algebraic, in character, *differential-geometric mechanism* of the same classical theory still remains intact, without having to resort to any carrier space at all (!), supporting that mechanism, as it was the case, so far. Therefore, a potential application of the same, even in the quantum régime.

Keywords: quantum gravity, singularities, relativistic quantum mechanics, abstract differential geometry (ADG), “physical geometry”, vector sheaf, Yang-Mills field, ADG-formalism, second quantization, Einstein’s equation (in vacuo), Yang-Mills equations, sheaf cohomology

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Dedicated to the memory of the late Professor Klaus Floret

1 Introduction

The two issues in the title of this article are only, *seemingly* (!), *different*, while, as we shall see, they are, in effect, in a very concrete sense, quite tautosemous. So, when we try to *quantize gravity*, we are inevitably confronted (cf. e.g. (3.14) below) with the so-called, thus far, “*singularities*”, that is, with the emerging “*infinities*” etc (referred, of course, always to *our* (!) calculations), something, that certainly remind us of the characteristic, in that context, relevant remarks of P.A.M. Dirac, already from 1975, see thus [7, p. 36], in that;

“...sensible mathematics involves neglecting a quantity when it turns out to be small not neglecting it just because it is infinitely (1.1) great and you do not want it”.

or even P.A.M. Dirac [8, p. 85], that:

(1.1') “Some day a *new quantum mechanics, a relativistic one*, will be discovered, in which *we will not have ... infinities at all* [(!).”

[Emphasis above is ours; this will also be, in principle, the case when referring to quotations, throughout the present work, unless otherwise is stated]. Accordingly, by coming back to our subject,

(1.2) *the first item, as in the title of this paper, is, in point of fact, reducible to the second one, or, equivalently, the famous problem of the quantization of gravity is virtually subject to that one of the so-called “singularities”.*

In this connection, we can still refer to the criticism of R.P. Feynman thereof [10, p. 166], pertaining thus to the use of the notion of “continuum” in the quantum deep, in that:

(1.3) *“...the theory that space is continuous is wrong, because we get ... infinities and other difficulties ... [while] the simple ideas of geometry, extended down into infinitely small ... are wrong!”*

Now, the problem here lies essentially with our blocked intension/endeavour to associate our (technical) theory (: “geometry”) with physis (: natural laws). In this regard, see also our previous relevant comments in A. Mallios [19], or even in [22]. On the other hand, within the above vein of ideas, we can also quote here C.J. Isham [15, p. 393], when remarking that;

(1.4) *“... at the Plank-length scale, classical differential geometry is simply incompatible with quantum theory ... [hence] one will not be able to use differential geometry in the true quantum gravity theory ...”*

Here again, as it was similarly the case with (1.3), *the problem is not with the (classical) differential geometry itself, even at the Plank-length scale (!)*, when mathematically speaking (there are no, in effect, such inherent restrictions on the Riemannian/Lorentzian, or on any other, whatsoever, type of “metric-geometry”, by its very definition, pertaining to the “metric”, read, to the “space”, we use), *but only with the extent to which we wish, in that context, to apply the framework (!) of that classical theory, as a model, for a mathematical-physical theory, to describe thus a physical situation; viz. the quantum domain, alias the physical laws governing that particular (physical) régime. So it is worthwhile to point out here, once more, that,*

(1.5) *the manner we try to apply, so far, the classical differential geometry (CDG) always refers to its standard framework, viz. to the theory of differential (i.e., smooth, or even C^∞ -)manifolds, and not (!), to its inherent (“leibnizian”, so to say) mechanism, as the latter aspect has been just exhibited, by the “abstract differential geometry” (ADG); the same*

still affords, as we shall see, a quite different perspective from that one of the classical case, concerning thus potential applicabilities of ADG, provided we have also suitably chosen, so to say, our “differentiable functions” (: “*generalized arithmetics*”, in the latter context; cf., for instance, (3.10) in the sequel).

The above diversifications from standard aspects, so far, of the same matter, will become progressively clearer, through the subsequent discussion.

Now, continuing further, within the previous point of view, as it concerns the applicability of the notion of the “*continuum*” (: space-time) in problems of quantum gravity we can still quote here A. Einstein himself, who, since 1916, already, has declared, indeed, in a pretty caustic manner, that;

(1.6) “... *continuum space-time* ... should be banned from theory as a *supplementary construction* not justified by the essence of the problem— a construction *which corresponds to nothing real* [(!)”.

See, for instance, J. Stachel [33, p. 280]. So we have actually been warned, already, either directly (Einstein), or indirectly (Feynman, Isham) for the *inappropriateness of combining classical differential geometry with quantum theory* (!).

On the other hand, R. Geroch (1968), trying to explain the situation one has with the “*singularities*” in general relativity, he further notes that;

(1.7) “... *general relativity differs* from [other field theories] *in one important respect*: ... one has [in those theories] a background (Minkowskian) metric to which the field quantities can be referred, [while] in general relativity *the “background metric” is the very field* whose singularities one wishes to describe”.

See R. Geroch [12]. Furthermore, we have a recent similar criticism to (1.7), as above, by J. Baez [1, p.v; Preface], as it actually concerns *quantization of gravity*, by remarking that:

(1.8) “A *fundamental problem with quantum gravity* ... is that *in ... general relativity there is no background geometry* to work with: the *geometry of spacetime itself becomes a dynamical variable*.”

Consequently, as an upshot of the preceding discussion, we do effectuate that;

(1.9) the fact that in *general relativity* one is compelled, by the very essence of the theory, to consider the “*geometry*” itself, as a “*dynamical variable*”, is a fundamental issue (problem) in *quantizing gravity*, the same being also intimately connected with the so-called “*singularities*” of the theory.

2 ADG, as a potential response

It is now our goal, by the subsequent account, to show that the afore-said, throughout the preceding discussion, *obstacles*, which appear when trying to cope with problems of *quantum gravity*, within the standard set-up (: differential–smooth–manifolds) of the classical differential geometry, *do not appear, at all*, when looking at the matter, *within the context of ADG* (: *abstract differential geometry*), according to the very definition of the latter: Indeed, it is thus a *basic moral* of the same point of view (ADG), that;

(2.1) to perform “*differential geometry*”, no “*space*” is virtually required (in the usual sense of the standard theory (CDG), viz. a smooth manifold), provided that one is equipped with a “*basic differential*”, ∂ , alias “*dx*”, along with the appropriate “*differential-geometric mechanism*”, that might be afforded thereby.

Thus, it is still a basic upshot of the *very context of ADG* (see also (2.1), along with (3.13) in the sequel) that *the problem* (see e.g. (1.8), as well as, (1.9), as above) of

(2.2) making the “*geometry*” into a “*dynamical variable*” is simply *begging the question*!

In this regard see also A. Mallios-I. Raptis [23], [24], [25]. On the other hand, continuing further, within the preceding vein of ideas, by turning back again to the situation, that is connected with the so-called “*Plank-length scale*” (cf. (1.4) above, along with the ensuing comments therein), we can still remark, yet, here too, by virtue of the same character of ADG (see also, for instance, A. Mallios [19, (9.34), and comments following it]), that,

(2.3) the (*physical*) “*geometry*”, one has in the “*Plank-length scale*” does not actually differ, in principle, as well as, in substance (nature), from that one, we have, anywhere else (“*physis is united*” (!), we suppose); yet, *nature*, viz. the *physical* “*geometry*”, still, in other words, what we perceive, as such, *is not, at all, our own*. Indeed, *the latter term* (: “*geometry*”) concerns, in point of fact, *simply* our own *technical* (!) (: *mathematical device* (in effect, “*cartesian*” (!), thus far) to describe (: to *model*) the former.

In this regard, cf. also here A. Mallios [22]. Indeed, as it was already hinted at in (2.3), we should still remark herewith, that:

(2.4) what we usually understand as (mean by a), “*physical geometry*”—we are thus *trapped* still, *by our own mathematical conception* of it, in that context—is, in point of fact, the “*cartesian*” one, either *globally* (e.g. *affine space*), or even *locally* (thus, manifold, e.g. the so-called “*space-time*”).

In this connection, see also A. Mallios [19, (8.5)]. Consequently,

what we actually perceive (: define), as “*space*”, is that one, which, in effect, may be called “*cartesian*” (or even, “*newtonian*”) one, hence, not, in anyway, the real “*physical*” one, which we may still name “*euclidean*” (see also loc. cit., as above). So the latter is, in point of fact, simply,

(2.5) (2.5.1) *that, what constitutes it (!),*

loc.cit.; (1.1), (1.4); viz., in other words, that, what we can still call,

(2.5.2) *“les objets géométriques”,*

in the sense of Leibniz (the same Ref.; (2.1)).

On the other hand, as already explained elsewhere (cf., for instance, the same Ref., as above), ADG is exactly referred to these “*objets géométriques*”, à la Leibniz, as before, the same being also, of course, the “*varying objects*”. Thus, ADG appears too to be *in accord with* the point of view of *general relativity*, let alone, without any need to resort to a particular “*background geometry (space)*”, to work with (!) (see, for example, (1.8) in the preceding). Indeed,

(2.6) *the (differential-)geometric mechanism*, in the formalism of ADG, does not depend on (emanate from) any “*background space*”, in the sense that the latter term is, at least, understood in the classical theory (CDG).

In this regard, see also A. Mallios [19, in particular, (6.1), or even (9.8), therein], yet, A. Mallios [22]. We further elaborate on (explain) our previous comments, as in (2.6) above, in a more technical manner, straightforwardly, by the subsequent Section.

3 ADG, technically speaking

As already hinted at, just before, we come now, by the subsequent discussion, to sustain the aspect, this being also another *fundamental issue* of the whole formalism of ADG, that:

(3.1) *within the setting of ADG, the “geometry” itself, in the sense that this notion is really understood, in that context (cf., for example, (3.2), as well as, (3.3) in the sequel) is already, by its very definition (ibid.), a “dynamical variable”, therefore, by itself, of a relativistic nature (!), while, as we shall also see, by the ensuing discussion, it still appears, as such, thus far, concerning our relevant equations, as well (cf. thus Section 5 in the sequel).*

Thus, to start with, by referring to “geometry”, within the *framework of ADG*, one virtually means the construction of a “*geometrical calculus*”, just, quite in the sense of Leibniz (see, for example, A. Mallios [19, beginning of Section 1]), referring thus exclusively to *interrelations* of what one considers, herewith, as “*geometrical objects*”, hence, for the case in focus, of the “*vector sheaves*” involved; therefore, by employing physical terms, the aforesaid “*calculus*” (being, as it is actually defined (!), of a “*differential-geometric*” character, in the classical sense of the latter term) *refers directly to the “elementary particles”* (alias, “*fields*”) themselves (see also (3.2) below). So, up to this point, we virtually consider the following “*identifications*”:

$$(3.2) \quad \begin{array}{l} \text{“geometric object”} \longleftrightarrow \text{vector sheaf} \\ \quad \quad \quad \longleftrightarrow \text{elementary particle} \\ \quad \quad \quad \longleftrightarrow \text{“field”}. \end{array}$$

Indeed, as we shall see, right below (cf. thus (3.3) in the sequel), the above will be appropriately supplemented, when further applying physical terminology. In this regard, see also A. Mallios [22], for a fuller account of the nowadays notion of “*geometry*”, yet, in perspective with *physics*.

On the other hand, the term “*interrelation*”, as applied in the foregoing, means, by its very definition, a *morphism* between the respective *sheaves* (alias, a “*sheaf-morphism*”, the most important of all, when, in particular, referring to a “*differential-geometric*” *syllabus*, within the relevant setup, being, what we call, an *A-connection* (cf. (3.5) in the sequel). In point of fact, *this* particular *morphism* appears, as we shall see (cf. thus (5.1), or even (5.19) in the sequel), *within the pertinent equations*, in the form, as we say, of an “*invariant morphism*”, something, of course, of *paramount physical importance* (cf., for instance, “*gauge principle*”); yet, technically speaking, in the form of an “*A-invariant morphism*”, which, for the case at issue, is the respective *curvature* (: “*field strength*”) of the *A-connection* concerned (loc. cit.).

Thus, to put the above into a better perspective, explaining also, at the same time, the previously applied terminology, we come first, as already promised, for that matter, to the following amendment (: supplement) of our previous

schematic version of the inherent situation herewith, as described in the preceding, at first sight, by (3.2). So one gets at the following associations (: identifications), in view of (3.2), this being, in effect, a *more intrinsic* (yet, in technical terms) *aspect* of the matter. That is, one has;

$$(3.3) \quad \begin{aligned} & \text{“geometric object”, à la Leibniz,} \longleftrightarrow \text{elementary particle} \longleftrightarrow \text{“field”} \\ & \longleftrightarrow \text{Yang-Mills field, viz., a pair,} \\ & (\mathcal{E}, D). \end{aligned} \tag{3.3.1}$$

We explain, right away, the above employed terminology, term by term. Thus, we have:

i) *“geometric object”* (à la Leibniz).— We have already mentioned elsewhere (see A. Mallios [19, Section 1]) that G. W. von Leibniz, just at his time, demanded a *“geometrical calculus”* (*“calcul géométrique”*, see, for instance, N. Bourbaki [4, Chap. I; p. 161 (Note historique), ft.1]) to be found, which should act directly on the *“geometrical objects”*, without the intervention of *coordinates*, that is, in other words, of any *“location of the objects in the “space”*”; of course, concerning the latter function, one certainly needs thereon a *“reference point”* (alias, an *“origin”* (!)). However, this *“fixation”*, in our case, *“accompanies”*, in effect, as we shall see (cf. thus, for instance, iv) below), viz. *“varies” with the* (geometrical) *object* at issue (: *vector sheaf*, cf. (3.2), hence, by definition, a *reference point–“space”– \mathcal{A}* , adjusted thus to the object, under consideration). Of course, the latter issue is of an extremely important significance, pertaining to a *“relativistic perspective”* of the whole matter (cf. also (3.1), as above).

ii) *elementary particle*.— Now, being primarily interested herewith in potential applications of the present point of view to *quantum gravity*, as also the title of this article indicates, it is natural, in principle, to *associate* (: in point of fact, to *identify*) the *“geometric objects”*, as above, *with* the *“elementary particles”*; in other words, the *geometrical objects*, yet, in the sense of Leibniz, *which* still, for that matter, *fill up the “space”*. In this connection, see also A. Mallios (loc. cit.), as well as, [20, (7.2), and subsequent remarks therein].

iii) *“field”*.— It is certainly natural to associate the *“ultimate constituents of the matter”* (: elementary particles) with the notion of a *“field”*, which is also considered (see, for instance, A. Einstein [9, p. 140]), as an *“independent, not further reducible fundamental concept”*, the same correspondence, as above, being still rooted on the classical *“duality”/identification*. Now, by further employing mathematical terminology, we come to the final association/identification, as indicated in (3.3) above, that is, to the fundamental notion, concerning, in effect, the whole account of ADG, namely, that one of a

iv) *Yang-Mills field*, (\mathcal{E}, D) .— Now, the terminology we apply herewith is quite technical, concerning actually, the intrinsic *formalism of ADG*, for which we refer to A. Mallios [16] [17], or even to [21]. So, for convenience, we recall that the pair

$$(\mathcal{E}, D), \quad (3.4)$$

as in (3.3.1) above, consists of a *vector sheaf* \mathcal{E} on an (arbitrary, in principle) topological space X , that is, of a *locally free \mathcal{A} -module* on X , of *finite rank* $n \in \mathbb{N}$, relative to an *algebra sheaf* \mathcal{A} on X , along with a given *\mathcal{A} -connection* D on \mathcal{E} ; now, the latter is, by definition, a *sheaf morphism*,

$$D : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1, \quad (3.5)$$

which is \mathbb{C} -linear (here the (*constant*) *sheaf* \mathbb{C} of the *complexes* is, by assumption, contained in \mathcal{A} , see thus (3.10) below), that also satisfies the pertinent herewith “*Leibniz condition*”: viz. one has the relation,

$$D(\alpha \cdot s) = \alpha \cdot D(s) + s \otimes \partial(\alpha), \quad (3.6)$$

for any (continuous) local sections $\alpha \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$, with U an open subset of X , such that

$$(\mathcal{A}, \partial, \Omega^1) \quad (3.7)$$

is a given *differential triad* on X . Furthermore, Ω^1 stands here for an *\mathcal{A} -module* on X (that occasionally might be too a vector sheaf on X), while

$$\partial : \mathcal{A} \longrightarrow \Omega^1 \quad (3.8)$$

is also a *morphism*, having analogous properties to D , as above (we call it, the *standard*, or even, the *basic \mathcal{A} -connection of \mathcal{A}*); so the corresponding herewith *Leibniz condition for ∂* is now reduced to the relation,

$$\partial(\alpha \cdot \beta) = \alpha \cdot \partial(\beta) + \beta \cdot \partial(\alpha), \quad (3.9)$$

valid, for any α, β in $\mathcal{A}(U)$, with $U \subseteq X$, as in the preceding. Yet, \mathcal{A} is, by hypothesis, a *unital and commutative \mathbb{C} -algebra sheaf* on X , such that one has (: canonical injection),

$$\mathbb{C} \underset{\longrightarrow_{\varepsilon}}{\subset} \mathcal{A}. \quad (3.10)$$

On the other hand, in the special case that the *rank of \mathcal{E}* , as before, *equals 1*, viz. when one has,

$$rk_{\mathcal{A}}\mathcal{E} \equiv rk\mathcal{E} = 1, \quad (3.11)$$

then \mathcal{E} is called, in particular, a *line sheaf* on X , that is also denoted by \mathcal{L} , while the corresponding pair, as in (3.4), by

$$(\mathcal{L}, D), \quad (3.12)$$

that is still named a *Maxwell field* on X . In this connection, we further note that the *electromagnetic field* is, of course (!), a Maxwell field, in the previous sense, that was also our primary motivation to the above employed terminology; however, see also Yu.I. Manin [29], or even [30], as well as, A. Mallios [21, Vol. I: Chapt. III]. In this context, we also remark that, in general, *bosons* are characterized (: identified with) *Maxwell fields*, while *fermions* are similarly associated with *Yang-Mills fields*, that is, with *pairs* (\mathcal{E}, D) , as in (3.4) above, *for which* one has $rk\mathcal{E} = n > 1$. However, for a fuller, as well as, a more precise account thereon, we still refer to A. Mallios [20], or even to [21, Vol. I: Chapt. II].

Thus, after the above brief technical account, we are next going to show, by the subsequent Section, that;

(3.13) based on the above interpretation of the notion of a “*field*”, and, still in conjunction with the very formalism of ADG, we are, in effect, able to look at “*the field itself, as a dynamical variable*”, a fact that, of course, we were always intensively looking for, thus far, when, in particular, confronted with problems of the “*quantum deep*”,

following thus, in that context, the *slogan* that,

(3.14) “*the field itself is* (to be considered, as it actually is (!), for that matter, as) *a dynamical variable*”.

So the application here of ADG affords the above possibility, as in (3.13), while also, *let alone*, that

(3.15) (viz., apart from having the situation, as described by (3.14)) *we are not, moreover, compelled to resort to any background “space”* (alias, “*geometry*”), “*to work with*” (cf. thus the relevant comments of J. Baez, as in (1.8) in the preceding).

On the other hand, the situation, as described, by the latter part of (3.15), was virtually the case (loc. cit.) in the *standard theory* (CDG), when referring, in particular, to the *quantization of the other forces of nature*, alas (!), *except gravity* (: general relativity).

Accordingly, the *shortage of an analogous situation* with that one, *as this was described by (3.15)*, when, in particular, referring to *general relativity*, within the *classical framework* (: CDG), while being especially confronted, in that context,

with problems of the *quantum deep* (let alone with those, pertaining to (3.14), as above, (viz. with “*infinities*” (!)), seems to be, thus far, a “*fundamental culprit*” of the whole issue.

On the other hand, based here, simply, on our *experience from* ADG, the following comments being, in point of fact, the *main moral*, thereby, one comes to the conclusion that:

the aforementioned *shortage* of the *classical theory* (CDG), as this, in particular, concerns *quantization of general relativity*, seems to arrive, as a result, thus far, of *our insistence on having*,

(3.16) (3.16.1) the *classical “dx”* (hence, the whole *standard differential-geometric mechanism* thereof, yet, another conclusion here of ADG (!)), *only* from a “*local presence*” of the classical cartesian \mathbb{R}^n , while, on the other hand, *we still insist to retain*, as well, *as a whole* (!), that particular local presence of *the same “space”*;

that is, *the* (smooth) *manifold* concept itself, concerning our calculations, (!), something, indeed, of a *paramount inconvenience*, pertaining to the aforesaid context.

We are going now, through the ensuing discussion in the following Section 4, to illuminate, as well as, further support the preceding, by referring directly to the nature and the type too of *fundamental differential equations* of the classical theory, that the latter acquire, when perceived *from the point of view of the abstract setup*, as above.

4 Differential equations, within the setting of ADG

Looking at the particular type of “*differential equations*”, that one can formulate, within the above *abstract framework*, as this is advocated by ADG, we are able, in principle, to remark here, yet, on the ground of a similar rationale, as before, that:

(4.1) *evolution* may be perceived, as an “*algebraic automorphism*” (cf., for instance, Feynman); that is, as something of a *relational character* (cf. also Sorkin), which, in turn, can still supply an “*analytic expression*”. So one can associate to it “*numbers*” (occasionally, in the most general sense of the term; here one can think, for instance, of something reminding “*(Gel’fand-)duality*”, thus, e.g., even of a “*generalized*” (!) *spectrum* of an appropriate operator, cf., for instance, in that connection,

Z. Daouldji-Malamou [5] or even [6], therefore, finally, through “*differentiation*” (Hamilton–Schrödinger), providing thus, yet, *algebraically* (!), the “*time operator*” (Heisenberg–Prigogine–Kähler–Hiley).

In this connection, see also B.J. Hiley [14], as well as, A. Mallios [19, (3.27)]. Hence, one thus arrives within the preceding framework, at the conclusion, that the

(4.2) (“*differential*”) equations acquire thus a “*dynamical character*”, more akin to “*second quantization*”, in point of fact, to the “*field*” itself, under consideration, and not merely to the vector states in the carrier space of a particular representation of CCR (: *first quantization*); in this context, the latter simply entails, in effect, the “*carrier space*”, thus, in turn, the supporting “*space-time manifold*” (alias, “*continuum*”), whose presense, however, creates again, as already noted in the foregoing, finally, an, indeed, *fundamental problem* for the whole set-up.

Consequently, as a really *instrumental outcome* of the preceding, one thus realizes that:

(4.3) based on ADG, we are able to refer to the equations of quantum field theory, directly, in terms of the fields themselves; therefore, without the intervention of any “*background space*”, which would provide, according to the classical pattern (CDG), the “*differential-geometric*” apparatus, employed in that framework.

The above constitutes, in effect, as already noted before, the *quintessence*, indeed, of the whole *potential applicability* of “ADG formalism” in problems of quantum field theory; let alone, of course, the fact that one is able, another upshot, as well, of the general theory of ADG, as it was also pointed out in the preceding, to incorporate (classical) “*singularities*” (: infinities, and the like) in (the (local) sections of) the structure sheaf \mathcal{A} .

Therefore, *equivalently*, by referring to our previous last comments, one thus concludes that;

(4.4) the “ADG formalism” can read over (or even, see through) “*singularities*”, in the standard (: CDG) sense of the latter term.

Yet, in other words, one can say, by still applying a language, akin to quantum field theory (cf., for instance, R. Haag [13, p. 326]), that:

(4.5) the “ADG formalism retains the information, one can (locally (!)) get, even through (or else, (locally) supplied (!) by) “*singularities*”.

In this connection, see also, for example, A. Mallios-I. Raptis [26], concerning an appropriate relevant formulation of the well-known “*Finkelstein* (coordinate) *singularities*” [11]. Yet, see A. Mallios [19, (0.6) and subsequent comments therein], for an early account of the same matter.

In toto, by summarizing the preceding, we can thus, finally, say that:

(4.6) the “*differential equations*”, that one obtains, within the *framework of ADG*, *pertaining*, thus directly *to the field itself*, by virtue of the (assumed) correspondences (3.3), hence, *being*, so to say, *in character*, “*second quantized ones*”,

(4.6.1) *represent*, in point of fact, *the very quantized equation(s) of the field* (viz. of the *elementary particle*), in focus.

In this connection, we can further remark that, the manner of understanding the *physis of elementary particles* (thus, of the “*fundamental entities*”), by virtue of the above correspondences (: identifications), as in (3.3) in the foregoing, may still be construed, as being in accord with recent tendencies of taking into account “*dynamic individuation of fundamental entities*” (see J. Stachel [34]). Now, in our case, as above, this can, very likely, be assigned to the notion of the pair,

$$(\mathcal{E}, D), \quad (4.7)$$

as the latter has been applied, throughout the preceding discussion (yet, cf. (3.3), herewith), along with the concomitant *invariance of the whole theory* (: ADG), *under the action of the group*

$$\text{Aut}\mathcal{E} \quad (4.8)$$

(*group sheaf of \mathcal{A} -automorphisms of \mathcal{E}*). Yet, *the latter notion might be perceived*, in point of fact, within our present *abstract perspective*, actually supplied by the ADG formalism, still (Klein) *as a* (“*variant*” (!)) “*space-time*” (see also A. Mallios [19, (3.23) and (3.26)]).

Next, we specialize the preceding, straightforwardly, by the subsequent Section, through concrete fundamental instances of the classical theory:

5 Concrete examples

As already said, our aim in this final Section of the present article is to illuminate the preceding account, by referring, in particular, to *fundamental examples* of the standard situation, thus far: Thus, we start with the following Subsection.

5.(a). Einstein’s equation (in vacuo).— The (“differential”) equation, referred to in the title of this Subsection, has actually, just, the *same form* with the homonymous one, *as in the classical case* (: CDG), however, now, *quite a different meaning* (!). We thus change point of view, as well as, the respective formalism, the latter being now, that one of the *abstract differential geometry* (: ADG), in conjunction with our perspective, in that context, as exhibited by (3.3) in the preceding. So the said equation has also herewith the familiar form,

$$\mathcal{R}ic(\mathcal{E}) = 0, \quad (5.1)$$

which thus in our case is the *Einstein’s equation (in vacuo)*. Now, concerning the technical part of the previous relation (5.1), we still refer to A. Mallios [18], or even, for a full account thereof, to the forthcoming 2-volume detailed treatment in A. Mallios [21, Vol. II: Chapt. IV; Section 3]. For convenience, however, of the ensuing discussion, herewith, we do recall, in brief, the following items about (5.1); that is, one thus sets:

$$\mathcal{R}ic(\mathcal{E}) = tr(R(\cdot, s)t) \equiv tr(R(D_{\mathcal{E}})(\cdot, s)t) : \mathcal{E}(U) \longrightarrow \mathcal{A}(U), \quad (5.2)$$

where s, t are *local* (continuous) *sections* of the *Yang-Mills field* concerned,

$$(\mathcal{E}, D \equiv D_{\mathcal{E}}), \quad (5.3)$$

in such a manner that the $\mathcal{A}(U)$ -*morphism*, as in (5.2) above, stands here for a “*local instance*” (viz., by restriction to a *local gauge* $U \subseteq X$ of \mathcal{E}) of the so-called *Ricci operator* of

$$\mathcal{E} \equiv (\mathcal{E}, D_{\mathcal{E}} \equiv D), \quad (5.4)$$

such that one further defines;

$$\mathcal{R}ic(\mathcal{E}) \equiv (\mathcal{R}ic_U(\mathcal{E}) \equiv \mathcal{R}ic(\mathcal{E})), \quad (5.5)$$

with U running over a given *local frame* of \mathcal{E} , the last relation yielding thus the first member of (the *equation*) (5.1), as an \mathcal{A} -*morphism* (as it actually entails any (“*differential*”) *equation*, whatsoever, cf. also (5.10) in the sequel) of the \mathcal{A} -*modules* (in fact, *vector sheaves*, see thus below) concerned, locally identified, through (5.2).

Now, the \mathcal{A} -*module* \mathcal{E} , as briefly indicated by (5.4) above, that is involved herewith, is, in point of fact, a “*Lorentz vector sheaf*” (loc. cit., Chapt. IV; (2.14), or even Note 3.1 therein) on a given *topological space* X , common base space, by definition, of all the \mathcal{A} -*modules* (*sheaves*) appeared throughout. Furthermore, within this same context, one assumes an appropriate “*differential triad*” on X ,

$$(\mathcal{A}, \partial, \Omega^1) \quad (5.6)$$

(see also Section 3 in the preceding for the relevant terminology applied here), while we still suppose that, in particular, one has;

$$\Omega^1 = \mathcal{E}^*, \quad (5.7)$$

the second member of (5.7) standing for the “*dual*” *vector sheaf* of \mathcal{E} (ibid. Chapt. IV; Section 3, see, in particular, Definition 3.1, along with the subsequent Scholium 3.1 therein).

On the other hand, by further looking at (5.1), as above, we also remark that *any field, that is* (see thus (3.2), or even (3.3.1) in the foregoing), *a pair, as in* (5.3) (but, *see also* (5.4), for an abbreviated form, of common usage too), *that satisfies* (5.1), thus, in other words, a “*solution of Einstein’s equation*”, appears in the latter equation, *by itself*, or even, precisely speaking, via its “*field strength*” (: *curvature*),

$$R(D_{\mathcal{E}}) \equiv R(D) \equiv R, \quad (5.8)$$

cf. thus (5.2) above.

Yet, in this regard, we should further remark that, based on the preceding (see thus (3.3.1), or even (5.4)), and “*dynamically speaking*”, so to say, we also assume, throughout the present discussion, the *basic correspondence* (: *identification*),

$$\text{field} \longleftrightarrow D_{\mathcal{E}} \equiv D, \quad (5.9)$$

as it concerns, in effect, a given pair (: a *Yang-Mills field*), as in (5.4). However, an \mathcal{A} -*connection* D , as above, is, by its very definition (cf. (3.5) in the preceding), only, a \mathbb{C} -*linear morphism*, therefore, *not a “tensor”*, thus, technically speaking, not an \mathcal{A} -*morphism* of the \mathcal{A} -modules concerned (ibid.), as it actually is, its *curvature* (: *field strength*), $R(D) \equiv R$, hence, the *appearance of the latter in the corresponding equations, describing the field* at issue: Indeed, something that we already hinted at in the preceding (cf. thus the pertinent comments, following (5.5) above), we should explicitly point out herewith the *fundamental principle*, in effect, that,

(5.10) *the (“differential”) equations, describing a field* (yet, otherwise, by obviously “*abusing language*” (!), herewith, the “*equations of a field*”), *are to be formulated, via “tensors”*; hence, in other words, in terms always of \mathcal{A} -*morphisms*, the equation itself *entailing* thus an \mathcal{A} -*morphism*, as well (*expressing*, by its very substance, for that matter, a *physical law*, that one, *determined by the “field”*, in focus).

In this connection, we can still say that the above may also be construed, as another upshot of the same “*principle of general covariance*”; in this regard, see

also, for instance, D. Bleecker [3, p. 50, Section 3.3, in particular, p. 52, Theorem 3.3.6]. Yet, within the same vein of ideas, we can further remark, in point of fact, that;

(5.11) when *considering the* (“differential”) *equations, describing fields* (: *natural laws*), as being *\mathcal{A} -morphisms* (see (5.10) above), this may be construed, in effect, as another, just, technical (!), *equivalent* expression of the same “*principle of general covariance*”.

On the other hand, by specializing our previous considerations in the preceding Section 4 (cf., for instance, (4.6) therein) to the case of the equation (5.1), thus, by further looking at the particular issues, involved in the same equation, one realizes that;

(5.12) *Einstein’s equation* (in vacuo) refers to *the field itself*, that is, *eo ipso* to the respective *quantum* (hence, for the case at issue, to the “*graviton*”). Therefore, it is, by its very formulation, already a *quantized equation*, and, as a matter of fact, a “*second quantized*” one, therefore, an *equation*, within the setting of *quantum field theory*.

Indeed, the *sheaf-theoretic character* of the framework, within which that equation has been formulated, provides also its *relativistic perspective*, being thus, at the same time, as already remarked (see comments following (5.5)), a *covariant* one, as well.

On the other hand, by further continuing our concrete specialization of the preceding (see thus our general comments on “*quantizing gravity*” in Section 1, or even (2.6) above) to the particular case, considered by the present Subsection, we can still remark that, what is to be viewed, herewith, as *of a particular significance*, especially *pertaining to* problems of “*quantum gravity*”, being also *in complete diversification* with the manner *we apply*, in that context, the *classical theory* (: CDG), see, for instance, (1.5), (1.9), or even (2.6) in the preceding, *is the following fact*, already mentioned, generally speaking, in the foregoing. Namely, one can still remark here that:

the whole formalism of ADG, according to its very definition,

(5.13.1) *is entirely “space-independent”*,

(5.13) in the usual sense of this term. That is, by applying herewith classical parlance (loc. cit.),

(5.13.2) one does not need any “*background geometry to work with*”,

while, at the same time, *this same “geometry”*, within the present *abstract set-up* (: ADG), being, in point of fact, represented by (alias, emanated from) the same *“structure sheaf of coefficients”*, \mathcal{A} (viz. our *“generalized arithmetics”*),

(5.13.3) *is actually still varied with us (!)*, as well,

according to the very definition of the objects, that are entangled in the equation, for instance, (5.1), as above. For,

(5.13.4) *“everything [there] boils down [locally (!)] to \mathcal{A} ”*.

Yet, as a result of the preceding, we can still say that;

(5.14) reflecting, within the framework of ADG, we realize that the *“observer”* (viz. *“we”*, to the extent that this is expressed, through our *“arithmetics”* \mathcal{A}) becomes a *“dynamical variable”*, as well, acquiring thus too, a *“relativistic character”*.

Thus, the following claim has here its relative position, being also in accord with previous similar considerations; that is, one can say that

(5.15) *“all is dynamical” (!)*

The above might also be related with J. Stachel’s, quite recently stated, *principle of dynamic individuation of the fundamental entities*”, see thus [34, p. 32].

Now, by still commenting on our previous remark in (5.13.4), as above, we further note that our last indication (*“locally”*) therein reminds us, of course, of a quite recent comment of R. Haag [13, p. 326] in that,

(5.16) *“all information characterizing the [quantum field] theory is strictly local”*,

this being also,

(5.17) *“the central message”* of nowadays Quantum Field Theory (loc. cit.).

In this connection, it is further worth mentioning here that the same author, as above (ibid.), refers to the aforesaid situation, about today QFT, while advocating a *sheaf-theoretic approach* to that theory, as being thus more akin to the *“local character”* of the latter (one thus considers here, neighborhoods), in contradistinction with the *“point-character”* of *fiber bundle* theory, (e.g. vector bundles), that has been employed, so far. On the other hand, by

(5.18) *applying ADG*, one also gets, via its overall *sheaf-theoretic character*, at a *“synthesis of the knowledge gained in . . . different approaches”*,

a fact, that was also in perspective, by the aforesaid author (loc. cit.). Thus, in the case of ADG, one has, for instance, the following *synthesis*:

$$\text{ADG} \longleftrightarrow \text{CDG} \quad (5.19.1)$$

(5.19) in a *new perspective*, since *no Calculus* is employed, *at all (!)*, plus

$$\textit{sheaf theory}, \quad (5.19.2)$$

in conjunction with *sheaf cohomology*.

We terminate the present discussion, by still pointing out, within the preceding framework, another, indeed, *definitive aspect of the formalism of ADG*, that we have also hinted at in the foregoing, within the abstract setting of our general commentary herein; namely, the

(5.20) *possibility of working*, within the framework of ADG, by employing functions, in point of fact, *local sections* of \mathcal{A} , that may have/incorporate a large, in effect, *the biggest, thus far (!)*, amount of “*singularities*”, in the classical sense of this term, *as if* (a significant, indeed, advantage of the aforesaid mechanism) the latter classical *anomalies* of the standard theory (CDG) *were not present, at all (!)*

Thus, to say it, once more, emphatically,

(5.21) *the formalism of ADG* can, indeed “*absorb*” the “*singularities*” of the classical theory,

by appropriately chosen \mathcal{A} (see also (4.4), (4.5) in the preceding). In this connection cf. A. Mallios-E.E. Rosinger [27], [28], for an early account of the subject, as well as, the recent work in A. Mallios-I. Raptis [26]; yet, cf. A. Mallios [19], along with A. Mallios [21, Vol. II: Chapt. IV; Subsection 5.(b)], concerning a *more general perspective* thereon; yet, the latter can actually be viewed, as the outcome of some recent *categorical considerations*, pertaining to the *formalism of ADG*, as presented in the relevant work of M. Papatriantafillou [31], [32].

5.(b). Yang-Mills equations.— As it was the case in the preceding with Einstein’s equation (in vacuo), see equa. (5.1) above, the equations in the title of the present Subsection, still retain, within the abstract set-up, employed herewith, the familiar form, they have, in the standard setting of the classical theory (CDG). Thus, the aforesaid equations preserve, here too, their classical

form, within, of course, the appropriate now formalism adapted to ADG. That is, one gets at the relations;

$$\delta_{\mathcal{E}nd\mathcal{E}}(R) = 0, \quad (5.22)$$

or even, *equivalently*,

$$\Delta_{\mathcal{E}nd\mathcal{E}}(R) = 0, \quad (5.23)$$

which thus constitute, within the *abstract setting of ADG*, the corresponding *Yang-Mills equations*.

Now, concerning the relevant technical details, connected with the previous equations, we still refer to A. Mallios [16], or even, for a complete account thereof, to A. Mallios [21, Vol. II: Chapt. III]. For convenience, of course, we simply recall (*loc. cit.*), that \mathcal{E} here stands for a *Yang-Mills field*, as in (5.3) above (*cf. also*, for instance, (5.4)), whose *field strength* is R (*cf.* (5.8)). Thus, here too, one realizes that;

(5.24) *a complete analogous rationale, as that one, we have exhibited above, pertaining to Einstein's equation (in vacuum), is also in force, referring now to the Yang-Mills equations, as in (5.22)/(5.23).*

In particular, see our previous remarks in (5.12), as well as, in (5.13), being, in that context, of a special significance, from a *quantum relativistic* point of view, in connection with nowadays aspects on the matter, as this was explained in the previous Subsection 5.(a), concerning, in particular, therein, Einstein's equation (in vacuo). Yet, in this connection, *cf. also*, for instance, the relevant critique of P.G. Bergmann [2], especially pertaining to an appropriate "*physicalization of geometry*" (!), the latter perspective being, in point of fact, quite akin to the *abstract point of view*, that has been also advocated, by the present study, as another *potential application of ADG* (see also [22]).

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